GEOMETRIC GENERALIZATIONS OF THE SQUARE SIEVE, WITH AN APPLICATION TO CYCLIC COVERS

ALINA BUCUR, ALINA CARMEN COJOCARU, MATILDE N. LALÍN, AND LILLIAN B. PIERCE
with an appendix by Joseph Rabinoff

Abstract. We formulate a general problem: given projective schemes $\mathbb{Y}$ and $\mathbb{X}$ over a global field K and a K-morphism η from $\mathbb{Y}$ to $\mathbb{X}$ of finite degree, how many points in $\mathbb{X}(K)$ of height at most B have a pre-image under η in $\mathbb{Y}(K)$? This problem is inspired by a well-known conjecture of Serre on quantitative upper bounds for the number of points of bounded height on an irreducible projective variety defined over a number field. We give a non-trivial answer to the general problem when $K = \mathbb{F}_q(T)$ and $\mathbb{Y}$ is a prime degree cyclic cover of $\mathbb{X} = \mathbb{P}^n_K$. Our tool is a new geometric sieve, which generalizes the polynomial sieve to a geometric setting over global function fields.

1. Introduction

We consider the following general problem: given a morphism between two projective schemes defined over a global field, how many points in the domain yield points with bounded height in the image? As we will outline, this problem is related to a well-known conjecture of Serre formulated over number fields; in our proposed formulation, it may be viewed as a general version of a question that arises in a wide array of problems in analytic number theory.

To be precise, let K be a global field of arbitrary characteristic and let $\mathcal{O}$ be a ring of integers in K. We denote by V_K the set of places of K. For each $v \in V_K$, we denote by $|\cdot|_v$ the associated valuation, normalized such that the product formula holds, namely, for every $x \in K^*$,

$$\prod_{v\in V_K} |x|_v = 1. \tag{1.1}$$

Using the notation

$$[x] := [x_0 : x_1 : \ldots : x_n] \in \mathbb{P}^n_K$$

for projective points, we consider the height function

$$\mathrm{ht}_K : \mathbb{P}^n_K \to (0,\infty),$$
$$\mathrm{ht}_K([x]) := \prod_{v\in V_K} \max\{|x_0|_v, \ldots, |x_n|_v\}, \tag{1.2}$$

and note that it gives rise to the height function

$$\mathrm{ht}_K : K \to [0,\infty),$$
$$\mathrm{ht}_K(x) := \begin{cases} \mathrm{ht}_K([x:1]) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{1.3}$$

General Problem

Given a global field K of arbitrary characteristic, a ring of integers $\mathcal{O}$ in K, projective schemes $\mathbb{X}/K$, $\mathbb{Y}/K$ over K with fixed models over $\mathcal{O}$, and a K-morphism $\eta : \mathbb{Y} \longrightarrow \mathbb{X}$, defined over $\mathcal{O}$ and of finite degree, find an upper bound for the cardinality of the set

$$\{[x] \in \mathbb{X}(\mathcal{O}) : \mathrm{ht}_K([x]) < B, \exists [y] \in \mathbb{Y}(\mathcal{O}) \text{ such that } \eta([y]) = [x]\}$$

Date: August 12, 2022.

that holds for every $B \geq 1$.

Note that upper bounds for the above cardinality are always given by one of the two cardinalities below,

$$\#\{[x] \in \mathbb{X}(K) : \mathrm{ht}_K([x]) < B\} \leq \#\{[x] \in \mathbb{P}^n_K : \mathrm{ht}_K([x]) < B\}.$$

For example, when K is a number field of degree d over $\mathbb{Q}$, by Schanuel's theorem (e.g. [Ser97, §2.5 p. 17]), there exists an explicit positive constant $C(K, n)$ such that, as $B \to \infty$,

$$\#\{[x] \in \mathbb{P}^n_K : \mathrm{ht}_K([x]) < B\} \sim C(K, n) B^{d(n+1)}. \tag{1.4}$$

As a second example, when K is the function field of an absolutely irreducible projective curve over $\mathbb{F}_q$, of genus g, as an immediate consequence of [Ser97, §2.5, Thm. p. 19], there exists an explicit positive constant $C(K, n)$ such that, as $b \to \infty$,

$$\#\left\{[x] \in \mathbb{P}^n_K : \mathrm{ht}_K([x]) < q^b\right\} \sim C(K, n) q^{(b-g)(n+1)}. \tag{1.5}$$

As usual when navigating between the number field and the function field settings, the parameter B in (1.4) was replaced by q^b in (1.5).

In our general problem, for nontrivial choices of $\mathbb{X}, \mathbb{Y}$, we seek a nontrivial upper bound, namely a bound that grows more slowly than the trivial bound, as $B \to \infty$ (respectively, as $q^b \to \infty$ as a function of b, or of q, or of both b and q).

1.1. **Serre's question.** Our General Problem has an antecedent in a well-known question of Serre [Ser97, §13.1 (4) p. 178], which we now recall.

Let $K/\mathbb{Q}$ be a number field of degree d, let $n \geq 1$ be an integer, and let V be an irreducible (non-linear) projective variety in $\mathbb{P}^n_K$. Serre seeks an upper bound in B for the cardinality of the set

$$\{[x] \in V(K) : \mathrm{ht}_K([x]) \leq B\}. \tag{1.6}$$

The trivial upper bound for (1.6) is $C(K, n)B^{d(n+1)}$, as mentioned in (1.4). In [Ser97, §13.1, Thm. 4 p. 178], Serre improves upon the trivial bound by showing that there exists a constant $0 < \gamma < 1$ such that, for all B,

$$\#\{[x] \in V(K) : \mathrm{ht}_K([x]) \leq B\} \ll_{n,K,V} (B^d)^{(n+1)-\frac{1}{2}} (\log B)^\gamma. \tag{1.7}$$

Serre deduces (1.7) from a result counting integral points on affine thin sets, which he proves using the large sieve. A variant due to Cohen [Coh81] of the result counting integral points on affine thin sets may also be used; however, Cohen's result leads to $\gamma = 1$. Serre then poses the question of whether (1.7) can be improved to

$$\#\{[x] \in V(K) : \mathrm{ht}_K([x]) \leq B\} \ll (B^d)^{(n+1)-1} (\log B)^c \tag{1.8}$$

for some $c \geq 0$, without specifying whether the implied $\ll$-constant might depend on any of n, K, V; see [Ser97, §13.1.3, p. 178]. Additionally, Serre notes that the logarithmic factor is necessarily present in certain cases.

Our General Problem is a generalization of Serre's question and, as a special case, encompasses a global function field version of Serre's question (1.8). The specific case of $K = \mathbb{F}_q(T)$ has been studied recently by Browning and Vishe, who proved an analogue of (1.7) by adapting Serre's argument, using a version of the large sieve inequality over function fields developed by Hsu [Hsu96]; see [BV15, Lemma 2.9] where their result is stated in an affine formulation. In particular, Browning and Vishe commented on the scarcity of results counting points of bounded height on geometrically irreducible (non-linear) varieties in the function field setting [BV15, p. 675]; this paper explores a particular class of such problems.

1.2. **Main goals.** The purpose of the present paper is to investigate the General Problem in a particular function field setting, and to go beyond the analogue of (1.7) in the case of prime degree cyclic covers. Precisely, our goals are two-fold:

(I) to provide a nontrivial upper bound for the General Problem when $K = \mathbb{F}_q(T)$, $\mathcal{O} = \mathbb{F}_q[T]$, $\mathbb{X} = \mathbb{P}^n_K$, $\mathbb{Y}$ is a prime degree cyclic cover of $\mathbb{X}$, and η is the natural projection;

(II) to accomplish (I) by developing a geometric sieve method which generalizes recent sieve methods (such as the square sieve of Heath-Brown and the polynomial sieve of Browning) that have been used to improve on Serre's bound (1.7) in the setting over $\mathbb{Q}$.

We will present our main results in the next two sections, according to the above two goals.

1.3. **Main results I: counting rational points.** We treat the General Problem in the following concrete case: $K = \mathbb{F}_q(T)$, $\mathcal{O} = \mathbb{F}_q[T]$, $\mathbb{X} = \mathbb{P}^n_K$, $\mathbb{Y}$ a prime degree cyclic cover of $\mathbb{X}$, and η the natural projection.

To be precise, let q be an odd rational prime power, $n \geq 1$ an integer, $m \geq 2$ an integer, and $\ell \geq 2$ a rational prime such that $\ell \mid m$. We set

$$\mathbb{X} := \mathbb{P}^n_{\mathbb{F}_q(T)}$$

and take $\mathbb{Y}$ as the projective scheme in the weighted projective space $\mathbb{P}^{n+1}_{\mathbb{F}_q(T)}\left(1, \ldots, 1, \frac{m}{\ell}\right)$ defined by the weighted projective model

$$\mathbb{Y}: \qquad X_{n+1}^{\ell} = F(X_0, \ldots, X_n) \tag{1.9}$$

for some polynomial $F \in \mathbb{F}_q[T][X_0, \ldots, X_n]$ of total degree m in $X_0, \ldots, X_n$. We take

$$\eta : \mathbb{Y} \longrightarrow \mathbb{P}^n_{\mathbb{F}_q(T)}$$

as the natural projection defined by

$$\eta([x_0 : x_1 : \ldots : x_n : x_{n+1}]) := [x_0 : x_1 : \ldots : x_n]. \tag{1.10}$$

Our interest is in estimating, from above and as a function of q^b (q fixed, $b \to \infty$), the counting function

$$N(\mathbb{Y}, \mathbb{F}_q(T), \eta; b) := \# \left\{ [x] \in \mathbb{P}^n_{\mathbb{F}_q(T)} : \mathrm{ht}_{\mathbb{F}_q(T)}[x] < q^b, \ \exists [y] \in \mathbb{Y}(\mathbb{F}_q(T)) \text{ such that } \eta([y]) = [x] \right\}, \tag{1.11}$$

or, equivalently, the counting function

$$\# \left\{ (x_0, \ldots, x_n) \in \mathbb{F}_q[T]^{n+1} : \ \deg_T(x_i) < b \ \forall 0 \leq i \leq n, \exists x_{n+1} \in \mathbb{F}_q[T] \text{ such that } x_{n+1}^{\ell} = F(x_0, \ldots, x_n) \right\},$$

where $\deg_T(x_i)$ denotes the degree of x_i as a polynomial in T.

Note that, in this setting, the trivial bound is

$$N(\mathbb{Y}, \mathbb{F}_q(T), \eta; b) \leq \# \left\{ [x] \in \mathbb{P}^n_{\mathbb{F}_q(T)} : \mathrm{ht}_{\mathbb{F}_q(T)}([x]) < q^b \right\} \leq \left(q^b \right)^{(n+1)} \tag{1.12}$$

(see (1.5) and the comment in equation (5.1) of §5).

In contrast, the function field analogue of Serre's conjecture (1.8) suggests that it might be possible to prove, under appropriate conditions on F, that there exists some constant c for which

$$N(\mathbb{Y}, \mathbb{F}_q(T), \eta; b) \ll_{\ell,m,n,q,F} \left(q^b \right)^{(n+1)-1} b^c, \tag{1.13}$$

with the implicit constant possibly depending on ℓ, m, n, q, F.

If F of degree $m \geq 2$ is such that $F(X_0, \ldots, X_n) = 0$ defines a nonsingular projective hypersurface, Browning and Vishe's work in [BV15] implies that

$$N(\mathbb{Y}, \mathbb{F}_q(T), \eta; b) \ll_{\ell,m,n} \left(q^b \right)^{(n+1)-1/2} b \log q. \tag{1.14}$$

This establishes a benchmark of roughly analogous strength to (1.7), and is the first improvement of the trivial bound (1.12). This can be derived by applying [BV15, Lemma 2.9] to count solutions on the affine model $X_{n+1}^{\ell} = F(X_0, \ldots, X_n)$ which is irreducible under the condition on F; see (3.1) to interpret the height function when applying this result. (While we focus exclusively on cyclic covers, we remark that Browning and Vishe's work applies more generally; see [BV15, p. 674] and [BV15, Lemmas 2.9, 2.10] for more general results counting points of bounded height on absolutely irreducible (non-linear) varieties in affine and projective settings, of equivalent strength to (1.7).)

Our first main theorem improves upon the trivial bound (1.12) as well as (1.14), and approaches, in the limit as $n \to \infty$ (upon omitting an analysis of how the limit impacts the dependence of the $\ll$-constant on n), the upper bound $\left(q^b\right)^{(n+1)-1}$ appearing in (1.13), as long as the defining polynomial F is such that $F(X_0, \ldots, X_n) = 0$ defines a nonsingular projective hypersurface.

Our first main theorem is:

Theorem 1.1 (Counting Rational Points on a Prime Degree Cyclic Cover of $\mathbb{P}^n_{\mathbb{F}_q(T)}$)**.** *Let q be an odd rational prime power, $n \geq 2$ an integer, $\ell \geq 2$ a rational prime, and $F \in \mathbb{F}_q[T][X_0, \ldots, X_n]$ a homogeneous polynomial of degree $m \geq 2$ in $X_0, \ldots, X_n$, with $\operatorname{char} \mathbb{F}_q \nmid m$. Assume that:*

(i) *$\ell \mid \gcd(m, q-1)$;*

(ii) *$F(X_0, \ldots, X_n) = 0$ defines a nonsingular projective hypersurface in $\mathbb{P}^n_{\overline{\mathbb{F}_q(T)}}$.*

Let $\mathbb{Y}$ be the projective scheme in the weighted projective space $\mathbb{P}^{n+1}_{\mathbb{F}_q(T)}\left(1, \ldots, 1, \frac{m}{\ell}\right)$ defined by the weighted projective model (1.9). Let η be the projection (1.10). Then for all $b \geq 1$ the quantity $N(\mathbb{X}, \mathbb{Y}, \mathbb{F}_q(T), \eta; b)$ defined in (1.11) satisfies the bound

$$N(\mathbb{Y}, \mathbb{F}_q(T), \eta; b) \ll_{\ell,m,n,q,F} \left(q^b\right)^{(n+1)-\frac{n+1}{n+2}} b^{\frac{n+1}{n+2}},$$

where the implicit constant depends on ℓ, m, n, q, and F.

Later on in Theorem 5.1, we will use the function mentioned in the displayed equation below (1.11) to state a version of Theorem 1.1 in terms of counting perfect ℓ-th power values of a homogeneous polynomial $F \in \mathbb{F}_q[T][X_0, \ldots, X_n]$. For more information on the way in which the implicit constant depends on F, see §9.

To put Theorem 1.1 in context, let us recall the current state of knowledge toward Serre's conjecture (1.8) when $K = \mathbb{Q}$. For $n = 1, 2$, Broberg [Bro09] proved a weak form of Serre's conjecture (with B^{ϵ} in place of a logarithmic factor) via the determinant method. For $n \geq 3$ and in the case of cyclic covers of degree ℓ, the power sieve argument presented by Munshi in [Mun09] leads to the upper bound

$$\ll_{\ell,m,n,F} B^{(n+1)-\frac{n}{n+1}} (\log B)^{\frac{n}{n+1}}, \tag{1.15}$$

where F is the defining polynomial of the cover and m is its degree. Recently, Bonolis [Bon21] refined the argument given in [Mun09] and obtained the upper bound

$$\ll_{\ell,m,n,F} B^{(n+1)-\frac{n+1}{n+2}} (\log B)^{\frac{n+1}{n+2}}. \tag{1.16}$$

(For clarity, we remark that Theorem 1.1 of [Mun09] states a bound of the strength (1.16), but the argument as written therein proves a result of the strength (1.15). At the suggestion of Munshi, Bonolis [Bon21] implemented nontrivial averaging in the relevant sieve inequality in order to prove (1.16) (as well as a more general result over $\mathbb{Q}$).) Our result in Theorem 1.1 is thus an analogue over $\mathbb{F}_q(T)$ of the result (1.16) over $\mathbb{Q}$. Note that, in the limit $n \to \infty$ and aside from an analysis of how the limit impacts the dependence of the $\ll$-constant on n, the upper bound (1.15) or (1.16) approaches one of the form conjectured in (1.8).

Over $\mathbb{Q}$, the best known result is due to Heath-Brown and the third author [HBP12], who proved Serre's conjecture (1.8) for all $n \geq 9$ in the case of cyclic covers, by combining a sieve method with the q-analogue of Van der Corput's method. It would be interesting to to adapt Heath-Brown and Pierce's q-analogue of Van der Corput method to the function field setting of Theorem 1.1.

1.4. **Main results II: geometric sieve inequalities.** Our approach to the General Problem proceeds via a sieve, formulated in the setting of the General Problem, with K a global field of arbitrary characteristic, $\mathcal{O}$ a ring of integers in K, $\mathrm{ht}_K : K \longrightarrow [0,\infty)$ the height function (1.3) constructed from valuations that satisfy the product formula (1.1), $\mathbb{X}/K$ and $\mathbb{Y}/K$ projective schemes over K with fixed models over $\mathcal{O}$, and $\eta : \mathbb{Y} \longrightarrow \mathbb{X}$ a K-morphism, defined over $\mathcal{O}$ and of finite degree.

As it will not result in any significant loss in sharpness of the results, we will, for convenience, count points on $\mathbb{X}$ in the affine sense. To clarify, using the notation

$$\underline{x} := (x_0, x_1, \dots, x_n) \in \mathbb{A}_K^{n+1}$$

for affine points, we will work with the height function in the affine space $\mathbb{A}_K^{n+1}$ given by

$$\mathrm{ht}_K(\underline{x}) := \max\left\{\mathrm{ht}_K(x_i) : \underline{x} = (x_0, \dots, x_n)\right\}, \tag{1.17}$$

focus on the set

$$\mathcal{A} = \mathcal{A}(B) := \{\underline{x} \in \mathbb{X}(\mathcal{O}) : \mathrm{ht}_K(\underline{x}) < B\}, \tag{1.18}$$

and seek an upper bound, in terms of B, for the cardinality of the set

$$\mathcal{S}(\mathcal{A}) = \mathcal{S}(\mathcal{A}(B)) := \left\{\underline{x} \in \mathcal{A} : \ \exists \underline{y} \in \mathbb{Y}(\mathcal{O}) \text{ such that } \eta(\underline{y}) = \underline{x}\right\}. \tag{1.19}$$

The typical sieve approach is to derive information about $\mathcal{S}(\mathcal{A})$ from reductions of η modulo primes. Towards this goal, for each finite place $v \in V_K$, we denote by $(\mathcal{O}_v, M_v)$ the associated discrete valuation ring and by $k_v := \mathcal{O}_v/M_v$ the associated residue field. For all but finitely many finite places v (in which case we refer to v as a place of good reduction for η), we denote by $\eta_v : \mathbb{Y} \to \mathbb{X}$ the reduction of η modulo v. We call η_v *ramified* at some $\underline{x} \in \mathbb{X}(\mathcal{O}_v)$ if $\underline{x} \pmod v \in \mathbb{X}(k_v)$ is a branch point for the function $\eta_v : \mathbb{Y} \to \mathbb{X}$. For each $\underline{x} \in \mathbb{X}(\mathcal{O})$, we define the set

$$V_K^{\mathrm{ram}}(\underline{x}, \eta) := \{v \in V_K : v \text{ of good reduction for } \eta \text{ and } \eta_v \text{ is ramified at } \underline{x} \pmod v\}.$$

Moreover, for any nonempty finite set $\mathcal{P} \subseteq V_K$ of finite places of good reduction for η, we define the subset

$$V_{\mathcal{P}}^{\mathrm{ram}}(\underline{x}) := V_K^{\mathrm{ram}}(\underline{x}, \eta) \cap \mathcal{P}. \tag{1.20}$$

With this notation, we prove the following sieve inequality:

Theorem 1.2. *(Geometric Sieve)*
Let K be a global field of arbitrary characteristic, $\mathcal{O}$ be the ring of integers in K, and $\mathrm{ht}_K : K \longrightarrow [0,\infty)$ the height function (1.3) *constructed from valuations that satisfy the product formula* (1.1)*. Let $\mathbb{X}/K$, $\mathbb{Y}/K$ be projective schemes over K with fixed models over $\mathcal{O}$ and let $\eta : \mathbb{Y} \longrightarrow \mathbb{X}$ be a K-morphism, defined over $\mathcal{O}$ and of finite degree. For an arbitrary $B > 0$, define the sets $\mathcal{A}$ and $\mathcal{S}(\mathcal{A})$ as in (1.18), (1.19). Then, for any real number $\alpha \geq 1$ and for any nonempty finite set $\mathcal{P} \subseteq V_K$ of finite places of good reduction for η,*

$$|\mathcal{S}(\mathcal{A})| \leq \frac{2}{|\mathcal{P}|} \sum_{\underline{x} \in \mathcal{A}} |V_{\mathcal{P}}^{\mathrm{ram}}(\underline{x})| + \frac{1}{|\mathcal{P}|^2} \sum_{\underline{x} \in \mathcal{A}} I_\alpha(\underline{x})^2, \tag{1.21}$$

where, for any $\underline{x} \in \mathbb{X}(\mathcal{O})$,

$$I_\alpha(\underline{x}) := \sum_{v \in \mathcal{P} \setminus V_{\mathcal{P}}^{\mathrm{ram}}(\underline{x})} \left(\alpha + \left(\left|\eta_v^{-1}(\underline{x} \pmod v)\right| - 1\right) \cdot \left(\deg \eta - \left|\eta_v^{-1}(\underline{x} \pmod v)\right|\right)\right).$$

Impetus for inequality (1.21) comes from a sequence of papers on sieve inequalities, beginning with the square sieve over $\mathbb{Q}$ of [HB84] (itself inspired by [Hoo78]), which was later developed into the power sieve over $\mathbb{Q}$ in [Mun09] and [Bra15], and into the square sieve over $\mathbb{F}_q(T)$ in [CD08]. The square and power sieves over $\mathbb{Q}$ were strengthened by being combined with the q-analogue of Van der Corput's method in [Pie06] and [HBP12]. Most recently, Browning [Bro15] expanded the square sieve over $\mathbb{Q}$ into a polynomial sieve over $\mathbb{Q}$, while Bonolis [Bon21] developed a related polynomial sieve over $\mathbb{Q}$ involving expansions via trace functions. It is Browning's work [Bro15] that motivates our approach towards generalizing the existing versions of the square sieve (over $\mathbb{Q}$ and over $\mathbb{F}_q(T)$) to a geometric sieve over global fields. We will make an explicit comparison to the square sieve, polynomial sieve, and other relatives in §2.4.

Since we will deduce Theorem 1.1 from the above sieve inequality, we now state the relevant consequence of Theorem 1.2 in the case of cyclic covers of prime degree that we derive by optimizing the choice of α.

Theorem 1.3. *(Geometric Sieve for Prime Degree Cyclic Covers of $\mathbb{P}^n_K$)*
Let K be a global field of arbitrary characteristic, let $\mathcal{O}$ be the ring of integers in K, and let $\mathrm{ht}_K : K \longrightarrow [0,\infty)$ be the height function (1.3) *constructed from valuations that satisfy the product formula* (1.1)*. Let $n, m \geq 1$ be integers, take $\mathbb{X} := \mathbb{P}^n_K$, and let $\mathbb{Y}$ be the projective scheme in the weighted projective space $\mathbb{P}^{n+1}_K\left(1,\ldots,1,\frac{m}{\ell}\right)$, defined by a model*

$$X_{n+1}^{\ell} = F(X_0,\ldots,X_n) \tag{1.22}$$

with ℓ prime such that $\ell \mid m$, and with $F \in \mathcal{O}[X_0,\ldots,X_n]$ homogeneous of degree m and having the property that the hypersurface in $\mathbb{P}^n_{\overline{K}}$ defined by

$$F(X_0,\ldots,X_n) = 0$$

is nonsingular. Let $\eta : \mathbb{Y} \longrightarrow \mathbb{X}$ be the cyclic map of degree ℓ defined over $\mathbb{A}^n_{\overline{K}}$ by

$$\eta(x_0,x_1,\ldots,x_n,x_{n+1}) := (x_0,x_1,\ldots,x_n). \tag{1.23}$$

For an arbitrary $B > 0$, define $\mathcal{A}$ and $\mathcal{S}(\mathcal{A})$ as in (1.18), (1.19). Then, for any nonempty finite set $\mathcal{P} \subseteq V_K$ of finite places of good reduction for η,

$$\begin{aligned}|\mathcal{S}(\mathcal{A})| \leq & \frac{|\mathcal{A}|}{|\mathcal{P}|}(\deg\eta - 1)^2 + \frac{2}{|\mathcal{P}|}\sum_{\underline{x}\in\mathcal{A}} |V_{\mathcal{P}}^{\mathrm{ram}}(\underline{x})| \\ & + \frac{1}{|\mathcal{P}|^2}\sum_{\substack{v_1,v_2\in\mathcal{P}\\ v_1\neq v_2}}\left|\sum_{\substack{\underline{x}\in\mathcal{A}\\ \underline{x}\notin\mathbb{X}_{\mathcal{O}}^{\mathrm{ram}}(v_1)\cup\mathbb{X}_{\mathcal{O}}^{\mathrm{ram}}(v_2)}} \left(\left|\eta_{v_1}^{-1}(\underline{x}\,(\mathrm{mod}\,v_1))\right| - 1\right)\cdot\left(\left|\eta_{v_2}^{-1}(\underline{x}\,(\mathrm{mod}\,v_2))\right| - 1\right)\right|,\end{aligned} \tag{1.24}$$

where, for each finite place $v \in V_K$ of good reduction for η,

$$\mathbb{X}_{\mathcal{O}}^{\mathrm{ram}}(v) := \left\{\underline{x} \in \mathbb{X}(\mathcal{O}) : \eta_v \textit{ is ramified at } \underline{x}\,(\mathrm{mod}\,v)\right\}. \tag{1.25}$$

On the right-hand side of inequality (1.24), the first term may be regarded as a main term for which a trivial upper bound suffices. We refer to the second term as the *ramified sieve term* and remark that it is similar in size to the main term. We refer to the third term as the *unramified sieve term* and remark that, in applications, the primary difficulty is to bound it nontrivially, and then to choose the sieving set $\mathcal{P}$ appropriately to balance the third term's contribution with that of the first term.

1.5. **Outline of the paper.** In §2, we prove Theorem 1.2 and Theorem 1.3. To prove Theorem 1.3, the essential point is to compute the optimal choice of α for which to apply Theorem 1.2. The rest of the paper focuses on proving Theorem 1.1, as follows. In §3, we recall notation and basic results related to the function field setting of Theorem 1.1. In §4, we present results on duals and reductions necessary in our analysis of the unramified sieve term. In §5, we reformulate Theorem 1.1 as an affine statement (Theorem 5.1) and bound all but the unramified sieve term in the inequality (1.24). (Here we apply a simple Schwartz-Zippel counting bound, for which we provide a proof in §10, for completeness.) The remainder of the work is focused on treating the unramified sieve term. In §6, we introduce background material on Fourier analysis on function fields and prove that the unramified sieve term can be stated as a mixed character sum. In §7, we state and verify the Weil-Deligne bounds we require for the unramified sieve term. In §8, we finally bound the unramified sieve term. In §9, we optimize the choice of the sieving set, thus completing the proof of Theorem 1.1. Finally, Appendix §11, written by Joseph Rabinoff, provides a self-contained account of certain standard facts about duals and reductions employed in §4.

Acknowledgements. We thank Dante Bonolis, Mihran Papikian, Joseph Rabinoff, and Melanie Wood for helpful conversations related to parts of this work. In addition, we thank the anonymous referee for exceptionally helpful recommendations with two significant effects. First, these recommendations simplified and strengthened the proof of Proposition 7.1 (i), so that our initial restriction that F in Theorem 1.1 be Dwork-regular could be weakened to assume that $F = 0$ is a nonsingular projective hypersurface. Second, these recommendations suggested the strategy to prove Lemma 8.2; this enabled averaging over prime elements in the sieving set, hence upgrading the main theorem from a result analogous (1.15) to the present result. We thank the referee for these astute and generous recommendations.

2. Derivation of the fundamental sieve inequalities

2.1. **Proof of Theorem 1.2.** We begin by proving the most general version of the sieve, as stated in Theorem 1.2. The proof follows the general framework of [HB84] and [Bro15].

For any fixed real number $\alpha \geq 1$, we consider the sum

$$\Sigma := \sum_{\underline{x} \in \mathcal{A}} \left(\sum_{v \in \mathcal{P} \setminus V_{\mathcal{P}}^{\mathrm{ram}}(\underline{x})} \left(\alpha + \left(|\eta_v^{-1}(\underline{x}(\mathrm{mod} v))| - 1 \right) \cdot \left(\deg \eta - |\eta_v^{-1}(\underline{x}(\mathrm{mod} v))| \right) \right) \right)^2$$

and note that each $\underline{x}$ therein is added with a non-negative weight. Moreover, observe that, if $\underline{x} \in \mathcal{S}(\mathcal{A})$, then the fiber above $\underline{x}$ has at least one element and at most $\deg \eta$ elements, that is,

$$1 \leq |\eta^{-1}(\underline{x})| \leq \deg \eta.$$

Therefore, in this case, for each $v \in \mathcal{P} \setminus V_{\mathcal{P}}^{\mathrm{ram}}(\underline{x})$, we have

$$\alpha + \left(|\eta_v^{-1}(\underline{x}(\mathrm{mod} v))| - 1 \right) \cdot \left(\deg \eta - |\eta_v^{-1}(\underline{x}(\mathrm{mod} v))| \right) \geq 1.$$

We deduce that, for every $\underline{x} \in \mathcal{S}(\mathcal{A})$,

$$\sum_{v \in \mathcal{P} \setminus V_{\mathcal{P}}^{\mathrm{ram}}(\underline{x})} \left(\alpha + \left(|\eta_v^{-1}(\underline{x}(\mathrm{mod} v))| - 1 \right) \cdot \left(\deg \eta - |\eta_v^{-1}(\underline{x}(\mathrm{mod} v))| \right) \right) \tag{2.1}$$

$$\geq \sum_{v \in \mathcal{P} \setminus V_{\mathcal{P}}^{\mathrm{ram}}(\underline{x})} 1 = |\mathcal{P}| - |V_{\mathcal{P}}^{\mathrm{ram}}(\underline{x})|, \tag{2.2}$$

which implies that

$$\begin{aligned}
\Sigma &\geq \sum_{\underline{x}\in\mathcal{S}(\mathcal{A})} (|\mathcal{P}| - |V_{\mathcal{P}}^{\mathrm{ram}}(\underline{x})|)^2 \\
&= |\mathcal{P}|^2 \cdot |\mathcal{S}(\mathcal{A})| + \sum_{\underline{x}\in\mathcal{S}(\mathcal{A})} (-2|\mathcal{P}| \cdot |V_{\mathcal{P}}^{\mathrm{ram}}(\underline{x})| + |V_{\mathcal{P}}^{\mathrm{ram}}(\underline{x})|^2) \\
&\geq |\mathcal{P}|^2 \cdot |\mathcal{S}(\mathcal{A})| - 2 \sum_{\underline{x}\in\mathcal{S}(\mathcal{A})} |\mathcal{P}| \cdot |V_{\mathcal{P}}^{\mathrm{ram}}(\underline{x})|. \qquad (2.3)
\end{aligned}$$

Rearranging the terms and using the non-negativity of $|V_{\mathcal{P}}^{\mathrm{ram}}(\underline{x})|$ to enlarge the sum over $\mathcal{S}(\mathcal{A})$ to $\mathcal{A}$, we deduce that

$$|\mathcal{S}(\mathcal{A})| \leq |\mathcal{P}|^{-2}\Sigma + 2|\mathcal{P}|^{-1} \sum_{\underline{x}\in\mathcal{A}} |V_{\mathcal{P}}^{\mathrm{ram}}(\underline{x})|,$$

which completes the proof of (1.21).

2.2. **Equivalent formulation of Theorem 1.2.** Recalling the notation $\mathbb{X}_{\mathcal{O}}^{\mathrm{ram}}(v)$ in (1.25), we see that the bound for $|\mathcal{S}(\mathcal{A})|$ may be rewritten by expanding the square inside Σ. This shows that

$$|\mathcal{S}(\mathcal{A})| \leq \frac{1}{|\mathcal{P}|^2} \sum_{v_1,v_2\in\mathcal{P}} \left| \sum_{i,j\in\{0,1,2\}} c_{i,j}(\alpha) S_{i,j}(v_1,v_2) \right| + \frac{2}{|\mathcal{P}|} \sum_{\underline{x}\in\mathcal{A}} |V_{\mathcal{P}}^{\mathrm{ram}}(\underline{x})|, \qquad (2.4)$$

in which, for all $v_1, v_2 \in \mathcal{P}$ and $i, j \in \{0, 1, 2\}$,

$$S_{i,j}(v_1,v_2) := \sum_{\substack{\underline{x}\in\mathcal{A} \\ \underline{x}\notin\mathbb{X}_{\mathcal{O}}^{\mathrm{ram}}(v_1)\cup\mathbb{X}_{\mathcal{O}}^{\mathrm{ram}}(v_2)}} \left|\eta_{v_1}^{-1}(\underline{x} \,(\mathrm{mod}\, v_1))\right|^i \cdot \left|\eta_{v_2}^{-1}(\underline{x} \,(\mathrm{mod}\, v_2))\right|^j \qquad (2.5)$$

and

$$c_{i,j}(\alpha) := \begin{cases} (\alpha - \deg\eta)^2 & \text{if } (i,j) = (0,0), \\ (\alpha - \deg\eta)(1 + \deg\eta) & \text{if } (i,j) = (1,0) \text{ or } (0,1), \\ (1 + \deg\eta)^2 & \text{if } (i,j) = (1,1), \\ -(\alpha - \deg\eta) & \text{if } (i,j) = (2,0) \text{ or } (0,2), \\ -(1 + \deg\eta) & \text{if } (i,j) = (2,1) \text{ or } (1,2), \\ 1 & \text{if } (i,j) = (2,2). \end{cases}$$

In this formulation, we see that (2.4) is analogous to the polynomial sieve in [Bro15, Thm. 1.1], except that our formulation omits the weight that is typically present in previous sieve inequalities of this type (see §2.4).

2.3. **Proof of Theorem 1.3: optimizing the choice of α.** The sieve inequality in Theorem 1.3 follows from inequality (1.21) when specialized to the case of prime degree cyclic covers of projective space, once we have computed the optimal choice of α to minimize the first term on the right-hand side of (1.21).

To compute this choice, recall that, since η is a cyclic map of prime degree ℓ, for each $v \in V_K$ of good reduction for η and for each $\underline{x} \in \mathbb{X}(\mathcal{O})$,

$$\left|\eta_v^{-1}(\underline{x} \,(\mathrm{mod}\, v))\right| = \begin{cases} 0 & \text{if } \underline{x} \,(\mathrm{mod}\, v) \notin \mathrm{Im}\,\eta_v, \\ 1 & \text{if } \underline{x} \,(\mathrm{mod}\, v) \in \mathrm{Im}\,\eta_v,\ \underline{x} \in \mathbb{X}_{\mathcal{O}}^{\mathrm{ram}}(v), \\ \ell & \text{if } \underline{x} \,(\mathrm{mod}\, v) \in \mathrm{Im}\,\eta_v,\ \underline{x} \notin \mathbb{X}_{\mathcal{O}}^{\mathrm{ram}}(v). \end{cases}$$

Note that in the above we used the primality of $\deg\eta$.

We deduce that, for each $v \in V_K$ of good reduction for η and for each $\underline{x} \in \mathbb{X}(\mathcal{O})$,

$$\left(\left|\eta_v^{-1}(\underline{x}\,(\mathrm{mod}\, v))\right| - 1\right)^2 = \begin{cases} 1 & \text{if } \underline{x}\,(\mathrm{mod}\, v) \notin \mathrm{Im}\,\eta_v, \\ 0 & \text{if } \underline{x}\,(\mathrm{mod}\, v) \in \mathrm{Im}\,\eta_v,\ \underline{x} \in \mathbb{X}_{\mathcal{O}}^{\mathrm{ram}}(v), \\ (\ell-1)^2 & \text{if } \underline{x}\,(\mathrm{mod}\, v) \in \mathrm{Im}\,\eta_v,\ \underline{x} \notin \mathbb{X}_{\mathcal{O}}^{\mathrm{ram}}(v). \end{cases} \tag{2.6}$$

As such, for each $v \in V_K$ of good reduction for η and for each $\underline{x} \in \mathbb{X}(\mathcal{O}) \backslash \mathbb{X}_{\mathcal{O}}^{\mathrm{ram}}(v)$,

$$\left(\left|\eta_v^{-1}(\underline{x}\,(\mathrm{mod}\, v))\right| - 1\right)^2 = (\ell - 1) + (\ell - 2)\left(\left|\eta_v^{-1}(\underline{x}\,(\mathrm{mod}\, v))\right| - 1\right). \tag{2.7}$$

Noting that $\underline{x} \in \mathcal{S}(\mathcal{A})$ implies $\underline{x}\,(\mathrm{mod}\, v) \in \mathrm{Im}\,\eta_v$, we infer that, for each $v \in V_K$ of good reduction for η and for each $\underline{x} \in \mathcal{S}(\mathcal{A})$, we have

$$\left|\eta_v^{-1}(\underline{x}\,(\mathrm{mod}\, v))\right| \geq 1. \tag{2.8}$$

Now, recall the notation of the model (1.22) for the prime degree cyclic cover $\mathbb{Y}$. Motivated by our upcoming expressions in terms of character sums in §6, for each $v \in V_K$ of good reduction for η and for each $\underline{x} \in \mathbb{X}(\mathcal{O})$, we set

$$\Psi_v(F(\underline{x})) := \left|\eta_v^{-1}(\underline{x}\,(\mathrm{mod}\, v))\right| - 1. \tag{2.9}$$

Then (2.7) implies that, for each $v \in V_K$ of good reduction for η and for each $\underline{x} \in \mathbb{X}(\mathcal{O}) \backslash \mathbb{X}_{\mathcal{O}}^{\mathrm{ram}}(v)$,

$$\Psi_v(F(\underline{x}))^2 = (\ell - 1) + (\ell - 2)\Psi_v(F(\underline{x})), \tag{2.10}$$

while (2.8) implies that, for each $v \in V_K$ of good reduction for η and for each $\underline{x} \in \mathcal{S}(\mathcal{A})$,

$$\Psi_v(F(\underline{x})) \geq 0.$$

We apply these observations in inequality (1.21) of Theorem 1.2. For this, fix $\alpha \geq 1$ and $\underline{x} \in \mathcal{S}(\mathcal{A})$, and expand

$$I_\alpha(\underline{x})^2 = \left(\sum_{v \in \mathcal{P} \backslash V_{\mathcal{P}}^{\mathrm{ram}}(\underline{x})} (\alpha + \Psi_v(F(\underline{x}))(\ell - 1 - \Psi_v(F(\underline{x})))) \right)^2$$

as

$$\begin{aligned} \sum_{v_1, v_2 \in \mathcal{P} \backslash V_{\mathcal{P}}^{\mathrm{ram}}(\underline{x})} & \Big(\alpha^2 + \alpha(\ell - 1)\left(\Psi_{v_1}(F(\underline{x})) + \Psi_{v_2}(F(\underline{x}))\right) \\ & -\alpha\left(\Psi_{v_1}(F(\underline{x}))^2 + \Psi_{v_2}(F(\underline{x}))^2\right) + (\ell-1)^2 \Psi_{v_1}(F(\underline{x}))\Psi_{v_2}(F(\underline{x})) \\ & -(\ell - 1)\left(\Psi_{v_1}(F(\underline{x}))^2\Psi_{v_2}(F(\underline{x})) + \Psi_{v_1}(F(\underline{x}))\Psi_{v_2}(F(\underline{x}))^2\right) + \Psi_{v_1}(F(\underline{x}))^2\Psi_{v_2}(F(\underline{x}))^2\Big). \end{aligned}$$

Applying (2.10), the above sum simplifies precisely to

$$\sum_{v_1, v_2 \in \mathcal{P} \backslash V_{\mathcal{P}}^{\mathrm{ram}}(\underline{x})} \left((\alpha - (\ell - 1))^2 + (\alpha - (\ell - 1))\left(\Psi_{v_1}(F(\underline{x})) + \Psi_{v_2}(F(\underline{x}))\right) + \Psi_{v_1}(F(\underline{x}))\Psi_{v_2}(F(\underline{x}))\right),$$

which reveals that the optimal choice of α to minimize $I_\alpha(\underline{x})^2$ is

$$\alpha := \ell - 1.$$

Using the above choice of α in (1.21) of Theorem 1.2, we obtain that

$$\begin{aligned} |\mathcal{S}(\mathcal{A})| & \leq \frac{1}{|\mathcal{P}|^2} \sum_{\underline{x} \in \mathcal{A}} I_{\ell - 1}(\underline{x})^2 + \frac{2}{|\mathcal{P}|} \sum_{\underline{x} \in \mathcal{A}} |V_{\mathcal{P}}^{\mathrm{ram}}(\underline{x})| \\ & = \frac{1}{|\mathcal{P}|^2} \sum_{\underline{x} \in \mathcal{A}} \sum_{v_1, v_2 \in \mathcal{P} \backslash V_{\mathcal{P}}^{\mathrm{ram}}(\underline{x})} \Psi_{v_1}(F(\underline{x}))\Psi_{v_2}(F(\underline{x})) + \frac{2}{|\mathcal{P}|} \sum_{\underline{x} \in \mathcal{A}} |V_{\mathcal{P}}^{\mathrm{ram}}(\underline{x})|. \end{aligned} \tag{2.11}$$

We leave the ramified sieve term as is, and treat the first term on the right-hand side as follows. By interchanging summations, we write this term as

$$\frac{1}{|\mathcal{P}|^2}\sum_{v_1,v_2\in\mathcal{P}}\sum_{\underline{x}\in\mathcal{A}\setminus\left(\mathbb{X}_{\mathcal{O}}^{\mathrm{ram}}(v_1)\cup\mathbb{X}_{\mathcal{O}}^{\mathrm{ram}}(v_2)\right)}\Psi_{v_1}(F(\underline{x}))\Psi_{v_2}(F(\underline{x})).$$

Distinguishing between $v_1 = v_2$ and $v_1 \neq v_2$, we obtain that the above expression equals

$$\frac{1}{|\mathcal{P}|^2}\sum_{v_1\in\mathcal{P}}\sum_{\underline{x}\in\mathcal{A}\setminus\mathbb{X}_{\mathcal{O}}^{\mathrm{ram}}(v_1)}\Psi_{v_1}(F(\underline{x}))^2+\frac{1}{|\mathcal{P}|^2}\sum_{\substack{v_1,v_2\in\mathcal{P}\\ v_1\neq v_2}}\sum_{\underline{x}\in\mathcal{A}\setminus\left(\mathbb{X}_{\mathcal{O}}^{\mathrm{ram}}(v_1)\cup\mathbb{X}_{\mathcal{O}}^{\mathrm{ram}}(v_2)\right)}\Psi_{v_1}(F(\underline{x}))\Psi_{v_2}(F(\underline{x})). \tag{2.12}$$

By (2.6), for each summand in the inner sum of the first double sum above we have

$$\Psi_{v_1}(F(\underline{x}))^2 \leq (\ell-1)^2,$$

so (2.12) is bounded from above by

$$(\ell-1)^2\frac{|\mathcal{A}|}{|\mathcal{P}|}+\frac{1}{|\mathcal{P}|^2}\sum_{\substack{v_1,v_2\in\mathcal{P}\\ v_1\neq v_2}}\left|\sum_{\underline{x}\in\mathcal{A}\setminus\left(\mathbb{X}_{\mathcal{O}}^{\mathrm{ram}}(v_1)\cup\mathbb{X}_{\mathcal{O}}^{\mathrm{ram}}(v_2)\right)}\Psi_{v_1}(F(\underline{x}))\Psi_{v_2}(F(\underline{x}))\right|. \tag{2.13}$$

Inserting this in (2.11) then proves the theorem.

2.4. **Comparison to square and polynomial sieves.** It is informative to compare Theorems 1.2 and 1.3 to their antecedents in a wide range of settings over $\mathbb{Q}$. These include the square sieve of Heath-Brown [HB84], which counts perfect square values of a polynomial; the power sieve in [Mun09], [Bra15], which counts perfect r-th power values of a polynomial; and stronger versions of the square or power sieve, combined with the q-analogue of Van der Corput's method, in [Pie06], [HBP12]. Moreover, our work is motivated by the polynomial sieve of Browning [Bro15], also developed in a different direction involving trace functions, by Bonolis [Bon21].

All of these works develop a sieve method to tackle a problem roughly of the following form (over $\mathbb{Q}$): given an appropriate polynomial $f(z;\underline{x})$ and a set of interest $\mathcal{A}$, how many $\underline{x}\in\mathcal{A}$ of bounded height have $f(z;\underline{x})$ solvable? This is clearly related to both our aims and our methods, now in the setting of function fields.

One of the most obvious differences is that we do not include a weight to count multiplicities of values. To understand the two main roles of the multiplicity-counting weight in the earlier settings, suppose one fixes a set $\mathcal{A}$ and aims to bound from above the quantity

$$S(\mathcal{A}) := \sum_{\substack{\underline{x}\in\mathcal{A}\\ f(z;\underline{x})\text{ soluble}}} w(\underline{x}) \tag{2.14}$$

for a fixed integer-coefficient polynomial $f(z;\underline{x}) = p_0(\underline{x})z^d + \cdots + p_d(\underline{x})$ of interest, with $\underline{x} = (x_0,\dots,x_n)$. In the most straightforward comparison to our setting, Browning's polynomial sieve then provides an upper bound for $S(\mathcal{A})$ in the form of an inequality like (2.4), but with $S_{i,j}(v_1,v_2)$ containing the coefficient $w(\underline{x})$ within the sum expressed in (2.5).

In the classical setting of the square sieve, in order to bound from above the number of perfect square values of a fixed polynomial, say $F(y_1,\dots,y_k)\in\mathbb{Z}[Y_1,\dots,Y_k]$ with $\underline{y}=(y_1,\dots,y_k)$ lying in a certain region $\mathcal{R}$, we would set $f(z;x)=z^2-x$ and choose $w(x)=\#\{\underline{y}\in\mathcal{R} : F(y_1,\dots,y_k)=x\}$. Then we would apply the sieve inequality, including the weight w, to bound (2.14) from above. Similarly for the r-power sieve and its variants in [Mun09], [HBP12], [Bra15], to count perfect r-power values of some polynomial $F(y_1,\dots,y_k)$ of interest, we would set $f(z;x)=z^r-x$ and define $w(x)$ as above. With the more recent polynomial sieve now available in [Bro15] (and furthermore in our present geometric treatment), this type of multiplicity-counting weight is no longer as relevant, since one could instead nominally set $w(x) = 1$ and instead choose $p_2(\underline{y}) = F(\underline{y})$ for the square sieve or more generally $p_r(\underline{y}) = F(\underline{y})$ for the r-power sieve. Indeed, in the application of [Bro15, p. 11], the weight is defined simply to be the indicator function of a finite set. (Bonolis [Bon21, p. 27] defines a weight that counts such solutions in a "smoothed" sense.)

This brings us to the second use of the weight: it is used to eliminate what we have here called the ramified sieve term, that is,

$$\frac{2}{|\mathcal{P}|} \sum_{\underline{x} \in \mathcal{A}} |V_{\mathcal{P}}^{\mathrm{ram}}(\underline{x})|.$$

Previous sieve lemmas in number field settings assume that $w(\underline{x})$ vanishes for $\underline{x}$ sufficiently large, e.g., that $w(\underline{x}) = 0$ if $|\underline{x}| \geq \exp(|\mathcal{P}|)$. In the classical settings, in the derivation of the sieve inequality, at step (2.1) the expression $|V_{\mathcal{P}}^{\mathrm{ram}}(\underline{x})|$ counted the number of primes in the sieving set $\mathcal{P}$ that divided a certain (nonzero) polynomial expression H in $\underline{x}$, so that for some degree D,

$$|V_{\mathcal{P}}^{\mathrm{ram}}(H(\underline{x}))| \leq \omega(H(\underline{x})) = \mathrm{o}(|\underline{x}|^D) = \mathrm{o}(|\mathcal{P}|),$$

under the relevant hypothesis assumed for the support of w. (Here, as is standard, ω counts the number of distinct prime factors, with $\omega(m) \ll \log m / \log\log(3m)$ for any integer $m \geq 1$.) In such a setting, we would conclude in place of (2.3) that

$$\Sigma \geq |\mathcal{P}|^2 \cdot |\mathcal{S}(\mathcal{A})| + |\mathcal{S}(\mathcal{A})| \cdot \mathrm{o}(|\mathcal{P}|^2),$$

so that the contribution of the ramified sieve term is dominated by the unramified sieve term, and hence omitted from the final sieve inequality (see e.g. [Bro15, Thm. 1.1]). In our present treatment, we prefer not to include any weight and hence we explicitly record the ramified sieve term, which must be bounded later.

Finally, in comparison to the polynomial sieve, we have emphasized not the size of the fiber $\left|\eta_v^{-1}(\underline{x} \,(\mathrm{mod}\, v))\right|$ but the quantity $\left|\eta_v^{-1}(\underline{x} \,(\mathrm{mod}\, v))\right| - 1$ as in (2.9). As we have seen in the argument of §2.3, at least in the case of cyclic morphisms, this normalization is more informative of the correct choice of α, rather than the expansion in terms of the fibers, which leads to the hard-to-interpret coefficients in (2.4).

3. Preliminaries on function field arithmetic

Since Theorem 1.1 is a result about $\mathbb{F}_q(T)$, we gather in this section standard function field notation and remarks pertinent to our forthcoming arguments.

We recall the general setting of Section 1: K is a global field of arbitrary characteristic; $\mathcal{O}$ is the ring of integers in K; V_K is the set of places of K; the valuations $|\cdot|_v$ associated to the places $v \in V_K$ are normalized such that the product formula (1.1) holds; ht_K is the height function on $\mathbb{P}^n_K$, defined in (1.2) and its companion height function on K, also denoted ht_K, is defined in (1.3).

We now focus on the particular global function field

$$K = \mathbb{F}_q(T)$$

and on the ring

$$\mathcal{O} = \mathcal{O}_K = \mathbb{F}_q[T],$$

where q is the power of an odd rational prime p. This setting will be kept throughout the remainder of the paper, unless explicitly specified otherwise.

In the setting of Theorem 1.1 and Theorem 5.1, we assume that $p \nmid m$, where m is the degree of the polynomial F therein. Furthermore, we assume that $\ell \mid (q-1)$, where ℓ is a rational prime defined as the degree of the cyclic cover η therein; this divisibility assumption ensures that $\mathbb{F}_q$ contains the ℓ-th roots of unity. Finally, the assumption $\ell \mid m$ ensures that the weighted projective space $\mathbb{P}^{n+1}_{\mathbb{F}_q(T)}\left(1, \ldots, 1, \frac{m}{\ell}\right)$ is well-defined.

Note that $K = \mathbb{F}_q(T)$ is the simplest instance of a global function field with field of constants $\mathbb{F}_q$, and that $\mathcal{O}_K$ is the ring of elements of K which have only $\frac{1}{T}$ as a pole. In particular, $\mathcal{O}_K$ is a Dedekind domain (and, actually, a Euclidean domain) and plays the role of the ring $\mathbb{Z}$ in the analogy between the arithmetic of $\mathbb{F}_q(T)$ and that of $\mathbb{Q}$.

As usual, we identify a place $v \in V_K$ with a generator of its associated unique maximal ideal. For our particular $K = \mathbb{F}_q(T)$, we have either that $v = \frac{1}{T}$, which we refer to as *the place at infinity* of K, or that any $v \in V_K \backslash \{\frac{1}{T}\}$, which we refer to as a *finite place* or simply as a *prime* of K, may be thought of as a monic irreducible polynomial in $\mathbb{F}_q[T]$. We recall that, for a nonzero polynomial $x \in \mathcal{O}_K = \mathbb{F}_q[T]$, we use $\deg_T(x)$ to denote its degree in T.

To simplify the exposition, we use the symbol ∞ for $\frac{1}{T}$ and the symbol π for an arbitrary prime of K (that is, a monic irreducible in $\mathcal{O}_K$). We denote by K_∞ the completion of K with respect to the topology defined by $|\cdot|_\infty$ and recall that

$$K_\infty = \mathbb{F}_q\left(\left(\frac{1}{T}\right)\right) = \left\{\sum_{j \le N} a_j T^j : N \in \mathbb{Z}, a_j \in \mathbb{F}_q \; \forall j \le N, a_N \neq 0\right\}.$$

We denote by k_π the residue field $\mathbb{F}_q[T]/(\pi)$ of π and by $\overline{k}_\pi$ a fixed algebraic closure of k_π, and recall that k_π is a finite field with $q^{\deg_T(\pi)}$ elements. We denote by $\mathrm{ord}_\pi(x)$ the power of π that exactly divides x.

The absolute values $|\cdot|_\infty$, $|\cdot|_\pi$ on K are defined by

$$|0|_\infty := 0, \quad \left|\frac{x}{y}\right|_\infty := q^{\deg_T x - \deg_T y},$$

and

$$|0|_\pi := 0, \quad \left|\frac{x}{y}\right|_\pi := q^{\mathrm{ord}_\pi(y) - \mathrm{ord}_\pi(x)},$$

where $x, y \in \mathbb{F}_q[T]\backslash\{0\}$. With these definitions, the associated valuations satisfy the product formula (1.1), the height function on $\mathbb{P}^n_K$ becomes

$$\mathrm{ht}_K([x_0 : x_1 : \ldots : x_n]) = \max\{|x_i|_\infty : 0 \le i \le n\}$$

for any $[x_0 : x_1 : \ldots : x_n] \in \mathbb{P}^n_K$, and the height function on K becomes

$$\mathrm{ht}_K(x) = |x|_\infty \tag{3.1}$$

for any $x \in K^*$.

Remark that, since $K = \mathbb{F}_q(T)$ has class number 1, for any $[x] = [x_0 : x_1 : \ldots : x_n] \in \mathbb{P}^n_K$ we can find $x'_0, x'_1, \ldots, x'_n \in \mathcal{O}_K$ with $\gcd(x'_0, x'_1, \ldots, x'_n) = 1$, such that $[x] = [x'_0 : x'_1 : \ldots : x'_n]$ and $\mathrm{ht}_K([x]) = \max\{|x'_i|_\infty : 0 \le i \le n\}$. Reasoning similarly, we can extend the height ht_K to $\mathbb{A}^{n+1}_K$ by setting $\mathrm{ht}_K((x_0, x_1, \ldots, x_n)) := \max\{\mathrm{ht}_K(x_i) : 0 \le i \le n\}$ for any $\underline{x} := (x_0, x_1, \ldots, x_n) \in \mathbb{A}^{n+1}_K$, which is consistent with (1.17).

In order to employ Fourier analysis on K, we extend $|\cdot|_\infty$ to K^{n+1}_∞ by

$$|(x_0, x_1, \ldots, x_n)|_\infty := \max\{|x_i|_\infty : 0 \le i \le n\}$$

and consider the additive character

$$\psi_\infty : K_\infty \longrightarrow \mathbb{C}^*,$$

$$\psi_\infty\left(\sum_{j \le N} a_j T^j\right) := \exp\left(\frac{2\pi i \, \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a_{-1})}{p}\right), \tag{3.2}$$

where $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$ is the trace map.

4. Preliminaries on duals and reductions

Our proof of Theorem 1.1 will be an application of the sieve inequality in Theorem 1.3 for $K = \mathbb{F}_q(T)$. To choose a sieving set $\mathcal{P}$ and to state the appropriate Weil-Deligne bounds needed to estimate the resulting unramified sieve term, we require certain standard facts about duals and reductions. An appendix by Joseph Rabinoff provides a self-contained reference for all the facts we require; we now summarize the consequences for our specific application.

Proposition 4.1. *Let q be an odd rational prime power, $n \geq 2$ an integer, and $H \in \mathbb{F}_q[T][X_0, \ldots, X_n]$ a homogeneous polynomial of degree $m := \deg_{\underline{X}}(H)$ with $m \geq 2$. Denote by W the projective hypersurface defined in $\mathbb{P}^n_{\mathbb{F}_q(T)}$ by*

$$W: \quad H(X_0, \ldots, X_n) = 0$$

and assume that W is smooth over $\mathbb{F}_q(T)$. Then the following properties hold.

(i) *W and its dual W^* are geometrically integral, with $\dim W = \dim W^* = n - 1$.*

(ii) *There exists an absolutely irreducible, homogeneous polynomial $H^* \in \mathbb{F}_q[T][X_0, \ldots, X_n]$, which depends only on H, such that W^* is defined in $\mathbb{P}^n_{\mathbb{F}_q(T)}$ by*

$$W^*: \quad H^*(X_0, \ldots, X_n) = 0.$$

(iii) *There exists a finite set of finite places $\mathcal{P} \subseteq \mathbb{F}_q[T]$, which depends only on H and H^*, such that:*

(iii.1) *for all finite places $\pi \notin \mathcal{P}$, $\deg_{\underline{X}}(H(\mathrm{mod}\, \pi)) = \deg_{\underline{X}}(H)$ and the projective variety W_π defined by the equation*

$$W_\pi: \quad H(X_0, \ldots, X_n) \equiv 0 \, (\mathrm{mod}\, \pi)$$

is a smooth and geometrically integral hypersurface;

(iii.2) *for all finite places $\pi \notin \mathcal{P}$, the dual variety $(W_\pi)^*$ is geometrically integral and is defined by the equation*

$$(W_\pi)^*: \quad H^*(X_0, \ldots, X_n) \equiv 0 \, (\mathrm{mod}\, \pi);$$

in particular, $(W_\pi)^* = (W^*)_\pi$ and the notation W^*_π is well-defined.

(iii.3) *for all finite places $\pi \notin \mathcal{P}$, $\dim W_\pi = \dim W^*_\pi = n - 1$.*

Proof. Assertion (i) is the statement of Proposition 11.2(1). As observed in the appendix, for any field k, a hypersurface in $\mathbb{P}^n_k$ is the zero set of a nonzero homogeneous polynomial in $k[X_0, \ldots, X_n]$. Thus implies that W^* is defined by some $H^* \in \mathbb{F}_q(T)[X_0, \ldots, X_n]$; after clearing denominators, we may assume $H^* \in \mathbb{F}_q[T][X_0, \ldots, X_n]$. Absolute irreducibility of H^* is equivalent to geometric irreducibility of W^*, so this proves (ii). Let $\mathcal{P}$ be the finite set S in Proposition 11.5. Then (iii.1) and (iii.2) amount to statements (2) and (3) of Proposition 11.5, using Lemma 11.1 and Proposition 11.2(1) for geometric integrality. Assertion (iii.3) is a consequence of (iii.1) and (iii.2). $\square$

5. Initiating the sieve to prove Theorem 1.1

We are now ready to start the proof of Theorem 1.1 on counting points on cyclic covers of $\mathbb{P}^n_K$ for $K = \mathbb{F}_q(T)$ and an arbitrary integer $n \geq 2$.

Recall that, in Theorem 1.3, we take $\mathbb{X} := \mathbb{P}^n_K$ and $\mathbb{Y}$ the projective scheme in the weighted projective space $\mathbb{P}^{n+1}_K\left(1, \ldots, 1, \frac{m}{\ell}\right)$ defined by a model $X^\ell_{n+1} = F(X_0, \ldots, X_n)$ with ℓ prime such that $\ell \mid m$, and with $F \in \mathcal{O}_K[X_0, \ldots, X_n]$ homogeneous of degree $m \geq 1$ and having the property that the hypersurface in $\mathbb{P}^n_K$ defined by $F(X_0, \ldots, X_n) = 0$ is smooth. Moreover, recall that $\eta : \mathbb{Y} \longrightarrow \mathbb{X}$ is the cyclic map of degree ℓ defined over $\mathbb{A}^n_K$ by $\eta(x_0, x_1, \ldots, x_n, x_{n+1}) := (x_0, x_1, \ldots, x_n)$.

Note that each element $[y]$ of $\mathbb{Y}$ is of the form $[y] = [x_0 : \ldots : x_n : x_{n+1}]$, where $[x_0 : \ldots x_n] \in \mathbb{P}^n_K$ and where $x_{n+1} \in K$ satisfies the equation

$$x_{n+1}^{\ell} = F(x_0, \ldots, x_n).$$

In order to bound the number of projective solutions, we will think of the above equation as an affine model. As such, it is equally natural to restate the result of Theorem 1.1 as:

Theorem 5.1. *Let q be an odd rational prime power, $n \geq 2$ an integer, $\ell \geq 2$ a rational prime, and $F \in \mathbb{F}_q[T][X_0, \ldots, X_n]$ a homogeneous polynomial of degree $m \geq 2$ in $X_0, \ldots, X_n$, with* $\operatorname{char} \mathbb{F}_q \nmid m$. *Assume the conditions:*

(i) $\ell \mid \gcd(m, q-1)$;

(ii) $F(X_0, \ldots, X_n) = 0$ *defines a nonsingular projective hypersurface in* $\mathbb{P}^n_{\overline{\mathbb{F}_q(T)}}$.

For every $b > 0$, let $M_n(F; b)$ denote the cardinality of the set

$$\left\{(x_0, \ldots, x_n) \in \mathbb{F}_q[T]^{n+1} : \ \deg_T(x_i) < b \ \forall 0 \leq i \leq n, \exists x_{n+1} \in \mathbb{F}_q[T] \text{ such that } x_{n+1}^{\ell} = F(x_0, \ldots, x_n)\right\}.$$

Then for any $b \geq 1$

$$M_n(F; b) \ll_{\ell,m,n,q,F} \left(q^b\right)^{(n+1)-\frac{n+1}{n+2}} b^{\frac{n+1}{n+2}},$$

in which the implicit constant depends on ℓ, m, n, q and F.

More precisely the implicit constant depends on F in terms of $\deg_T(F)$ and $\deg_T(F^*)$ as well as a finite exceptional set of primes determined by F; see §9. (Here, and in all that follows, given a polynomial $H \in \mathbb{F}_q[T][X_0, \ldots, X_n]$, $\deg_T H$ refers to the maximum degree in T of any coefficient of H.)

5.1. **Choosing the sieving set.** We prove Theorem 5.1 (and hence Theorem 1.1) by means of the geometric sieve derived in Theorem 1.3. To phrase our goal as a sieve problem, from here onwards, unless otherwise stated, we keep the setting of Theorem 5.1 and the notation of §3, and work with the sets

$$\begin{aligned}
\mathcal{A} &:= \left\{\underline{x} = (x_0, \ldots, x_n) \in \mathcal{O}_K^{n+1} : \ \deg_T(x_i) < b \ \forall 0 \leq i \leq n\right\}, \\
\mathcal{S}_F(\mathcal{A}) &:= \left\{\underline{x} \in \mathcal{A} : \ \exists y \in \mathcal{O}_K \text{ such that } y^{\ell} = F(\underline{x})\right\}
\end{aligned}$$

for a fixed arbitrary integer $b > 0$. Note that the trivial upper bound for $|\mathcal{S}_F(\mathcal{A})|$ is

$$|\mathcal{A}| \leq q^{b(n+1)}. \tag{5.1}$$

To improve upon this bound using Theorem 1.3, we seek a suitable sieving set $\mathcal{P}$ of primes π of K, which we describe below.

We consider the smooth projective hypersurface

$$W(F) : \quad F(X_0, \ldots, X_n) = 0,$$

whose projective dual we denote by $W(F)^*$. By parts (i), (ii) of Proposition 4.1, there exists a homogeneous polynomial $F^* \in \mathcal{O}_K[X_0, \ldots, X_n]$, absolutely irreducible over K, that defines $W(F)^*$, that is,

$$W(F)^* : \quad F^*(X_0, \ldots, X_n) = 0.$$

Recall that by part (iii) of Proposition 4.1 there exists a finite set of finite primes

$$\mathcal{P}_{\text{exc}} = \mathcal{P}_{\text{exc}}(F) \tag{5.2}$$

satisfying that proposition.

Upon fixing some positive integer Δ, we define the sieving set of primes as

$$\mathcal{P} := \{\pi \in \mathcal{O}_K : \deg_T(\pi) = \Delta, \pi \notin \mathcal{P}_{\text{exc}}\}. \tag{5.3}$$

Later on in (9.4), we will choose $\Delta = \Delta(n, b)$ optimally in terms of n and b; see also the observation below (5.4).

To understand the growth of $|\mathcal{P}|$, let us recall the Prime Polynomial Theorem [Ros02, Thm. 2.2, p. 14]:

$$\left|\#\{\pi \in \mathcal{O}_K : \deg_T(\pi) = \Delta\} - \frac{q^{\Delta}}{\Delta}\right| \leq \frac{q^{\frac{\Delta}{2}}}{\Delta} + q^{\frac{\Delta}{3}}. \tag{5.4}$$

We infer from (5.4) that, upon taking $\Delta \approx \lfloor (n+1)b/(n+2) \rfloor$ (see the precise choice in (9.4)), we have that

$$|\mathcal{P}| \geq \frac{q^{\Delta}}{\Delta} - \left(\frac{q^{\frac{\Delta}{2}}}{\Delta} + q^{\frac{\Delta}{3}}\right) - |\mathcal{P}_{\text{exc}}| \geq \frac{q^{\Delta}}{2\Delta} \tag{5.5}$$

for all b that are sufficiently large relative to n and $|\mathcal{P}_{\text{exc}}|$. We denote the least such b by

$$b(n, q, F)$$

and refer the reader to §9.2 for more details. While we still allow $\Delta > 0$ to be arbitrary, from now on we assume the lower bound (5.5) for $|\mathcal{P}|$.

Our task for this particular sieve problem is to bound non-trivially the right-hand side of the resulting sieve inequality of Theorem 1.3. Recall that there are three terms – the main term, the ramified term, and the unramified term. In this section, we bound the first two terms quite easily, and in the remaining sections we focus on bounding the third term.

We make the notational convention that, for $\underline{x} := (x_0, \ldots, x_n) \in \mathcal{O}_K^{n+1}$, we write

$$\deg_T(\underline{x}) < b$$

to mean

$$\deg_T(x_i) < b \;\forall 0 \leq i \leq n.$$

5.2. **Upper bound for the main sieve term.** Assuming Δ is chosen and b is sufficiently large, relative to n and $|\mathcal{P}_{\text{exc}}(F)|$, so that (5.5) holds (again, see §9.2 for the explicit requirements on b), then by (5.1) and (5.5), the first term on the right-hand side of (1.24) is bounded above by

$$\frac{|\mathcal{A}|}{|\mathcal{P}|}(\ell - 1)^2 \leq 2\Delta q^{b(n+1)-\Delta}(\ell - 1)^2 \ll_\ell \Delta q^{b(n+1)-\Delta}. \tag{5.6}$$

5.3. **Upper bound for the ramified sieve term.** For the second term on the right-hand side of (1.24), note that for each prime $\pi \in \mathcal{O}_K$ of good reduction for η, the condition that η_π is ramified at $\underline{x} \,(\mathrm{mod}\, \pi)$ means that the extension of residue fields defined by $\underline{x} \,(\mathrm{mod}\, \pi)$ is inseparable, which cannot happen in our case since the residue fields are finite, or that η_π has a ramification index at $(\underline{x}, y) \,(\mathrm{mod}\, \pi)$ larger than 1, which is equivalent to $y \,(\mathrm{mod}\, \pi)$ being zero. Then,

$$\frac{1}{|\mathcal{P}|}\sum_{\underline{x} \in \mathcal{A}} |V_{\mathcal{P}}^{\text{ram}}(\underline{x})| = \frac{1}{|\mathcal{P}|}\sum_{\substack{\underline{x} \in \mathcal{O}_K^{n+1} \\ \deg_T(\underline{x}) < b}} \#\left\{\pi \in \mathcal{O}_K : \deg_T(\pi) = \Delta, \pi \notin \mathcal{P}_{\text{exc}},\ F(\underline{x}) \equiv 0 \,(\mathrm{mod}\, \pi)\right\}. \tag{5.7}$$

We will bound each term according to whether $F(\underline{x}) \neq 0$ or $F(\underline{x}) = 0$. First we observe that for any nonconstant polynomial $u \in \mathcal{O}_K$ we have

$$\#\left\{\pi \in \mathcal{O}_K : \pi \mid u\right\} \leq \deg_T(u).$$

We also restrict the degree of π, so more precisely, let $\omega_\Delta(G)$ denote the number of distinct primes π of degree Δ that divide a fixed $G(T) \in \mathbb{F}_q[T]$. Then

$$\Big(\prod_{\substack{\pi \mid G \\ \deg_T(\pi) = \Delta}} \pi\Big) \mid G.$$

Taking norms, we have $q^{\Delta\omega_\Delta(G)} \leq |G|$ and therefore,

$$\omega_\Delta(G) \leq \frac{\deg_T(G)}{\Delta}.$$

Thus, after we fix $\underline{x} \in \mathcal{O}_K^{n+1}$ with $\deg_T(\underline{x}) < b$ and $F(\underline{x}) \neq 0$,

$$\begin{aligned}
\#\{\pi \in \mathcal{O}_K : \deg_T(\pi) = \Delta, \pi \notin \mathcal{P}_{\mathrm{exc}},\ F(\underline{x}) \equiv 0 \,(\mathrm{mod}\,\pi)\} &\leq \frac{1}{\Delta}\deg_T(F(\underline{x})) \\
&\leq \frac{1}{\Delta}(\deg_T(F) + \deg_{\underline{X}}(F)\cdot\deg_T(\underline{x})) \\
&= \frac{1}{\Delta}(\deg_T(F) + mb).
\end{aligned}$$

Here we emphasize, $m = \deg_{\underline{X}}(F)$ is the degree of $F \in \mathcal{O}_K[X_0,\dots,X_n] = \mathbb{F}_q[T, X_0, \dots, X_n]$ in $X_0,\dots,X_n$ and $\deg_T(F)$ is the maximum degree of (any coefficient of) F in T.

Applying the above observations to each summand with $F(\underline{x}) \neq 0$ on the right-hand side of (5.7), trivially counting the number of $\underline{x}$ in the sum, and applying (5.5) to bound $1/|\mathcal{P}|$ from above, we deduce that this contribution to the ramified sieve term is

$$\ll_{m,n,\deg_T(F),}\ bq^{b(n+1)-\Delta},$$

for all $b \geq b(n,q,F)$.

On the other hand, the contribution to (5.7) from those $\underline{x}$ such that $F(\underline{x}) = 0$ is

$$\frac{|\mathcal{P}|}{|\mathcal{P}|}\#\{\underline{x} \in \mathcal{O}_K^{n+1} : \deg_T(\underline{x}) < b, F(\underline{x}) = 0\} \leq \deg_{\underline{X}}(F)q^{bn},$$

by the trivial bound (see e.g. Lemma 10.1, which we include at the end of the paper for completeness). Thus in total the ramified sieve term is

$$\ll_{m,n,\deg_T(F)}\ bq^{b(n+1)-\Delta} + q^{bn},$$

for all $b \geq b(n,q,F)$.

5.4. **Remaining work.** So far we have proved the upper bound

$$\begin{aligned}
|\mathcal{S}_F(\mathcal{A})| \ll_{m,n,\deg_T(F)}\ & bq^{b(n+1)-\Delta} + q^{bn} \\
&+ \frac{1}{|\mathcal{P}|^2}\sum_{\substack{\pi_1,\pi_2\in\mathcal{P}\\ \pi_1\neq\pi_2}}\left|\sum_{\substack{\underline{x}\in\mathcal{O}_K^{n+1}\\ \deg_T(\underline{x})<b\\ F(\underline{x})\not\equiv 0\,(\mathrm{mod}\,\pi_1\pi_2)}}\left(\left|\eta_{\pi_1}^{-1}(\underline{x}\,(\mathrm{mod}\,\pi_1))\right| - 1\right)\left(\left|\eta_{\pi_2}^{-1}(\underline{x}\,(\mathrm{mod}\,\pi_2))\right| - 1\right)\right|. \quad (5.8)
\end{aligned}$$

Recall that our goal is to prove an upper bound for $\mathcal{S}_F(\mathcal{A})$ that improves on the trivial upper bound $q^{b(n+1)}$ recorded in (5.1). The first term on the right-hand side of (5.8) will be nontrivial as long as $\Delta > 0$. Since the trivial upper bound for the unramified sieve term, the last term on the right-hand side of (5.8), is $\ll q^{b(n+1)}$, which is as large as the aforementioned trivial bound (5.1), we seek any bound for the unramified sieve term that improves upon this. Ultimately, we will choose Δ to balance our upper-estimates for the terms on the right-hand side of (5.8).

To tackle the unramified sieve term, we will break our treatment into two main steps: in §6, we expand the term into a sum of complete character sums; in §8, we apply Weil-Deligne bounds (proved in §7) to each of these sums, and then use point-counting results to average over $\pi_1, \pi_2 \in \mathcal{P}$, and sum up the resulting Weil-Deligne bounds.

6. Expansion of unramified sieve term as a mixed character sum

In this section we recall the basic notions of Fourier analysis in our function field setting and use them to expand the unramified sieve term as a mixed character sum. For ease of reference, we first record the main outcome of this section and then proceed with rigorous definitions and derivations.

For any element $\underline{w} \in \mathcal{O}_K^{n+1}$, prime $\pi \in \mathcal{O}_K$, and non-principal multiplicative character χ_π of $\mathcal{O}_K$, of modulus π, we define a mixed character sum relative to a polynomial $G \in \mathcal{O}_K[X_0, \dots, X_n]$ by

$$S_G(\underline{w}, \chi_\pi) := \sum_{\underline{a} \,(\mathrm{mod}\, \pi) \in k_\pi^{n+1}} \chi_\pi(G(\underline{a}))\, \psi_\infty\left(-\frac{\underline{w}\cdot\underline{a}}{\pi}\right), \tag{6.1}$$

where the additive character $\psi_\infty(\cdot/\pi)$ is defined in (3.2).

Proposition 6.1. *Let q be an odd rational prime power, $n \geq 2$ an integer, $\ell \geq 2$ a rational prime, and $F \in \mathcal{O}_K[X_0, \dots, X_n]$ a homogeneous polynomial of degree $m \geq 2$ in $X_0, \dots, X_n$, with* $\mathrm{char} K \nmid m$, *where, as before, $K = \mathbb{F}_q(T)$. Assume $\ell \mid \gcd(m, q-1)$. Let $b, \Delta > 0$ be integers and assume that*

$$b < 2\Delta.$$

Defining $\mathcal{P}$ as in (5.3), for all primes $\pi_1, \pi_2 \in \mathcal{P}$ with $\pi_1 \neq \pi_2$, the unramified sieve term can be expanded as

$$\begin{aligned}\sum_{\substack{\underline{x} \in \mathcal{O}_K^{n+1} \\ \deg_T(\underline{x}) < b \\ F(\underline{x}) \not\equiv 0 \,(\mathrm{mod}\, \pi_1\pi_2)}} &\left(\left|\eta_{\pi_1}^{-1}(\underline{x} \,(\mathrm{mod}\, \pi_1))\right| - 1\right)\left(\left|\eta_{\pi_2}^{-1}(\underline{x} \,(\mathrm{mod}\, \pi_2))\right| - 1\right) \\ &= q^{-(n+1)(2\Delta - b)} \sum_{\substack{\chi_{\pi_1} \neq \chi_0 \\ \chi_{\pi_2} \neq \chi_0}} \sum_{\substack{\underline{x} \in \mathcal{O}_K^{n+1} \\ \deg_T(\underline{x}) < 2\Delta - b}} S_F\left(\bar{\pi}_2\underline{x}, \chi_{\pi_1}\right) S_F\left(\bar{\pi}_1\underline{x}, \chi_{\pi_2}\right),\end{aligned} \tag{6.2}$$

in which, for each π_i, the sum is over all non-principal characters χ_{π_i} of order ℓ.

For later reference, we remark that the left-hand side of (6.2) is unchanged if we omit the condition $F(\underline{x}) \not\equiv 0 \,(\mathrm{mod}\, \pi_1\pi_2)$. Indeed, upon observing that

$$\eta_{\pi_i}^{-1}(\underline{x} \,(\mathrm{mod}\, \pi_i)) = \{z \in k_{\pi_i} : z^m = F(\underline{x})\},$$

it follows that $(|\eta_{\pi_i}^{-1}(\underline{x} \,(\mathrm{mod}\, \pi_i))| - 1) = 0$ whenever $F(\underline{x}) = 0$.

We note that for any polynomial $F \in \mathcal{O}_K[X_0, \dots, X_n]$, the trivial bound for $S_F(\underline{w}, \chi_\pi)$, valid for every prime $\pi \in \mathcal{O}_K$ and every $\underline{w} \in \mathcal{O}_K^{n+1}$, is

$$|S_F(\underline{w}, \chi_\pi)| \leq |\pi|_\infty^{n+1} = q^{(n+1)\deg_T(\pi)}. \tag{6.3}$$

(Of course, this is far from sharp.) We then see that for all primes $\pi_1, \pi_2 \in \mathcal{P}$ with $\pi_1 \neq \pi_2$, the trivial bound for the right-hand side of (6.2) is

$$\ll \ell^2 q^{(n+1)2\Delta},$$

which is larger than the trivial bound for the left-hand side whenever $b \leq 2\Delta$, as will occur in our ultimate choice for Δ. The transformation is nevertheless worthwhile, since we have passed from an incomplete (multiplicative) character sum on the left-hand side to a sum of complete (mixed) character sums on the right-hand side, to which we can apply Weil-Deligne bounds.

We will not try to average nontrivially over the characters χ_{π_i} or over $\underline{x}$; for our current scope, it will suffice to prove a nontrivial bound for the individual sums S_F, which we will return to in §7.

6.1. **Expansion in terms of multiplicative characters.** We will now rewrite the unramified sieve term in (5.8) in terms of characters. Let us fix primes $\pi_1, \pi_2 \in \mathcal{O}_K$ with $\pi_1 \neq \pi_2$ (the condition $\pi_1, \pi_2 \in \mathcal{P}$ need only be specified later). For any $\underline{x} \in \mathcal{O}_K^{n+1}$, we will rewrite each of the quantities $(|\eta_{\pi_i}^{-1}(\underline{x} \,(\mathrm{mod}\, \pi_i))| - 1)$ as a character sum by using the following proposition, whose proof we defer to Section 6.4.

Proposition 6.2. *Let q be an odd rational prime power and, as before, take $K = \mathbb{F}_q(T)$. Fix a rational prime ℓ with $\ell \mid (q-1)$. For any prime $\pi \in \mathcal{O}_K$ and any $a \in \mathcal{O}_K$, we have*

$$\#\left\{ y \,(\mathrm{mod}\, \pi) \in k_\pi : a \equiv y^\ell \,(\mathrm{mod}\, \pi) \right\} = \sum_{\chi_\pi} \chi_\pi(a),$$

where χ_π runs over all multiplicative characters on $\mathcal{O}_K$ of modulus π and order dividing ℓ, whose definition is extended from k_π^* to $\mathcal{O}_K$ via the rule

$$\forall x \in \mathcal{O}_K \text{ with } \pi \mid x, \;\; \chi_\pi(x) := \begin{cases} 0 & \text{if } \chi_\pi \text{ is non-principal,} \\ 1 & \text{if } \chi_\pi \text{ is principal.} \end{cases} \tag{6.4}$$

By Proposition 6.2, for each of π_1, π_2 and any $\underline{x} \in \mathcal{O}_K^{n+1}$,

$$\begin{aligned} \left|\eta_{\pi_i}^{-1}(\underline{x} \,(\mathrm{mod}\, \pi_i))\right| &= \#\left\{ y \,(\mathrm{mod}\, \pi_i) \in k_{\pi_i} : F(x_0, \ldots, x_n) \equiv y^\ell \,(\mathrm{mod}\, \pi_i) \right\} \\ &= \sum_{\chi_{\pi_i}} \chi_{\pi_i}(F(x_0, \ldots, x_n)), \end{aligned} \tag{6.5}$$

where χ_{π_i} runs over all multiplicative characters on $\mathcal{O}_K$ of modulus π_i and order dividing ℓ, extended to $\mathcal{O}_K$ as in (6.4). From this, we can write

$$\left|\eta_{\pi_i}^{-1}(\underline{x} \,(\mathrm{mod}\, \pi_i))\right| - 1 = \sum_{\chi_{\pi_i} \neq \chi_0} \chi_{\pi_i}(F(x_0, \ldots, x_n)), \tag{6.6}$$

in which the sum is now over all (non-principal) characters of order ℓ and modulus π_i.

6.2. **Fourier expansion in terms of additive characters.** Next, we perform a Fourier expansion in terms of the additive character defined in Section 3. We defer the proof of the required proposition to Section 6.5.

Proposition 6.3. *Let q be an odd rational prime power and, as before, take $K = \mathbb{F}_q(T)$. For all primes $\pi, \pi' \in \mathcal{O}_K$ with $\pi \neq \pi'$, for all $\chi_\pi, \chi_{\pi'}$ non-principal multiplicative characters of $\mathcal{O}_K$ of moduli π, π' (respectively), for all $G \in \mathcal{O}_K[X_0, \ldots, X_n]$, and for all integers b such that*

$$0 < b < \deg_T(\pi\pi'),$$

we have

$$\sum_{\substack{\underline{x} \in \mathcal{O}_K^{n+1} \\ \deg_T(\underline{x}) < b}} \chi_\pi(G(\underline{x}))\chi_{\pi'}(G(\underline{x})) = q^{-(n+1)(\deg_T(\pi\pi') - b)} \sum_{\substack{\underline{x} \in \mathcal{O}_K^{n+1} \\ \deg_T(\underline{x}) < \deg_T(\pi\pi') - b}} S_G\left(\bar{\pi}'\underline{x}, \chi_\pi\right) \, S_G\left(\bar{\pi}\underline{x}, \chi_{\pi'}\right),$$

where $\bar{\pi}', \bar{\pi}' \in \mathcal{O}_K$ are determined by the congruences $\pi\bar{\pi} \equiv 1 \,(\mathrm{mod}\, \pi')$, $\pi'\bar{\pi}' \equiv 1 \,(\mathrm{mod}\, \pi)$, and the sum $S_G(\underline{w}, \chi_\pi)$ is defined in (6.1).

6.3. **Application to the unramified sieve term.** By (6.6) and Proposition 6.3, we observe that, for any primes $\pi_1, \pi_2 \in \mathcal{P}$ with $\pi_1 \neq \pi_2$,

$$\sum_{\substack{\underline{x} \in \mathcal{O}_K^{n+1} \\ \deg_T(\underline{x}) < b}} \left(\left| \eta_{\pi_1}^{-1}(\underline{x} \,(\mathrm{mod}\, \pi_1)) \right| - 1 \right) \left(\left| \eta_{\pi_2}^{-1}(\underline{x} \,(\mathrm{mod}\, \pi_2)) \right| - 1 \right)$$
$$= q^{-(n+1)(2\Delta - b)} \sum_{\substack{\chi_{\pi_1} \neq \chi_0 \\ \chi_{\pi_2} \neq \chi_0}} \sum_{\substack{\underline{x} \in \mathcal{O}_K^{n+1} \\ \deg_T(\underline{x}) < 2\Delta - b}} S_F\left(\bar{\pi}_2 \underline{x}, \chi_{\pi_1} \right) S_F\left(\bar{\pi}_1 \underline{x}, \chi_{\pi_2} \right). \quad (6.7)$$

Here we must assume that

$$b < 2\Delta \quad (6.8)$$

in order to apply Proposition 6.3. We keep this assumption from here onwards. Finally, recall the sum over $\underline{x}$ in (5.8) actually restricts to those $\underline{x}$ for which $F(\underline{x}) \not\equiv 0 \,(\mathrm{mod}\, \pi_1\pi_2)$. But as remarked immediately below (6.2), the left-hand side of (6.2) does not change when the restriction $F(\underline{x}) \not\equiv 0 \,(\mathrm{mod}\, \pi_1\pi_2)$ is removed. Thus, (6.7) is actually equal to the unramified sieve term. As such, Proposition 6.1 is proved, as long as we verify Proposition 6.2 and Proposition 6.3.

6.4. **Proof of Proposition 6.2.** As in the setting of the proposition, let q be an odd rational prime power, take $K = \mathbb{F}_q(T)$, let ℓ be a rational prime with $\ell \mid (q-1)$, and let $\pi \in \mathcal{O}_K$ be a prime. Since $\ell \mid (q-1)$, then $\ell \mid (q^{\deg_T(\pi)} - 1) = |k_\pi^*|$. We can then consider the set of multiplicative characters

$$\chi_\pi : k_\pi^* \to \mathbb{C}$$

of order dividing ℓ on k_π^*, which forms a group of order ℓ. We can extend χ_π to be defined over $\mathcal{O}_K$ by setting $\chi_\pi(a) = 0$ when $\pi \mid a$. These characters can be given explicitely by ℓ-power symbols, as described in [Ros02, ch. 3]. It is known that

$$\chi_\pi(a) = 1 \;\Leftrightarrow\; \pi \nmid a \text{ and } X^\ell \equiv a \,(\mathrm{mod}\, \pi) \text{ is solvable}; \quad (6.9)$$

see [Ros02, Prop. 3.1, p. 24].

The character sums are given by

$$\sum_{\chi_\pi} \chi_\pi(a) = \begin{cases} 1 & \text{if } \pi \mid a, \\ \ell & \text{if } \pi \nmid a \text{ and } X^\ell \equiv a \,(\mathrm{mod}\, \pi) \text{ is solvable}, \\ 0 & \text{if } \pi \nmid a \text{ and } X^\ell \equiv a \,(\mathrm{mod}\, \pi) \text{ is not solvable}, \end{cases} \quad (6.10)$$

where χ_π runs over all the multiplicative characters of $\mathcal{O}_K$ of modulus π and order dividing ℓ. See [Ros02, Prop. 4.2, p. 35]. Consequently,

$$\sum_{\chi_\pi} \chi_\pi(a) = \#\left\{ y \,(\mathrm{mod}\, \pi) \in k_\pi : a \equiv y^\ell \,(\mathrm{mod}\, \pi) \right\},$$

which completes the proof of Proposition 6.2.

6.5. **Proof of Proposition 6.3.** As before, let q be an odd rational prime power and take $K = \mathbb{F}_q(T)$. We first prove a lemma about detecting congruences with the additive character ψ_∞.

Lemma 6.4. *Let $u \in \mathcal{O}_K$ and $\underline{a} \in \mathcal{O}_K^{n+1}$. Let b be an integer such that*

$$0 < b < \deg_T(u).$$

Then
$$\sum_{\substack{\underline{x}\in\mathcal{O}_K^{n+1}\\ \deg_T(\underline{x})<b\\ \underline{x}\equiv\underline{a}\,(\mathrm{mod}\,u)}} 1 = q^{-(n+1)(\deg_T(u)-b)} \sum_{\substack{\underline{x}\in\mathcal{O}_K^{n+1}\\ \deg_T(\underline{x})<\deg_T(u)-b}} \psi_\infty\left(-\frac{\underline{x}}{u}\cdot\underline{a}\right).$$

Proof. We rewrite the left-hand side by using the following indicator functions:
$$w_{\infty,b}:\mathcal{O}_K\longrightarrow\{0,1\},$$
$$w_{\infty,b}(x):=\begin{cases}1 & \text{if } \deg_T(x)<b,\\ 0 & \text{otherwise,}\end{cases}$$
and
$$w_b:\mathcal{O}_K^{n+1}\longrightarrow\{0,1\},$$
$$w_b(\underline{x}):=\prod_{1\le i\le n} w_{\infty,b}(x_i).$$
With these definitions, we write
$$\sum_{\substack{\underline{x}\in\mathcal{O}_K^{n+1}\\ \deg_T(\underline{x})<b\\ \underline{x}\equiv\underline{a}\,(\mathrm{mod}\,u)}} 1 = \sum_{\underline{x}'\in\mathcal{O}_K^{n+1}} w_b(\underline{a}+\underline{x}'u).$$

To understand the above sum, we use Fourier analysis on K_∞^{n+1}, as detailed in [BV15, Section 2]. By [BV15, Lemma 2.1], we obtain
$$\begin{aligned}\sum_{\underline{x}'\in\mathcal{O}_K^{n+1}} w_b(\underline{a}+\underline{x}'u) &= \sum_{\underline{x}''\in\mathcal{O}_K^{n+1}}\int_{K_\infty^{n+1}} w_b(\underline{a}+\underline{s}u)\ \psi_\infty(\underline{x}''\cdot\underline{s})\ d\underline{s}\\ &= \sum_{\underline{x}''\in\mathcal{O}_K^{n+1}}\int_{\left\{\underline{s}\in K_\infty^{n+1}:\ |\underline{a}+\underline{s}u|_\infty<q^b\right\}} \psi_\infty(\underline{x}''\cdot\underline{s})\ d\underline{s}.\end{aligned}$$
Making the change of variables
$$\underline{t}:=\underline{a}+\underline{s}\ u,$$
for which
$$d\underline{t}=q^{(n+1)\deg_T(u)}\ d\underline{s},$$
by [BV15, Lemma 2.3] we infer that
$$\begin{aligned}&\int_{\left\{\underline{s}\in K_\infty^{n+1}:\ |\underline{a}+\underline{s}u|_\infty<q^b\right\}} \psi_\infty(\underline{x}''\cdot\underline{s})\ d\underline{s}\\ &\quad= q^{-(n+1)\deg_T(u)}\int_{\left\{\underline{t}\in K_\infty^{n+1}:\ |\underline{t}|_\infty<q^b\right\}}\psi_\infty\left(\underline{x}''\cdot\frac{\underline{t}-\underline{a}}{u}\right)\ d\underline{t}\\ &\quad= q^{-(n+1)\deg_T(u)}\psi_\infty\left(-\frac{\underline{x}''\cdot\underline{a}}{u}\right)\int_{\left\{\underline{t}\in K_\infty^{n+1}:\ |\underline{t}|_\infty<q^b\right\}}\psi_\infty\left(\frac{\underline{x}''}{u}\cdot\underline{t}\right)\ d\underline{t}.\end{aligned}$$
By [BV15, Lemma 2.2],
$$\int_{\left\{\underline{t}\in K_\infty^{n+1}:\ |\underline{t}|_\infty<q^b\right\}}\psi_\infty\left(\frac{\underline{x}''}{u}\cdot\underline{t}\right)\ d\underline{t}=\begin{cases} q^{(n+1)b} & \text{if } \left|\frac{\underline{x}''}{u}\right|_\infty<q^{-b},\\ 0 & \text{otherwise.}\end{cases}$$

Hence

$$\sum_{\underline{x}'\in\mathcal{O}_K^{n+1}} w_b(\underline{a}+\underline{x}'u) \;=\; q^{-(n+1)(\deg_T(u)-b)} \sum_{\substack{\underline{x}''\in\mathcal{O}_K^{n+1}\\ \deg_T(\underline{x}'')<\deg_T(u)-b}} \psi_\infty\left(-\frac{\underline{x}''}{u}\cdot\underline{a}\right).$$

□

Now we prove Proposition 6.3. Let $\pi,\pi'\in\mathcal{O}_K$ be primes with $\pi\neq\pi'$, let $\chi_\pi,\chi_{\pi'}$ be non-principal multiplicative characters of $\mathcal{O}_K$ of moduli π, π' (respectively), and let $G\in\mathcal{O}_K[X_0,\ldots,X_n]$. Fix an integer b such that

$$0<b<\deg_T(\pi_1\pi_2).$$

We partition $\mathcal{O}_K^{n+1}$ according to the residue classes modulo $\pi\pi'$:

$$\sum_{\substack{\underline{x}\in\mathcal{O}_K^{n+1}\\ \deg_T(\underline{x})<b}} \chi_\pi(G(\underline{x}))\chi_{\pi'}(G(\underline{x})) \;=\; \sum_{\underline{a}\,(\mathrm{mod}\,\pi\pi')\in(\mathcal{O}_K/(\pi\pi'))^{n+1}} \chi_\pi(G(\underline{a}))\chi_{\pi'}(G(\underline{a})) \sum_{\substack{\underline{x}\in\mathcal{O}_K^{n+1}\\ \deg_T(\underline{x})<b\\ \underline{x}\equiv\underline{a}\,(\mathrm{mod}\,\pi\pi')}} 1.$$

By Lemma 6.4, the above double sum equals

$$q^{-(n+1)(\deg_T(\pi\pi'))-b} \sum_{\underline{a}\,(\mathrm{mod}\,\pi\pi')\in(\mathcal{O}_K/(\pi\pi'))^{n+1}} \chi_\pi(G(\underline{a}))\chi_{\pi'}(G(\underline{a})) \sum_{\substack{\underline{x}\in\mathcal{O}_K^{n+1}\\ \deg_T(\underline{x})<\deg_T(\pi\pi')-b}} \psi_\infty\left(-\frac{\underline{x}}{\pi\pi'}\cdot\underline{a}\right). \tag{6.11}$$

Recall that $\pi\neq\pi'$. Then, on one hand, there exist uniquely determined $\overline{\pi}'\,(\mathrm{mod}\,\pi)\in k_\pi$, $\overline{\pi}\,(\mathrm{mod}\,\pi')\in k_{\pi'}$ such that

$$\pi\overline{\pi}\equiv 1\,(\mathrm{mod}\,\pi'),$$
$$\pi'\overline{\pi}'\equiv 1\,(\mathrm{mod}\,\pi);$$

on the other hand, by the Chinese Remainder Theorem, there exist uniquely determined elements $\underline{a}_1\,(\mathrm{mod}\,\pi)\in k_\pi^{n+1}$, $\underline{a}_2\,(\mathrm{mod}\,\pi')\in k_{\pi'}^{n+1}$ such that

$$\underline{a}\equiv\underline{a}_1\pi'+\underline{a}_2\pi\,(\mathrm{mod}\,\pi\pi').$$

Consequently,

$$\begin{aligned}&\sum_{\underline{a}\,(\mathrm{mod}\,\pi\pi')\in(\mathcal{O}_K/(\pi\pi'))^{n+1}} \chi_\pi(G(\underline{a}))\chi_{\pi'}(G(\underline{a})) \sum_{\substack{\underline{x}\in\mathcal{O}_K^{n+1}\\ \deg_T(\underline{x})<\deg_T(\pi\pi')-b}} \psi_\infty\left(-\frac{\underline{x}}{\pi\pi'}\cdot\underline{a}\right)\\ =&\sum_{\substack{\underline{a}_1\,(\mathrm{mod}\,\pi)\in k_\pi^{n+1}\\ \underline{a}_2\,(\mathrm{mod}\,\pi')\in k_{\pi'}^{n+1}}} \chi_\pi\left(G(\underline{a}_1\pi')\right)\chi_{\pi'}\left(G(\underline{a}_2\pi)\right) \sum_{\substack{\underline{x}\in\mathcal{O}_K^{n+1}\\ \deg_T(\underline{x})<\deg_T(\pi\pi')-b}} \psi_\infty\left(-\frac{\underline{x}\cdot\underline{a}_1}{\pi}\right)\,\psi_\infty\left(-\frac{\underline{x}\cdot\underline{a}_2}{\pi'}\right).\end{aligned}$$

Using that $\pi\neq\pi'$, we make the change of variables $\underline{a_1}\mapsto\underline{a_1}\overline{\pi}'\,(\mathrm{mod}\,\pi)$ and $\underline{a_2}\mapsto\underline{a_2}\overline{\pi}\,(\mathrm{mod}\,\pi')$ and deduce that the above expression equals

$$\sum_{\substack{\underline{a}_1\,(\mathrm{mod}\,\pi)\in k_\pi^{n+1}\\ \underline{a}_2\,(\mathrm{mod}\,\pi')\in k_{\pi'}^{n+1}}} \chi_\pi\left(G(\underline{a}_1)\right)\chi_{\pi'}\left(G(\underline{a}_2)\right) \sum_{\substack{\underline{x}\in\mathcal{O}_K^{n+1}\\ \deg_T(\underline{x})<\deg_T(\pi\pi')-b}} \psi_\infty\left(-\frac{(\overline{\pi}'\underline{x})\cdot\underline{a}_1}{\pi}\right)\,\psi_\infty\left(-\frac{(\overline{\pi}\underline{x})\cdot\underline{a}_2}{\pi'}\right).$$

This completes the proof of Proposition 6.3.

7. The Weil-Deligne bounds

We now state and prove the Weil-Deligne estimates that will be used to bound the unramified sieve term in the form of the expansion in Proposition 6.1.

Proposition 7.1. *(Weil-Deligne bounds)*
Let q be the power of an odd rational prime and take $K = \mathbb{F}_q(T)$. Let $n \geq 1$ be an integer and $H \in \mathcal{O}_K[X_0, \dots, X_n]$ a homogeneous polynomial of degree $m \geq 2$ such that the projective hypersurface defined by $H(X_0, \dots, X_n) = 0$ in $\mathbb{P}^n_K$ is nonsingular. Assume that $\mathrm{char}\, \mathbb{F}_q \nmid m$. *Denote by W, $\mathcal{W}$ the nonsingular projective hypersurface, respectively the affine hypersurface, defined by*

$$H(X_0, \dots, X_n) = 0.$$

Denote by $\mathcal{P}_{\mathrm{exc}}(H) \subseteq \mathcal{O}_K$ the finite set of exceptional primes introduced in Proposition 4.1. For a prime $\pi \notin \mathcal{P}_{\mathrm{exc}}(H)$, denote by W_π, $\mathcal{W}_\pi$ the nonsingular projective hypersurface, respectively the affine hypersurface, defined by

$$H(X_0, \dots, X_n) \equiv 0 \,(\mathrm{mod}\, \pi),$$

and denote by W_π^*, $\mathcal{W}_\pi^*$ their duals. Fix a rational prime ℓ with $\ell \mid (q-1)$. Then, for every prime $\pi \notin \mathcal{P}_{\mathrm{exc}}(H)$, for every (non-principal) multiplicative character χ_π of $\mathcal{O}_K$ of modulus π and order ℓ, and for every $\underline{w} \in \mathcal{O}_K^{n+1}$, the mixed character sum $S_H(\underline{w}, \chi_\pi)$ defined in (6.1) satisfies the bounds:

(i) *provided $\underline{w} \equiv \underline{0} \,(\mathrm{mod}\, \pi)$,*

$$|S_H(\underline{w}, \chi_\pi)| \ll_{n, \deg_{\underline{X}}(H)} q^{\frac{(n+2)\deg_T(\pi)}{2}};$$

(ii) *provided $\underline{w} \not\equiv \underline{0} \,(\mathrm{mod}\, \pi)$,*

$$|S_H(\underline{w}, \chi_\pi)| \ll_{n, \deg_{\underline{X}}(H)} q^{\frac{(n+2)\deg_T(\pi)}{2}};$$

(iii) *provided $\underline{w} \not\equiv \underline{0} \,(\mathrm{mod}\, \pi)$ and $\underline{w} \notin \mathcal{W}_\pi^*$,*

$$|S_H(\underline{w}, \chi_\pi)| \ll_{n, \deg_{\underline{X}}(H)} q^{\frac{(n+1)\deg_T(\pi)}{2}}.$$

The proof of Proposition 7.1 is based on estimates for character sums of polynomials in several variables, pioneered by Deligne [Del74] and further generalized by Katz [Kat02], [Kat07], Rojas-León, and others. In the proof we will call upon the following definition:

Definition 7.2. *Let k be a finite field and let $d, r \geq 1$ be integers. Let $f \in k[X_1, \dots, X_r]$ be a polynomial of degree d, which we write as*

$$f = f_d + f_{d-1} + \cdots + f_0$$

for uniquely determined homogeneous polynomials $f_i \in k[X_1, \dots, X_r]$ with $\deg_{\underline{X}}(f_i) = i$. We call f a Deligne polynomial over *k if:*

(i) $\mathrm{char}\, k \nmid d$*;*
(ii) *the equation $f_d = 0$ defines a smooth, degree d hypersurface in $\mathbb{P}^{r-1}_k$.*

Relative to the trivial bound for $|S_H(\underline{w}, \chi_\pi)|$ given in (6.3), we see that the non-trivial bounds given in cases (i) and (ii) provide square-root cancellation in all but one of the $n+1$ variables of the sum, while case (iii) provides square-root cancellation in all $n+1$ variables. We think of cases (i) and (ii) as exceptional "zero" or "bad" cases, respectively, and of case (iii) as the "good" case. In our application in §8, we use that for a fixed prime π, as the parameter $\underline{w}$ varies over $\mathcal{O}_K^{n+1}$, case (iii) is generic, with cases (i) and (ii) being rare.

7.1. **Proof of part (i) of Proposition 7.1.** We consider the case $\underline{w} \equiv \underline{0} \,(\bmod\, \pi)$ and seek to bound

$$|S_H\left(\underline{0}, \chi_\pi\right)| = \left| \sum_{\underline{a}\,(\bmod\,\pi) \in k_\pi^{n+1}} \chi_\pi(H(\underline{a})) \right|. \tag{7.1}$$

Our main tool is the following estimate for multiplicative character sums, which is a special case of a much more general result in [RL05].

Theorem 7.3. *(special case of* [RL05, Thm. 1.1(a)]*)*
Let k be a finite field, $r \geq 1$ an integer, and $\chi : k^* \longrightarrow \mathbb{C}^*$ a non-principal multiplicative character, extended to k by $\chi(0) := 0$. Let $Y = \mathbb{P}_k^r$, let $H \in k[X_0, \ldots, X_r]$ be a homogeneous polynomial of degree m, and let $Z \in k[X_0, \ldots, X_r]$ be a homogeneous polynomial of degree e. Assume that $(e, m) = 1$, $(e, \mathrm{char} k) = 1$, and χ^e is nontrivial. Let H denote the hypersurface $H = 0$ in $\mathbb{P}^r$ and similarly let Z denote the hypersurface $Z = 0$ in $\mathbb{P}^r$. Assume that $Y \cap H \cap Z$ has codimension 2 in Y, and let δ denote the dimension of the singular locus of $Y \cap H \cap Z \subset \mathbb{P}^r$. Set $V = Y - (H \cup Z)$ and define $f : V \to k^*$ by $f(\underline{X}) = H(\underline{X})^e / Z(\underline{X})^m$. Then

$$\left| \sum_{\underline{x} \in V(k)} \chi(f(\underline{x})) \right| \leq 3(3 + e + m)^{r+2} |k|^{\frac{r+\delta+2}{2}}.$$

Corollary 7.4. *Let k be a finite field, $n \geq 1$ an integer, and $\chi : k^* \longrightarrow \mathbb{C}^*$ a non-principal multiplicative character, extended to k by $\chi(0) := 0$. Let $H \in k[X_0, \ldots, X_n]$ be a homogeneous polynomial of degree $m \geq 1$ such that the projective hypersurface defined by $H(X_0, \ldots, X_n) = 0$ in $\mathbb{P}_k^n$ is nonsingular. Then*

$$\left| \sum_{\underline{x} \in k^{n+1}} \chi(H(\underline{x})) \right| \leq 3(m + 4)^{n+3} |k|^{\frac{n+2}{2}}.$$

Proof of Corollary 7.4. Note that $H(\underline{x})$ is not a well-defined map on a projective variety, since for example $\underline{x} = c\underline{x}$ as a projective point, for each $c \in k^*$, while $H(c\underline{x}) = c^m H(\underline{x}) \neq H(\underline{x})$ unless $c^m = 1$ in k. The setting of Theorem 7.3 corrects for this, as follows. We set $Y = \mathbb{P}^{n+1}$ in the variables $X_0, \ldots, X_n, T$, so that $r = n+1$. We define $H(X_0, \ldots, X_n, T) = H(X_0, \ldots, X_n)$ with degree m and $Z(X_0, \ldots, X_n, T) = T$ so that $e = 1$. Then $f(\underline{X}) = H(\underline{X})/T^m = H(c\underline{x})/(cT)^m$ for any $c \in k^*$, so is well-defined as a polynomial map on $V = \mathbb{P}^{n+1} - ((H = 0) \cup (T = 0))$. Furthermore, since $\chi(0) = 0$, and using the homogeneity described above,

$$\sum_{(\underline{x}, t) \in V(k)} \chi(f(\underline{x}, t)) = \sum_{(\underline{x}, t) \in \mathbb{P}_k^{n+1} \setminus (t=0)} \chi(H(\underline{x})/t^d) = \sum_{(\underline{x}, 1) \in \mathbb{P}_k^{n+1}} \chi(H(\underline{x})) = \sum_{\underline{x} \in k^{n+1}} \chi(H(\underline{x})).$$

Note that

$$\mathbb{P}^{n+1} \cap (H = 0) \cap (T = 0) = \mathbb{P}^n \cap (H = 0),$$

so that in the notation of the theorem, $\delta = -1$ since by assumption $H = 0$ is nonsingular as a projective hypersurface in $\mathbb{P}^n$. Moreover, the codimension of $\mathbb{P}^{n+1} \cap (H = 0) \cap (T = 0)$ in $\mathbb{P}^{n+1}$ is 2, as required. Hence by Theorem 7.3, the corollary holds.

$\square$

The corollary immediately implies (i) in Proposition 7.1, upon taking H as in the proposition, and any $\pi \notin \mathcal{P}_{\mathrm{exc}}(H)$, with $k = k_\pi$ and χ_π as in the proposition.

7.2. **Proof of part (ii) of Proposition 7.1.** We consider the case $\underline{w} \not\equiv \underline{0} \pmod{\pi}$ and seek to bound

$$|S_H(\underline{w}, \chi_\pi)| = \left| \sum_{\underline{a} \pmod{\pi} \in k_\pi^{n+1}} \chi_\pi(H(\underline{a}))\psi_\infty\left(-\frac{\underline{w}\cdot\underline{a}}{\pi}\right)\right|.$$

Our main tool is the following estimate for non-singular additive character sums, due to Deligne:

Theorem 7.5. *(*[Del74, Thm. 8.4 p. 302]*)*
Let k be a finite field, $r \geq 1$ an integer, and $\psi : (k, +) \longrightarrow (\mathbb{C}^*, \cdot)$ a non-trivial additive character. Let $g \in k[X_1, \ldots, X_r]$ be a polynomial of degree $d \geq 2$. Assume that g is a Deligne polynomial over k. Then

$$\left|\sum_{\underline{a}\in k^r} \psi(g(\underline{a}))\right| \leq (d-1)^r |k|^{\frac{r}{2}}.$$

We apply Theorem 7.5 to the finite field k_π, the integer $r = n+1$, the character ψ_∞, and each of $q^{\deg_T(\pi)} - 1$ instances of polynomials g, derived from H, as explained in what follows.

Recall that the Gauss sum

$$\tau(\chi_\pi) := \sum_{\alpha\in k_\pi} \chi_\pi(\alpha)\psi_\infty\left(\frac{\alpha}{\pi}\right)$$

satisfies the Riemann Hypothesis (see, for example, [IK04, Prop 11.5 p. 275]):

$$|\tau(\chi_\pi)| = q^{\frac{\deg_T(\pi)}{2}}.$$

Note that for each $\underline{a} \pmod{\pi} \in k_\pi^{n+1}$ such that $H(\underline{a}) \not\equiv 0 \pmod{\pi}$, we have a bijection

$$\begin{array}{ccc} k_\pi^* & \longrightarrow & k_\pi^*, \\ \alpha & \mapsto & \alpha H(\underline{a})^{-1}. \end{array}$$

Then, using properties of the Gauss sum (e.g. [IR90, Prop. 8.2.2 p. 92]), we obtain

$$\begin{aligned}
S_H(\underline{w}, \chi_\pi) &= \frac{\chi_\pi(-1)\tau(\chi_\pi)\tau\left(\overline{\chi_\pi}\right)}{q^{\deg_T(\pi)}} \sum_{\underline{a} \pmod{\pi}\in k_\pi^{n+1}} \chi_\pi\left(H(\underline{a})\right)\psi_\infty\left(-\frac{\underline{w}\cdot\underline{a}}{\pi}\right) \\
&= \frac{\chi_\pi(-1)\tau(\chi_\pi)\tau\left(\overline{\chi_\pi}\right)}{q^{\deg_T(\pi)}} \sum_{\substack{\underline{a} \pmod{\pi}\in k_\pi^{n+1} \\ H(\underline{a})\not\equiv 0 \pmod{\pi}}} \chi_\pi\left(H(\underline{a})\right)\psi_\infty\left(-\frac{\underline{w}\cdot\underline{a}}{\pi}\right) \\
&= \frac{\chi_\pi(-1)\tau(\chi_\pi)}{q^{\deg_T(\pi)}} \sum_{\alpha\in k_\pi^*} \sum_{\substack{\underline{a} \pmod{\pi}\in k_\pi^{n+1} \\ H(\underline{a})\not\equiv 0 \pmod{\pi}}} \chi_\pi\left(\alpha^{-1}H(\underline{a})\right)\psi_\infty\left(\frac{\alpha - \underline{w}\cdot\underline{a}}{\pi}\right) \\
&= \frac{\chi_\pi(-1)\tau(\chi_\pi)}{q^{\deg_T(\pi)}} \sum_{\beta\in k_\pi^*} \chi_\pi(\beta) \sum_{\underline{a} \pmod{\pi}\in k_\pi^{n+1}} \psi_\infty\left(\frac{\beta H(\underline{a}) - \underline{w}\cdot\underline{a}}{\pi}\right). \qquad (7.2)
\end{aligned}$$

In the last identity, in order to sum back in the contribution from $H(\underline{a}) \equiv 0 \pmod{\pi}$, we used that

$$\sum_{\beta\in k_\pi^*} \chi_\pi(\beta) \sum_{\substack{\underline{a} \pmod{\pi}\in k_\pi^{n+1} \\ H(\underline{a})\equiv 0 \pmod{\pi}}} \psi_\infty\left(\frac{\beta H(\underline{a}) - \underline{w}\cdot\underline{a}}{\pi}\right) = \left(\sum_{\beta\in k_\pi^*}\chi_\pi(\beta)\right)\left(\sum_{\substack{\underline{a} \pmod{\pi}\in k_\pi^{n+1} \\ H(\underline{a})\equiv 0 \pmod{\pi}}} \psi_\infty\left(\frac{-\underline{w}\cdot\underline{a}}{\pi}\right)\right) = 0,$$

which follows from the orthogonality of the characters χ_π. By taking absolute values in (7.2), we deduce that

$$|S_H(\underline{w},\chi_\pi)| \leq q^{-\frac{\deg_T(\pi)}{2}} \sum_{\beta\in k_\pi^*} \left| \sum_{\underline{a}\in k_\pi^{n+1}} \psi_\infty\left(\frac{\beta H(\underline{a}) - \underline{w}\cdot\underline{a}}{\pi}\right)\right|. \tag{7.3}$$

We estimate the inner sum above using Theorem 7.5 for each of the polynomials over k_π defined by the congruence

$$g(X_0,\dots,X_n) \equiv \beta H(X_0,\dots,X_n) - \underline{w}\cdot(X_0,\dots,X_n) \pmod{\pi}$$

and for the additive character ψ_∞. Using that $\beta \neq 0$, $\deg_{\underline{X}}(H) = m \geq 2$, and $\pi \notin \mathcal{P}_{\rm exc}(H)$, we obtain that $\deg_{\underline{X}}(g) = m$. Let us write $g = g_0 + \cdots + g_m$, where $g_0, \dots, g_m$ are the uniquely determined homogeneous polynomials in $k_\pi[X_0,\dots,X_n]$ such that $\deg_{\underline{X}}(g_i) = i$. Then $g_m \equiv \beta H \pmod{\pi}$. By hypothesis, $H = 0$ is nonsingular in $\mathbb{P}^n_{\overline{K}}$ and $\pi \notin \mathcal{P}_{\rm exc}(H)$ so that W_π is nonsingular; also char $\mathbb{F}_q \nmid \deg(H)$. Thus $H(\bmod\,\pi)$ is a Deligne polynomial over k_π, and hence g also is. By (7.3) and Theorem 7.5, we deduce that

$$|S_H(\underline{w},\chi_\pi)| \leq q^{-\frac{\deg_T(\pi)}{2}} \sum_{\beta\in k_\pi^*} (m-1)^{n+1} q^{\frac{(n+1)\deg_T(\pi)}{2}} < (m-1)^{n+1} q^{\frac{(n+2)\deg_T(\pi)}{2}},$$

which completes the proof for case (ii).

7.3. **Proof of part (iii) of Proposition 7.1.** We consider the case $\underline{w} \not\equiv \underline{0} \pmod{\pi}$, $\underline{w} \notin \mathcal{W}_\pi^*$, and seek to bound

$$|S_H(\underline{w},\chi_\pi)| = \left| \sum_{\underline{a}\,(\bmod\,\pi)\in k_\pi^{n+1}} \chi_\pi(H(\underline{a}))\psi_\infty\left(-\frac{\underline{w}\cdot\underline{a}}{\pi}\right)\right|$$

in such a way that we improve upon the bound in part (ii). Our main tool is the following estimate for non-singular mixed character sums, again due to Katz:

Theorem 7.6. *(*[Kat07, Thm. 1.1 p. 3]*)*
Let k be a finite field, $r \geq 1$ an integer, $\chi : (k^*,\cdot) \longrightarrow (\mathbb{C}^*,\cdot)$ a non-principal multiplicative character, extended to k by $\chi(0) := 0$, and $\psi : (k,+) \longrightarrow (\mathbb{C}^*,\cdot)$ a non-trivial additive character. Let $f,g \in k[X_1,\dots,X_r]$ be polynomials of degrees $d,e \geq 1$ with leading homogeneous forms f_d of degree d, g_e of degree e (respectively). Assume that:

(i) *f is a Deligne polynomial;*
(ii) *g is a Deligne polynomial;*
(iii) *in case $r \geq 2$, the smooth hypersurfaces in $\mathbb{P}^{r-1}_k$ defined by $f_d = 0$ and by $g_e = 0$ are transverse, in the sense that their intersection is smooth and of codimension 2 in $\mathbb{P}^{r-1}_k$.*

Then

$$\left|\sum_{\underline{a}\in k^r} \chi(f(\underline{a}))\psi(g(\underline{a}))\right| \ll_{r,d,e} |k|^{\frac{r}{2}}.$$

We apply Theorem 7.6 to the finite field k_π, the integer $r = n+1$, the characters ψ_∞, χ_π, and the polynomials over k_π defined by the congruences

$$\begin{aligned} f(X_0,\dots,X_n) &\equiv H(X_0,\dots,X_n) \pmod{\pi}, \\ g(X_0,\dots,X_n) &\equiv -w_0X_0 - w_1X_1 - \cdots - w_nX_n \pmod{\pi}. \end{aligned}$$

Note that $f(\bmod\,\pi)$ is homogeneous of degree $\deg_{\underline{X}}(H) = m \geq 2$ (because $\pi \notin \mathcal{P}_{\rm exc}(H)$) and that $g(\bmod\,\pi)$ is homogeneous of degree 1 (because $\underline{w} \not\equiv 0 \pmod{\pi}$). Recalling that char $\mathbb{F}_q \nmid \deg_{\underline{X}}(H)$ by hypothesis, we deduce that $f(\operatorname{mod}\pi)$ and $g(\operatorname{mod}\pi)$ are indeed Deligne polynomials. Since $\underline{w}\notin\mathcal{W}_\pi^*$, they are also transverse. Thus Theorem 7.6 applies, giving

$$|S_H(\underline{w},\chi_\pi)|\ll_{n,m} q^{\frac{(n+1)\deg_T(\pi)}{2}}.$$

With this, we completed the verification of all Weil-Deligne bounds of Proposition 7.1.

8. Application of the Weil-Deligne bounds to the unramified sieve term

In this section, we apply the Weil-Deligne bounds of §7 to the unramified sieve term (given by the right hand side of (6.2)).

Proposition 8.1. *Let q be an odd rational prime power, $n\geq 2$ an integer, $\ell\geq 2$ a rational prime, and $F\in\mathcal{O}_K[X_0,\dots,X_n]$ a homogeneous polynomial of degree $m\geq 2$ in $X_0,\dots,X_n$, with* $\operatorname{char}K\nmid m$*, where, as before, $K=\mathbb{F}_q(T)$. Assume the conditions:*

- (i) $\ell\mid\gcd(m,q-1)$*;*
- (ii) *the projective hypersurface $F(X_0,\dots,X_n)=0\subset\mathbb{P}^n_K$ is nonsingular.*

Let $b,\Delta>0$ be integers such that

$$\Delta<b<2\Delta. \tag{8.1}$$

Defining $\mathcal{P}$ as in (5.3),

$$\frac{q^{-(n+1)(2\Delta-b)}}{|\mathcal{P}|^2}\sum_{\substack{\pi_1,\pi_2\in\mathcal{P}\\ \pi_1\neq\pi_2}}\sum_{\substack{\chi_{\pi_1}\neq\chi_0\\ \chi_{\pi_2}\neq\chi_0}}\sum_{\substack{\underline{x}\in\mathcal{O}_K^{n+1}\\ \deg_T(\underline{x})<2\Delta-b}}|S_F(\bar{\pi}_2\underline{x},\chi_{\pi_1})\,S_F(\bar{\pi}_1\underline{x},\chi_{\pi_2})|$$
$$\ll_{\ell,m,n\,\deg_T(F^*)} b^2q^{n\Delta}+q^{(n+1)\Delta}+q^{(n+1)b-\Delta}. \tag{8.2}$$

To prove Proposition 8.1, we will estimate the absolute value of the innermost term according to different cases of $\underline{x}$ suggested by Proposition 7.1.

8.1. Proof of Proposition 8.1: dissecting the inner sum into cases.

Let $\pi_1,\pi_2\in\mathcal{P}$ with $\pi_1\neq\pi_2$ and let $\chi_{\pi_1},\chi_{\pi_2}\neq\chi_0$ be fixed. We dissect the set $\{\underline{x}\in\mathcal{O}_K^{n+1}:\deg_T(\underline{x})<2\Delta-b\}$ into subsets suggested by Proposition 7.1, as follows.

For each $i\in\{1,2\}$, we consider the smooth projective hypersurface

$$W(F)_{\pi_i}:\quad F(X_0,\dots,X_n)\equiv 0\,(\operatorname{mod}\pi_i).$$

By part (iii) of Proposition 4.1, the reduction modulo π_i of F^* remains absolutely irreducible and defines the projective dual $W(F)^*_{\pi_i}$, that is,

$$W(F)^*_{\pi}:\quad F^*(X_0,\dots,X_n)\equiv 0\,(\operatorname{mod}\pi_i).$$

We denote by

$$\mathcal{W}(F)_{\pi_i},\ \mathcal{W}(F)^*_{\pi_i}$$

the *affine* varieties defined by the homogenous polynomials $F\,(\operatorname{mod}\pi_i)$, $F^*\,(\operatorname{mod}\pi_i)$ (respectively). We partition $\{\underline{x}\in\mathcal{O}_K^{n+1}:\deg_T(\underline{x})<2\Delta-b\}$ according to the following cases:

- (C1) $\bar{\pi}_1\underline{x}\notin\mathcal{W}(F)^*_{\pi_2}$, $\bar{\pi}_2\underline{x}\notin\mathcal{W}(F)^*_{\pi_1}$ (good-good);
- (C2) $\bar{\pi}_1\underline{x}\notin\mathcal{W}(F)^*_{\pi_2}$, $\bar{\pi}_2\underline{x}\in\mathcal{W}(F)^*_{\pi_1}$ (good-bad);
- (C2') $\bar{\pi}_1\underline{x}\in\mathcal{W}(F)^*_{\pi_2}$, $\bar{\pi}_2\underline{x}\notin\mathcal{W}(F)^*_{\pi_1}$ (bad-good);
- (C3) $\bar{\pi}_1\underline{x}\in\mathcal{W}(F)^*_{\pi_2}$, $\bar{\pi}_2\underline{x}\in\mathcal{W}(F)^*_{\pi_1}$ (bad-bad).

In the above partition, the case $\bar{\pi}_1\underline{x} \in \mathcal{W}(F)^*_{\pi_2}$ includes the subcase in which $\bar{\pi}_1\underline{x} \equiv \underline{0} \,(\mathrm{mod}\, \pi_2)$; similarly, the case $\bar{\pi}_2\underline{x} \in \mathcal{W}(F)^*_{\pi_1}$ includes the subcase in which $\bar{\pi}_2\underline{x} \equiv \underline{0} \,(\mathrm{mod}\, \pi_1)$. Thus the condition $\bar{\pi}_1\underline{x} \notin \mathcal{W}(F)^*_{\pi_2}$ implies that $\bar{\pi}_1\underline{x} \not\equiv \underline{0} \,(\mathrm{mod}\, \pi_2)$; similarly, $\bar{\pi}_2\underline{x} \notin \mathcal{W}(F)^*_{\pi_1}$ implies that $\bar{\pi}_2\underline{x} \not\equiv \underline{0} \,(\mathrm{mod}\, \pi_1)$. We will treat each case separately; we begin with the bad-bad case (C3), which is most difficult. Here we will crucially use nontrivial averaging over the primes π_1, π_2 in order to produce an efficient upper bound. The good-good case sums over $\underline{x}$ and π more trivially, but this is allowable because of the square-root cancellation achieved in case (iii) of Proposition 7.1. The strategy for bounding the good-bad case is a hybrid of these two methods. (The case (C2') is analogous to the case (C2), and thus we only explicitly describe the treatment of (C2).)

8.2. **Proof of Proposition 8.1: the bad-bad case (C3).** We break the left-hand side of (8.2) into two sums $\Sigma_1 + \Sigma_2$, according to whether the sum of $\underline{x} \in \mathcal{O}_K^{n+1}$ takes place over the $\underline{x}$ satisfying $F^*(\underline{x}) \neq 0$ or $F^*(\underline{x}) = 0$; that is to say,

$$\Sigma_1 = \frac{q^{-(n+1)(2\Delta-b)}}{|\mathcal{P}|^2} \sum_{\substack{\pi_1,\pi_2\in\mathcal{P}\\ \pi_1\neq\pi_2}} \sum_{\substack{\chi_{\pi_1}\neq\chi_0\\ \chi_{\pi_2}\neq\chi_0}} \sum_{\substack{\underline{x}\in\mathcal{O}_K^{n+1}\\ \deg_T(\underline{x})<2\Delta-b\\ F^*(\underline{x})\neq 0,\text{ case (C3)}}} |S_F(\bar{\pi}_2\underline{x}, \chi_{\pi_1})\, S_F(\bar{\pi}_1\underline{x}, \chi_{\pi_2})|$$

$$\Sigma_2 = \frac{q^{-(n+1)(2\Delta-b)}}{|\mathcal{P}|^2} \sum_{\substack{\pi_1,\pi_2\in\mathcal{P}\\ \pi_1\neq\pi_2}} \sum_{\substack{\chi_{\pi_1}\neq\chi_0\\ \chi_{\pi_2}\neq\chi_0}} \sum_{\substack{\underline{x}\in\mathcal{O}_K^{n+1}\\ \deg_T(\underline{x})<2\Delta-b\\ F^*(\underline{x})= 0,\text{ case (C3)}}} |S_F(\bar{\pi}_2\underline{x}, \chi_{\pi_1})\, S_F(\bar{\pi}_1\underline{x}, \chi_{\pi_2})|.$$

Within each term, the subscript case (C3) means that we restrict to those $\pi_1, \pi_2, \underline{x}$ such that case (C3) holds. In Σ_1, we note that by applying cases (i) and (ii) of Proposition 7.1,

$$|\Sigma_1| \ll_{\ell,m,n} \frac{q^{-(n+1)(2\Delta-b)}q^{(n+2)\Delta}}{|\mathcal{P}|^2} \sum_{\substack{\underline{x}\in\mathcal{O}_K^{n+1}\\ \deg_T(\underline{x})<2\Delta-b\\ F^*(\underline{x})\neq 0}} \sum_{\substack{\pi_1,\pi_2\in\mathcal{P}\\ \pi_1\neq\pi_2\\ \text{case (C3)}}} 1. \tag{8.3}$$

Here we used that for each π_i, the number of characters of order ℓ modulo π_i is ℓ. To bound this efficiently, we will use the fact that $F^*(\underline{x}) \neq 0$ in order to show that relatively few pairs of π_1, π_2 can correspond to the bad-bad case.

In contrast, in Σ_2, since $F^*(\underline{x}) = 0$, then certainly $\underline{x}$ is "bad" for all π_i, since $F^*(\underline{x}) \equiv 0 \,(\mathrm{mod}\, \pi_1)$ for all $\pi_i \in \mathcal{P}$. Thus, by applying cases (i) and (ii) of Proposition 7.1, we write

$$|\Sigma_2| \ll_{\ell,m,n} q^{-(n+1)(2\Delta-b)}q^{(n+2)\Delta} \sum_{\substack{\underline{x}\in\mathcal{O}_K^{n+1}\\ \deg_T(\underline{x})<2\Delta-b\\ F^*(\underline{x})=0}} 1. \tag{8.4}$$

The heart of the argument in the bad-bad case is thus to count efficiently those $\underline{x}$ for which $F^*(\underline{x}) = 0$. We return to this momentarily.

8.2.1. *Bounding* Σ_1. To bound Σ_1, we begin with (8.3). Case (C3) requires that $\bar{\pi}_1\underline{x} \in \mathcal{W}(F)^*_{\pi_2}$, so that by homogeneity $\pi_2 | F^*(\underline{x})$, and analogously $\pi_1 | F^*(\underline{x})$. Consequently, the innermost sum over $\pi_1 \neq \pi_2$ in (8.3) is bounded by

$$\#\{\pi_1 \neq \pi_2 \in \mathcal{P} : \pi_1\pi_2 | F^*(\underline{x})\} \leq (\omega(F^*(\underline{x})))^2,$$

where we let $\omega(y)$ denote the number of distinct prime divisors of an element $y \in \mathcal{O}_K$. Now let $\deg_T(F^*)$ denote the largest degree of T that appears in a coefficient of F^*. Then for $\underline{x} \in \mathcal{O}_K^{n+1}$

with $\deg_T(\underline{x}) < 2\Delta - b$ and $F^*(\underline{x}) \neq 0$,

$$\omega(F^*(\underline{x})) \leq \deg_T(F^*) + \deg_{\underline{X}} F^* \cdot \deg_T(\underline{x}) \leq \deg_T(F^*) + m(m-1)^{n-1}(2\Delta - b), \tag{8.5}$$

where in the last inequality we applied Proposition 11.2 (3) to bound $\deg_{\underline{X}}(F^*)$. We also note that $2\Delta - b < b$ under the hypothesis (8.1). In conclusion,

$$|\Sigma_1| \ll_{\ell,m,n,\deg_T(F^*)} b^2 q^{-(n+1)(2\Delta-b)} q^{-2\Delta} q^{(n+2)\Delta} q^{(n+1)(2\Delta-b)} \ll_{\ell,m,n,\deg_T(F^*)} b^2 q^{n\Delta}.$$

8.2.2. *Bounding* Σ_2. Now we turn to bounding Σ_2, starting from (8.4). The following lemma is the main tool for bounding Σ_2 in the bad-bad case (C3), as well as for bounding an analogous sum in the good-bad cases (C2) and (C2'); we defer its proof to §8.5. (It is also possible to apply [BV15, Lemma 2.9]; we nevertheless include the more flexible lemma below, in case of independent interest.)

Lemma 8.2. *Let $G \in \mathcal{O}_K[X_0, \dots, X_n]$ be an irreducible homogeneous polynomial of degree $\deg_{\underline{X}} G \geq 2$, and let $L \geq N \geq 1$ be such that there is an irreducible polynomial π of degree L such that $G(\underline{X})$ remains irreducible in k_π. Then*

$$\sum_{\substack{\underline{x} \in \mathcal{O}_K^{n+1} \\ \deg_T(\underline{x}) < N \\ G(\underline{x}) = 0}} 1 \ll_{n,\deg_{\underline{X}}(G)} q^{(n+1)N-L} + q^{(n-1)L}.$$

We remark that arguing as in the proof of Proposition 4.1, Part (iii.1) implies that for all but finitely many π, $G(\underline{X})$ remains irreducible in k_π. Thus, choosing L sufficiently large relative to a fixed polynomial G of interest, the conditions of Lemma 8.2 will be met.

We are now ready to bound Σ_2. Apply Lemma 8.2 to (8.4) with $G = F^*$ (recall $\deg_{\underline{X}}(F^*) \ll_{m,n} 1$ by Proposition 11.2 (3)), and the choices $N = 2\Delta - b$ and $L = \Delta$ (note that $N \leq L$ by (8.1)). We obtain

$$\begin{aligned} |\Sigma_2| &\ll_{\ell,m,n} q^{(n+1)(b-2\Delta)} q^{(n+2)\Delta} \left(q^{(n+1)(2\Delta-b)-\Delta} + q^{(n-1)\Delta} \right) \\ &\ll_{\ell,m,n} q^{(n+1)\Delta} + q^{(n+1)b-\Delta}. \end{aligned}$$

This is valid as long as there is a prime $\pi \in \mathcal{O}_K$ of degree Δ for which F^* is irreducible modulo π. Under the hypothesis of Proposition 8.1, this this will be true for all Δ sufficiently large, say

$$\Delta \geq L_0(F^*) \tag{8.6}$$

for a finite parameter provided by Proposition 4.1, Part (iii.1); we will ensure this with our final choice of Δ in (9.4); see §9. Combining the bounds for Σ_1 and Σ_2, we obtain that (under (8.6)) the total contribution of the bad-bad case (C3) to the left-hand side of (8.2) is

$$\ll_{\ell,m,n,\deg_T(F^*)} b^2 q^{n\Delta} + q^{(n+1)\Delta} + q^{(n+1)b-\Delta}. \tag{8.7}$$

Note that, in order for this to be strictly better than the trivial bound $\ll q^{(n+1)b}$, we must have that

$$\Delta < b. \tag{8.8}$$

Combined with (6.8), this motivates the hypothesis $\Delta < b < 2\Delta$ we currently assume.

8.3. **Proof of Proposition 8.1: the good-good case (C1).** When $\underline{x}$ is in case (C1), we apply part (iii) of Proposition 7.1 to estimate each of the character sums $S_F(\bar{\pi}_2\underline{x}, \chi_{\pi_1})$, $S_F(\bar{\pi}_1\underline{x}, \chi_{\pi_2})$. We obtain

$$\frac{q^{-(n+1)(2\Delta-b)}}{|\mathcal{P}|^2} \sum_{\substack{\pi_1,\pi_2\in\mathcal{P}\\ \pi_1\neq\pi_2}} \sum_{\substack{\chi_{\pi_1}\neq\chi_0\\ \chi_{\pi_2}\neq\chi_0}} \sum_{\substack{\underline{x}\in\mathcal{O}_K^{n+1}\\ \deg_T(\underline{x})<2\Delta-b\\ \underline{x}\text{ in case (C1)}}} |S_F(\bar{\pi}_2\underline{x}, \chi_{\pi_1}) S_F(\bar{\pi}_1\underline{x}, \chi_{\pi_2})| \ll_{\ell,m,n} q^{(n+1)(b-\Delta)} N_1,$$

where

$$N_1 := \max_{\pi_1\neq\pi_2\in\mathcal{P}} \#\left\{\underline{x}\in\mathcal{O}_K^{n+1} : \deg_T(\underline{x}) < 2\Delta-b, \bar{\pi}_2\underline{x}\notin\mathcal{W}(F)^*_{\pi_1}, \bar{\pi}_1\underline{x}\notin\mathcal{W}(F)^*_{\pi_2}\right\},$$

and we again used that for each π_i, the number of characters of order ℓ modulo π_i is ℓ. Note that

$$N_1 \leq \#\left\{\underline{x}\in\mathcal{O}_K^{n+1} : \deg_T(\underline{x}) < 2\Delta-b\right\} \leq q^{(2\Delta-b)(n+1)}.$$

Thus the total contribution of $\underline{x}$ in case (C1) into the unramified sieve term is

$$\ll_{\ell,m,n} q^{(n+1)\Delta},$$

which we note is comparable to a term in the bad-bad contribution (8.7).

8.4. **Proof of Proposition 8.1: cases (C2) and (C2').** As in the case (C3), we break the sum in (8.2) into two sums $\Sigma_1+\Sigma_2$, according to whether the sum of $\underline{x}\in\mathcal{O}_K^{n+1}$ takes place over the $\underline{x}$ satisfying $F^*(\underline{x})\neq 0$ or $F^*(\underline{x})=0$. We define

$$\Sigma_1 = \frac{q^{-(n+1)(2\Delta-b)}}{|\mathcal{P}|^2} \sum_{\substack{\pi_1,\pi_2\in\mathcal{P}\\ \pi_1\neq\pi_2}} \sum_{\substack{\chi_{\pi_1}\neq\chi_0\\ \chi_{\pi_2}\neq\chi_0}} \sum_{\substack{\underline{x}\in\mathcal{O}_K^{n+1}\\ \deg_T(\underline{x})<2\Delta-b\\ F^*(\underline{x})\neq 0,\text{ case (C2)}}} |S_F(\bar{\pi}_2\underline{x}, \chi_{\pi_1}) S_F(\bar{\pi}_1\underline{x}, \chi_{\pi_2})|.$$

The sum Σ_2 is formally defined analogously, with the condition $F^*(\underline{x})\neq 0$ replaced by the condition $F^*(\underline{x})=0$. However, note that under the conditions of case (C2) the sum in Σ_2 is empty. Indeed, if $F^*(\underline{x})=0$ then *all* primes are bad (that is to say, $\underline{x}\in\mathcal{W}(F)^*_\pi$ for all π in the sieving set), whereas in case (C2) at least one prime is good (since by assumption $\overline{\pi}_1\underline{x}\notin\mathcal{W}(F)^*_{\pi_2}$).

It only remains to bound Σ_1, which we accomplish by using cases (i) and (ii) of Propositions 7.1 and (8.5) as before, to conclude that

$$|\Sigma_1| \ll_{\ell,m,n,\deg_T(F^*)} bq^{\frac{(2n-1)\Delta}{2}}.$$

This is clearly dominated by the bad-bad contribution (8.7). The same arguments for Σ_1, Σ_2 apply for the (C2') contribution.

To recap, the work above for cases (C1), (C2), (C2'), (C3), has shown that the unramified sieve term satisfies the bound claimed in Proposition 8.1. All that remains is to prove the counting result in Lemma 8.2, to which we now turn.

8.5. **Proof of Lemma 8.2.** Let π be as in the statement of the proposition. We have that

$$\sum_{\substack{\underline{x}\in\mathcal{O}_K^{n+1}\\ \deg_T(\underline{x})<N\\ G(\underline{x})=0}} 1 \leq \sum_{\substack{\underline{x}\in\mathcal{O}_K^{n+1}\\ \deg_T(\underline{x})<N\\ G(\underline{x})=0\,(\mathrm{mod}\,\pi)}} 1.$$

We complete the sum, by counting for each $\alpha \in k_\pi^{n+1}$ such that $G(\alpha) = 0 \,(\mathrm{mod}\, \pi)$, those $\underline{x}$ with $\deg_T(\underline{x}) < N$ such that $\underline{x} = \underline{\alpha} \,(\mathrm{mod}\, \pi)$:

$$\sum_{\substack{\underline{x} \in \mathcal{O}_K^{n+1} \\ \deg_T(\underline{x}) < N \\ G(\underline{x}) = 0 \,(\mathrm{mod}\, \pi)}} 1 = \sum_{\substack{\underline{\alpha} \in k_\pi^{n+1} \\ G(\underline{\alpha}) = 0 \,(\mathrm{mod}\, \pi)}} \sum_{\substack{\underline{x} \in \mathcal{O}_K^{n+1} \\ \deg_T(\underline{x}) < N}} \frac{1}{q^{(n+1)L}} \sum_{\underline{\beta} \in k_\pi^{n+1}} \psi_\infty \left(\frac{\underline{\beta} \cdot (\underline{\alpha} - \underline{x})}{\pi} \right)$$

$$= \frac{1}{q^{(n+1)L}} \sum_{\underline{\beta} \in k_\pi^{n+1}} \sum_{\substack{\underline{x} \in \mathcal{O}_K^{n+1} \\ \deg_T(\underline{x}) < N}} \psi_\infty \left(\frac{-\underline{\beta} \cdot \underline{x}}{\pi} \right) \sum_{\substack{\underline{\alpha} \in k_\pi^{n+1} \\ G(\underline{\alpha}) = 0 \,(\mathrm{mod}\, \pi)}} \psi_\infty \left(\frac{\underline{\beta} \cdot \underline{\alpha}}{\pi} \right), \tag{8.9}$$

where the additive character $\psi_\infty(\cdot/\pi)$ is defined in (3.2) and we have applied Lemma 6.4. We remark that

$$\sum_{\substack{\underline{x} \in \mathcal{O}_K^{n+1} \\ \deg_T(\underline{x}) < N}} \psi_\infty \left(\frac{-\underline{\beta} \cdot \underline{x}}{\pi} \right) = \prod_{j=0}^{n} \left(\sum_{\substack{x_j \in \mathcal{O}_K \\ \deg_T(x_j) < N}} \psi_\infty \left(\frac{-\beta_j x_j}{\pi} \right) \right) = \prod_{j=0}^{n} \left(\sum_{\substack{x_j \in \mathcal{O}_K \\ \deg_T(x_j) < N}} \psi_\infty \left(-\operatorname{Tr}_{k_\pi/\mathbb{F}_q}(\beta_j x_j) \right) \right).$$

Thus we can work one coordinate at a time. Fix any $\beta_j \in k_\pi$. Write $k_\pi = \mathbb{F}_q[\rho]$, where ρ is a root of π. Then $\{1, \rho \dots, \rho^{L-1}\}$ is a basis of k_π as an $\mathbb{F}_q$-vector space. Thus, $x \in \mathcal{O}_K$ with $\deg_T(x) < N$ can be expressed as $x = a_0 + a_1\rho + \dots + a_{N-1}\rho^{N-1}$ with the coefficients $a_\ell \in \mathbb{F}_q$ uniquely determined; this leads to

$$\sum_{\substack{x_j \in \mathcal{O}_K \\ \deg_T(x_j) < N}} \psi_\infty \left(-\operatorname{Tr}_{k_\pi/\mathbb{F}_q}(\beta_j x_j) \right) = \prod_{\ell=0}^{N-1} \left(\sum_{a_\ell \in \mathbb{F}_q} \psi_\infty \left(-\operatorname{Tr}_{k_\pi/\mathbb{F}_q} a_\ell (\rho^\ell \beta_j) \right) \right). \tag{8.10}$$

Also notice that we have for each fixed ℓ that

$$\sum_{a_\ell \in \mathbb{F}_q} \psi_\infty \left(-\operatorname{Tr}_{k_\pi/\mathbb{F}_q} a_\ell (\rho^\ell \beta_j) \right) = \begin{cases} q & \text{if } \operatorname{Tr}_{k_\pi/\mathbb{F}_q}(\rho^\ell \beta_j) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Combining the above with (8.10), we obtain

$$\sum_{\substack{x_j \in \mathcal{O}_K \\ \deg_T(x_j) < N}} \psi_\infty \left(-\operatorname{Tr}_{k_\pi/\mathbb{F}_q}(\beta_j x_j) \right) = \begin{cases} q^N & \text{if } \operatorname{Tr}_{k_\pi/\mathbb{F}_q}(\rho^\ell \beta_j) = 0 \text{ for } \ell = 0, \dots, N-1, \\ 0 & \text{otherwise.} \end{cases}$$

Define for any $0 \le N \le L$,

$$S_N = \{\beta \in k_\pi \,:\, \operatorname{Tr}_{k_\pi/\mathbb{F}_q}(\rho^\ell \beta) = 0, \text{ for } \ell = 0, \dots, N-1\}.$$

We claim that $\#S_N = q^{L-N}$. To see this, consider for a given $0 \le \ell \le L-1$,

$$H_\ell := \{\beta \in k_\pi \,:\, \operatorname{Tr}_{k_\pi/\mathbb{F}_q}(\rho^\ell \beta) = 0\}.$$

Note that H_ℓ is a hyperplane in the vector space k_π for any $\ell = 0, \dots, L-1$. We will prove that $S_L = \cap_{\ell=0}^{L-1} H_\ell = \{0\}$. Indeed, if $\gamma \in S_L$, then, since $\{1, \rho, \dots, \rho^{L-1}\}$ is a basis for k_π, linearity of trace implies that $\operatorname{Tr}_{k_\pi/\mathbb{F}_q}(\gamma y) = 0$ for all $y \in k_\pi$. Now the trace pairing is non-degenerate if and only if the extension is separable (see, for example, [Jan96, Ch.1, Sec.5.2]), and since $\mathbb{F}_q$ is perfect, we conclude that $\gamma = 0$. Since $\#S_0 = q^L$ and $\#S_L = 1$, we conclude that each hyperplane $H_{\ell+1}$ lowers the dimension by exactly 1 when going from S_ℓ to $S_{\ell+1} = S_\ell \cap H_\ell$ as long as $\ell + 1 \le L$. Once we reach $\ell \ge L$, the dimension remains 0. Thus, in particular for a given $N \le L$, we conclude that $\#S_N = q^{L-N}$.

Back to the identity in (8.9), we have shown that

$$\sum_{\substack{\underline{x}\in\mathcal{O}_K^{n+1}\\ \deg_T(\underline{x})<N\\ G(\underline{x})=0\,(\mathrm{mod}\,\pi)}} 1 = \frac{q^{(n+1)N}}{q^{(n+1)L}} \sum_{\underline{\beta}\in k_\pi^{n+1}} \mathbf{1}_{\underline{\beta}\in S_N^{n+1}} \sum_{\substack{\underline{\alpha}\in k_\pi^{n+1}\\ G(\underline{\alpha})=0\,(\mathrm{mod}\,\pi)}} \psi_\infty\left(\frac{\underline{\beta}\cdot\underline{\alpha}}{\pi}\right)$$

$$\ll_{n,\deg_{\underline{X}}(G)} \frac{q^{(n+1)N}}{q^{(n+1)L}} |S_N|^{n+1} \max_{\substack{\underline{\beta}\in k_\pi^{n+1}\\ \underline{\beta}\neq 0}} \left| \sum_{\substack{\underline{\alpha}\in k_\pi^{n+1}\\ G(\underline{\alpha})=0\,(\mathrm{mod}\,\pi)}} \psi_\infty\left(\frac{\underline{\beta}\cdot\underline{\alpha}}{\pi}\right)\right| + q^{(n+1)N-L}.$$

In the last term, corresponding to $\underline{\beta}=0$, we have applied the Lang-Weil bound [LW54, Thm. 1] to count

$$\{\underline{\alpha}\in k_\pi^{n+1} : G(\underline{\alpha}) = 0(\mathrm{mod}\,\pi)\} \ll_{n,\deg_{\underline{X}}(G)} q^{nL}.$$

We now consider the additive character sum above. Given $\underline{\beta}\neq 0$, we start by writing

$$\sum_{\substack{\underline{\alpha}\in k_\pi^{n+1}\\ G(\underline{\alpha})=0\,(\mathrm{mod}\,\pi)}} \psi_\infty\left(\frac{\underline{\beta}\cdot\underline{\alpha}}{\pi}\right) = \sum_{\gamma\in k_\pi} \psi_\infty\left(\frac{\gamma}{\pi}\right) \sum_{\substack{\underline{\alpha}\in k_\pi^{n+1}\\ G(\underline{\alpha})=0\,(\mathrm{mod}\,\pi)\\ \underline{\alpha}\cdot\underline{\beta}=\gamma\,(\mathrm{mod}\,\pi)}} 1. \tag{8.11}$$

Consider the sum over $\underline{\alpha}$. This sum is independent of γ for $\gamma\in k_\pi^*$. Indeed, for $\gamma,\gamma_0\in k_\pi^*$, write $\gamma_0=\delta\gamma$. Setting $d=\deg_{\underline{X}} G$, and using the homogeneity of $G(\underline{X})$, this gives

$$\sum_{\substack{\underline{\alpha}\in k_\pi^{n+1}\\ G(\underline{\alpha})=0\,(\mathrm{mod}\,\pi)\\ \underline{\alpha}\cdot\underline{\beta}=\gamma\,(\mathrm{mod}\,\pi)}} 1 = \sum_{\substack{\underline{\alpha}\in k_\pi^{n+1}\\ \delta^d G(\underline{\alpha})=0\,(\mathrm{mod}\,\pi)\\ \underline{\alpha}\cdot\underline{\beta}=\gamma\,(\mathrm{mod}\,\pi)}} 1 = \sum_{\substack{\underline{\alpha}\in k_\pi^{n+1}\\ G(\delta\underline{\alpha})=0\,(\mathrm{mod}\,\pi)\\ \underline{\alpha}\cdot\underline{\beta}=\gamma\,(\mathrm{mod}\,\pi)}} 1$$

$$= \sum_{\substack{\underline{\alpha_0}\in k_\pi^{n+1}\\ G(\underline{\alpha_0})=0\,(\mathrm{mod}\,\pi)\\ \underline{\alpha_0}\cdot\underline{\beta}=\delta\gamma\,(\mathrm{mod}\,\pi)}} 1 = \sum_{\substack{\underline{\alpha_0}\in k_\pi^{n+1}\\ G(\underline{\alpha_0})=0\,(\mathrm{mod}\,\pi)\\ \underline{\alpha_0}\cdot\underline{\beta}=\gamma_0\,(\mathrm{mod}\,\pi)}} 1.$$

Combining the above observation with (8.11), we obtain

$$\sum_{\substack{\underline{\alpha}\in k_\pi^{n+1}\\ G(\underline{\alpha})=0\,(\mathrm{mod}\,\pi)}} \psi_\infty\left(\frac{\underline{\beta}\cdot\underline{\alpha}}{\pi}\right) = - \sum_{\substack{\underline{\alpha}\in k_\pi^{n+1}\\ G(\underline{\alpha})=0\,(\mathrm{mod}\,\pi)\\ \underline{\alpha}\cdot\underline{\beta}=1\,(\mathrm{mod}\,\pi)}} 1 + \sum_{\substack{\underline{\alpha}\in k_\pi^{n+1}\\ G(\underline{\alpha})=0\,(\mathrm{mod}\,\pi)\\ \underline{\alpha}\cdot\underline{\beta}=0\,(\mathrm{mod}\,\pi)}} 1.$$

We conclude by applying the next lemma.

Lemma 8.3. *Let $G\in\mathcal{O}_K[X_0,\dots,X_n]$ be an irreducible homogeneous polynomial of degree $\deg_{\underline{X}} G\geq 2$, and let π be an irreducible polynomial of degree L. Fix $\underline{\beta}\in k_\pi^{n+1}$, $\underline{\beta}\neq(0,\dots,0)$. Then*

$$\sum_{\substack{\underline{\alpha}\in k_\pi^{n+1}\\ G(\underline{\alpha})=0\,(\mathrm{mod}\,\pi)\\ \underline{\alpha}\cdot\underline{\beta}=1\,(\mathrm{mod}\,\pi)}} 1 \ll_{n,\deg_{\underline{X}}(G)} q^{(n-1)L}$$

and

$$\sum_{\substack{\underline{\alpha}\in k_\pi^{n+1}\\ G(\underline{\alpha})=0\,(\mathrm{mod}\,\pi)\\ \underline{\alpha}\cdot\underline{\beta}=0\,(\mathrm{mod}\,\pi)}} 1 \ll_{n,\deg_{\underline{X}}(G)} q^{(n-1)L}.$$

Proof of Lemma 8.3. We consider an embedding of k_π^{n+1} in $\mathbb{P}_{k_\pi}^{n+1}$ by adding an extra coordinate T and interpreting k_π^{n+1} as the subset where $T=1$.

$$\sum_{\substack{\underline{\alpha}\in k_\pi^{n+1}\\ G(\underline{\alpha})=0\,\mathrm{mod}\,\pi\\ \underline{\alpha}\cdot\underline{\beta}=1\,\mathrm{mod}\,\pi}} 1 = \sum_{\substack{(\underline{\alpha},T)\in\mathbb{P}^{n+1}\\ G(\underline{\alpha})=0\,\mathrm{mod}\,\pi\\ \underline{\alpha}\cdot\underline{\beta}=T\,\mathrm{mod}\,\pi\\ T\neq 0}} 1 \le \sum_{\substack{(\underline{\alpha},T)\in\mathbb{P}^{n+1}\\ G(\underline{\alpha})=0\,\mathrm{mod}\,\pi\\ \underline{\alpha}\cdot\underline{\beta}=T\,\mathrm{mod}\,\pi}} 1 \ll_{n,\deg_{\underline{X}}(G)} q^{(n-1)L},$$

where in the last identity we applied the fact that G is irreducible and has degree at least 2, so that we can apply the result of Lang-Weil for a variety of codimension 2 [LW54, Thm 1].

For the second inequality, write for $\underline{\alpha}\in k_\pi^{n+1}$, $\underline{\alpha}=a\underline{\gamma}$ with $a\in k_\pi$ and $\underline{\gamma}\in\mathbb{P}_{k_\pi}^n$. This gives

$$\sum_{\substack{\underline{\alpha}\in k_\pi^{n+1}\\ G(\underline{\alpha})=0\,\mathrm{mod}\,\pi\\ \underline{\alpha}\cdot\underline{\beta}=0\,\mathrm{mod}\,\pi}} 1 = 1 + \sum_{a\neq 0\,\mathrm{mod}\,k_\pi} \sum_{\substack{\underline{\gamma}\in\mathbb{P}^n\\ G(\underline{\gamma})=0\,\mathrm{mod}\,\pi\\ \underline{\gamma}\cdot\underline{\beta}=0\,\mathrm{mod}\,\pi}} 1 \ll_{n,\deg_{\underline{X}}(G)} 1+(q^L-1)q^{(n-2)L}$$

by a second application of Lang-Weil for a variety of codimension 2, and we conclude. □

This concludes the proof of Lemma 8.2, and the proof of Proposition 8.1 is complete.

9. Completing the proof of Theorem 5.1: choice of parameters

It is time to wrap up the proof of Theorem 5.1 (and Theorem 1.1). Putting together all our estimates for the main sieve term, the ramified sieve term, and the unramified sieve term, we obtain that, for fixed q,n,ℓ,F as in the statement of the aforementioned theorems, for any sufficiently large positive integer b and for any positive integer Δ chosen such that (5.5) holds and

$$\Delta < b < 2\Delta,$$

we have

$$|\mathcal{S}_F(\mathcal{A})| \ll_{\ell,m,n,\deg_T(F),\deg_T(F^*)} bq^{(n+1)b-\Delta}+q^{nb}+b^2q^{n\Delta}+q^{(n+1)\Delta}+q^{(n+1)b-\Delta} \ll bq^{(n+1)b-\Delta}+q^{(n+1)\Delta}, \tag{9.1}$$

where in the last inequality we applied the fact that $\Delta<b$ so that $nb<(n+1)b-\Delta$ and we imposed the additional assumption

$$b^2\le q^\Delta, \tag{9.2}$$

which we will verify momentarily. Our remaining goal is to choose Δ optimally such that the resulting upper bound for $|\mathcal{S}_F(\mathcal{A})|$ improves upon the trivial bound $q^{(n+1)b}$. In what follows, we address the choice of Δ, after which we address the existence of a minimal $b(n,q,F)$ such that our previous working assumptions on Δ hold for all $b\ge b(n,q,F)$ (namely $\Delta<b<2\Delta$, and the assumptions (5.5), (8.6), (9.2)).

9.1. **Choice of $\Delta=\Delta(n,b)$.** Looking at the right-most side of (9.1), we see that an initial choice Δ_0 of Δ could be made such that the two terms are balanced, that is,

$$bq^{(n+1)b-\Delta_0}=q^{(n+1)\Delta_0},$$

which is equivalent to choosing

$$\Delta_0 := \frac{(n+1)b+\log_q b}{n+2}. \tag{9.3}$$

Since the above Δ_0 is not necessarily an integer, while Δ, as the degree of a polynomial, must be an integer, we write Δ_0 in terms of its integral and fractional parts, and choose Δ to be the former:

$$\Delta_0 = \left\lfloor \frac{(n+1)b + \log_q b}{n+2} \right\rfloor + \left\{ \frac{(n+1)b + \log_q b}{n+2} \right\} = \Delta + \delta_0,$$

where, we highlight,

$$\Delta = \Delta(n,b) := \left\lfloor \frac{(n+1)b + \log_q b}{n+2} \right\rfloor \in \mathbb{N} \quad \text{(including zero)} \tag{9.4}$$

and

$$\delta_0 := \left\{ \frac{(n+1)b + \log_q b}{n+2} \right\} \in [0,1).$$

With this choice of Δ, we obtain:

$$\begin{aligned}
|\mathcal{S}_F(\mathcal{A})| \quad &\ll_{\ell,m,n,\deg_T(F),\deg_T(F^*)} \quad bq^{(n+1)b-\Delta} + q^{(n+1)\Delta} \\
&\leq 2bq^{(n+1)b-\Delta} \quad \text{(since } (n+1)\Delta = (n+2)\Delta - \Delta \leq (n+2)\Delta_0 - \Delta = (n+1)b - \Delta + \log_q b) \\
&= 2b\left(q^b\right)^{(n+1)-\frac{\Delta}{b}} \\
&< 2b\left(q^b\right)^{(n+1)-\frac{n+1}{n+2}-\frac{\log_q b}{b(n+2)}+\frac{1}{b}},
\end{aligned}$$

since

$$-\frac{\Delta}{b} = -\frac{n+1}{n+2} - \frac{\log_q b}{b(n+2)} + \frac{\delta_0}{b} < -\frac{n+1}{n+2} - \frac{\log_q b}{b(n+2)} + \frac{1}{b}.$$

We recognize the bound above as $\ll b^{1-\frac{1}{n+2}}\left(q^b\right)^{(n+1)-\frac{n+1}{n+2}} q$. Recalling that q is fixed, we conclude that

$$|\mathcal{S}_F(\mathcal{A})| \ll_{\ell,m,n,q,\deg_T(F),\deg_T(F^*)} b^{\frac{n+1}{n+2}}\left(q^b\right)^{(n+1)-\frac{n+1}{n+2}}, \tag{9.5}$$

under the assumption that $b \geq b(n,q,F)$ is such that the inequalities $\Delta < b < 2\Delta$ and (5.5), (8.6), and (9.2) hold for $\Delta = \Delta(n,b)$.

9.2. **Choice of $b(n,q,F)$.** First, let us note that, as a first constraint, b must be chosen sufficiently large to ensure that $\Delta(n,b) \neq 0$; it suffices to have $b \geq b_1$ for some $b_1 = b_1(n)$ chosen such that

$$\left\lfloor \frac{(n+1)b_1 + \log_q b_1}{n+2} \right\rfloor \geq \left\lfloor \frac{(n+1)b_1}{n+2} \right\rfloor \neq 0. \tag{9.6}$$

Next, recall that, in (5.5) of §5, we introduced the assumption

$$\frac{q^\Delta}{\Delta} - \left(\frac{q^{\frac{\Delta}{2}}}{\Delta} + q^{\frac{\Delta}{3}}\right) - |\mathcal{P}_{\text{exc}}(F)| \geq \frac{q^\Delta}{2\Delta}, \tag{9.7}$$

which gives rise to a second set of constraints on $b = b(n,q,F)$. Observe that, if we choose b such that

$$|\mathcal{P}_{\text{exc}}(F)| \leq \frac{q^\Delta}{4\Delta} \tag{9.8}$$

and

$$\frac{q^{\frac{\Delta}{2}}}{\Delta} + q^{\frac{\Delta}{3}} \leq \frac{q^\Delta}{4\Delta}, \tag{9.9}$$

then

$$\frac{q^\Delta}{\Delta} - \left(\frac{q^{\frac{\Delta}{2}}}{\Delta} + q^{\frac{\Delta}{3}}\right) - |\mathcal{P}_{\text{exc}}(F)| \geq \frac{q^\Delta}{\Delta} - \frac{q^\Delta}{4\Delta} - \frac{q^\Delta}{4\Delta} = \frac{q^\Delta}{2\Delta},$$

which ensures (9.7). Since $\mathcal{P}_{\mathrm{exc}}(F)$ is a finite set and its cardinality depends on q and F, we can find $b_{2,1} = b_{2,1}(q, F)$ such that, for any $b \geq b_{2,1}$, (9.8) holds for $\Delta = \Delta(n, b)$ as chosen in (9.4). Observe that

$$q^{\frac{\Delta}{2}} + \Delta q^{\frac{\Delta}{3}} \leq q^{\frac{\Delta}{2}} + \Delta q^{\frac{\Delta}{2}} < q^{\frac{\Delta}{2}} + b q^{\frac{\Delta}{2}} = (b+1) q^{\frac{\Delta}{2}},$$

where we used that $\Delta < b$. Thus, to ensure (9.9), it suffices to choose $b_{2,2} = b_{2,2}(n, q)$ such that, for any $b \geq b_{2,2}$, we have

$$4(b+1) \leq q^{\frac{1}{2}\left\lfloor \frac{(n+1)b}{n+2} \right\rfloor}.$$

In addition, remark that the above condition ensures (9.2).

Next we have to ensure (8.6) holds, namely $\Delta \geq L_0(F^*)$. This similarly will hold as long as $b \geq b_3$ for some $b_3(F)$ sufficiently large.

Now note that the inequalities $\Delta < b < 2\Delta$ give rise to a final set of constraints for b. With our final choice (9.4) of Δ, these inequalities become

$$\left\lfloor \frac{(n+1)b + \log_q b}{n+2} \right\rfloor < b < 2 \left\lfloor \frac{(n+1)b + \log_q b}{n+2} \right\rfloor. \tag{9.10}$$

We claim that, for any $n \geq 1$, the inequalities (9.10) hold for any $b \geq b_4$ for some $b_4 = b_4(n)$. In what follows, we verify this claim.

Let $b \geq 3(n+2)$. Dividing b with quotient and remainder by $n+2$, we find uniquely determined non-negative integers b_0, r_0 such that

$$b = (n+2)b_0 + r_0 \tag{9.11}$$

and

$$0 \leq r_0 \leq n+1. \tag{9.12}$$

Note that

$$b_0 \geq 3.$$

Rewriting b using (9.11), we obtain that (9.10) is equivalent to

$$(n+1)b_0 + \left\lfloor \frac{(n+1)r_0 + \log_q b}{n+2} \right\rfloor < (n+1)b_0 + b_0 + r_0 < 2(n+1)b_0 + 2 \left\lfloor \frac{(n+1)r_0 + \log_q b}{n+2} \right\rfloor. \tag{9.13}$$

The left-hand side inequality in (9.13) is equivalent to

$$\left\lfloor \frac{(n+1)r_0 + \log_q b}{n+2} \right\rfloor < b_0 + r_0;$$

for this it suffices that

$$\frac{(n+1)r_0 + \log_q b}{n+2} < b_0 + r_0 = \frac{b - r_0}{n+2} + r_0 = \frac{b + (n+1)r_0}{n+2},$$

which certainly holds.

The right-hand side inequality in (9.13) is equivalent to

$$b_0 + r_0 < (n+1)b_0 + 2 \left\lfloor \frac{(n+1)r_0 + \log_q b}{n+2} \right\rfloor;$$

this will certainly hold if

$$r_0 < n b_0 + 2 \left\lfloor \frac{(n+1)r_0}{n+2} \right\rfloor.$$

Since $b_0 \geq 3$ and $0 \leq r_0 \leq n+1$, the above inequality will hold if

$$n + 1 < 3n,$$

which holds for any $n \geq 1$.

In conclusion, provided $n \geq 1$, there exists a positive integer $b(n,q,F)$ such that, for any $b > b(n,q,F)$, the inequalities $\Delta < b < 2\Delta$, (5.5), (8.6), and (9.2) hold for $\Delta = \Delta(n,b)$, so that the sieve process has proved (9.5). On the other hand, for each $b \leq b(n,q,F)$, we can apply the trivial bound

$$\mathcal{S}_F(\mathcal{A}) \leq (q^b)^{n+1} \leq q^{(n+1)b(n,q,F)} \ll_{n,q,F} 1.$$

Thus by enlarging the implicit constant in (9.5) if necessary, it holds for all b. This completes the proof of Theorem 1.1 (and of Theorem 5.1).

10. Counting bound

For completeness, we record below a simple counting lemma, which can be considered a "trivial bound" (sometimes also called the Schwartz-Zippel bound); we applied this in §5.3.

Lemma 10.1. *Let A be a domain, $n \geq 1$ an integer, and $G \in A[X_0, \dots, X_n]$ a homogeneous polynomial of degree $e \geq 1$ in $X_0, \dots, X_n$. Then, for any finite subset $S \subseteq A$, we have*

$$\#\left\{(\gamma_0, \dots, \gamma_n) \in S^{n+1} : G(\gamma_0, \dots, \gamma_n) = 0\right\} \leq e|S|^n. \tag{10.1}$$

We recall the standard proof, which proceeds by induction on n (e.g., see [HB02, Thm. 1] for a version of this result when $A = \mathbb{Z}$).

Proof. In what follows, L is the field of fractions of A and $\overline{L}$ is a fixed algebraic closure of L. In our argument below, it suffices that $G(X_0, \dots, X_n) \in \overline{L}[X_0, \dots, X_n]$, that is, we do not need to assume that the coefficients of G are in A.

Since G is homogenous of degree $e \geq 1$, for each $0 \leq i \leq e$ there exists a homogenous polynomial $G_i \in \overline{L}[X_1, \dots, X_n]$, of degree $e - i$, such that

$$G(X_0, \dots, X_n) = \sum_{0 \leq i \leq e} X_0^i \, G_i(X_1, \dots, X_n).$$

Take i_0 to be the maximal index $0 \leq i \leq e$ such that G_i is not identically zero. Thus,

$$G(X_0, \dots, X_n) = \sum_{0 \leq i \leq i_0} X_0^i \, G_i(X_1, \dots, X_n).$$

Our goal is to show that the number of solutions in S^{n+1} to the equation

$$\sum_{0 \leq i \leq i_0} x_0^i \, G_i(x_1, \dots, x_n) = 0 \tag{10.2}$$

is at most $e|S|^n$. We prove this statement by induction on n.

When $n = 1$, (10.2) becomes the equation

$$\sum_{0 \leq i \leq i_0} x_0^i \, G_i(x_1) = 0, \tag{10.3}$$

whose degree is i_0. Choosing $\gamma_1 \in S$ to be a root of the polynomial $G_{i_0}(X_1)$, provided such a root exists, we note that (10.3) may be satisfied by the pair $(x_0, x_1) = (\gamma_0, \gamma_1)$ for any $\gamma_0 \in S$. Since $G_{i_0}(X_1)$ has at most $\deg_{X_1}(G_1) = e - i_0$ roots in $\overline{L}$, there are at most $e - i_0$ choices for γ_1. There are at most $|S|$ choices for γ_0. As such, in this case, there are at most $(e - i_0)|S|$ possible solutions $(\gamma_0, \gamma_1) \in S^2$ of (10.3). Choosing $\gamma_1 \in S$ to not be a root of the polynomial $G_{i_0}(X_1)$, provided such γ_1 exists, we see that (10.3) is a degree i_0 equation with unknown x_0. Viewed over $\overline{L}$, this equation has i_0 solutions $x_0 = \gamma_0$. In total, in this case, there are at most $|S| i_0$ solutions $(\gamma_0, \gamma_1) \in S^2$ of (10.3). Altogether, we obtain that (10.3) has at most $(e - i_0)|S| + i_0|S| = e|S|$ solutions in S^2.

When $n \geq 2$, we make the inductive hypothesis that

$$\#\left\{(\gamma_1, \dots, \gamma_n) \in S^n : G'(\gamma_1, \dots, \gamma_n) = 0\right\} \leq \deg_{\underline{X}}(G')|S|^{n-1} \tag{10.4}$$

for any homogenous polynomial $G'(X_1, \ldots, X_n) \in \overline{L}[X_1, \ldots, X_n]$. In particular, we assume that (10.4) holds for $G_{i_0}(X_1, \ldots, X_n)$. Choosing $(\gamma_1, \ldots, \gamma_n) \in S^n$ to be a root of the polynomial $G_{i_0}(X_1, \ldots, X_n)$, provided it exists, we note that (10.2) might be satisfied by $(x_0, x_1, \ldots, x_n) = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ for any $\gamma_0 \in S$. Since $\deg_{\underline{X}}(G_{i_0}) = e - i_0$, by the induction hypothesis we know that there are at most $(e - i_0)|S|^{n-1}$ roots $(\gamma_1, \ldots, \gamma_n) \in S^n$. As such, in this case, there are at most $(e - i_0)|S|^n$ solutions $(\gamma_0, \gamma_1, \ldots, \gamma_n) \in S^{n+1}$ of (10.2). Choosing $(\gamma_1, \ldots, \gamma_n) \in S^n$ to not be a root of the polynomial $G_{i_0}(X_1, \ldots, X_n)$, provided it exists, we see that (10.2) gives rise to the degree i_0 equation

$$\sum_{0 \leq i \leq i_0} x_0^i \; G_i(\gamma_1, \ldots, \gamma_n) = 0,$$

with unknown x_0. Viewed over $\overline{L}$, this equation has at most i_0 solutions $x_0 = \gamma_0$. In total, in this case, there are at most $i_0|S|^n$ solutions $(\gamma_0, \gamma_1, \ldots, \gamma_n) \in S^{n+1}$ of (10.2). Altogether, we obtain that (10.2) has at most $(e - i_0)|S|^n + i_0|S|^n = e|S|^n$ solutions in S^{n+1}. □

Funding. This work was initiated at the Women In Numbers 3rd workshop organized at the Banff International Research Station (Alberta, Canada) in Spring 2014. We thank the workshop organizers, Ling Long, Rachel Pries, and Katherine Stange, for providing us with a focused and stimulating research environment for starting this work. We thank the Department of Mathematics, Statistics, and Computer Science at the University of Illinois at Chicago (USA) for sponsoring and hosting a visit of the authors to continue this work. AB has been partially supported by NSF grant DMS-2002716, Simons Foundation collaboration grants No. 244988 and No. 524015, and by Institute for Advanced Study, which includes funding from NSF grant DMS-1638352. ACC has been partially supported by the Simons Collaboration Grants No. 318454 and No. 709008. ML has been partially supported by the Natural Sciences and Engineering Research Council of Canada under Discovery Grant 355412-2013 and the Fonds de recherche du Québec - Nature et technologies, Projet de recherche en équipe 256442 and 300951. LBP thanks the Hausdorff Center for Mathematics for a productive research environment, and has been partially supported by NSF CAREER grant DMS-1652173, a Sloan Research Fellowship, by the Charles Simonyi Endowment and NSF Grant No. 1128155 at the IAS, and by a Joan and Joseph Birman Fellowship.

11. Appendix: Projective Duality in Fibers for Smooth Hypersurfaces by Joseph Rabinoff

In this appendix we gather some results from the literature on projective duality for smooth hypersurfaces. In the case of global fields, we can "spread out from the generic fiber" to conclude that the same results hold at all but finitely many places. We will use the language of algebraic varieties to the extent possible, and our proofs will use only elementary facts from commutative algebra, but as we will be working with non-algebraically closed fields and coefficient rings like $\mathbb{Z}$, some basic scheme theory will be required to make certain constructions precise.

Let k be a field, let $\bar{k}$ be an algebraic closure, and let $\mathbb{P}^n_k$ be the n-dimensional projective space over k, with homogeneous coordinate ring $k[X_0, \ldots, X_n]$. A *hypersurface* in $\mathbb{P}^n_k$ is the zero set of a nonzero homogeneous polynomial $H \in k[X_0, \ldots, X_n]$. Equivalently, a hypersurface is a closed subvariety (or subscheme) of pure dimension $n-1$ (see the proof of [Har77, Proposition II.6.4]). A hypersurface W defined by $H \in k[X_0, \ldots, X_n]$ is *smooth* if the homogeneous polynomials $H, \partial H/\partial X_0, \ldots, \partial H/\partial X_n$ have no common zeros in $\mathbb{P}^n_{\bar{k}}$. The hypersurface W is *integral* if H is an irreducible polynomial over k, and it is *geometrically integral* if H is irreducible over $\bar{k}$.

Lemma 11.1. *Let $n \geq 2$ and let $W \subset \mathbb{P}^n_k$ be a smooth hypersurface. Then W is geometrically integral.*

Proof. Suppose that W is defined by $H \in k[X_0, \ldots, X_n]$. Then the hypersurface $W_{\bar{k}} \subset \mathbb{P}^n_{\bar{k}}$ defined by $H \in \bar{k}[X_0, \ldots, X_n]$ is again smooth, since $H, \partial H/\partial X_0, \ldots, \partial H/\partial X_n$ have no common zeros *in* $\bar{k}$. This implies that $W_{\bar{k}}$ is irreducible, since any two irreducible components would intersect in a singular point [Har77, Theorem I.7.2]. This means that $W_{\bar{k}}$ is a nonsingular variety [Har77, Example 10.0.3], which is necessarily defined by an irreducible polynomial. $\square$

In the following we assume $n \geq 2$. Let $W \subset \mathbb{P}^n_k$ be the smooth hypersurface defined by $H \in k[X_0, \ldots, X_n]$. The tangent space to W at a rational point $P \in W(k)$ is the hyperplane $T_P(W)$ defined by

$$\frac{\partial H}{\partial X_0}(P)\, X_0 + \cdots + \frac{\partial H}{\partial X_n}(P)\, X_n = 0.$$

This is in fact a hyperplane, as $H(P) = 0$ implies $\partial H/\partial X_i(P) \neq 0$ for some i. This construction can be improved in the following way. Let $\mathbb{P}^{n*}_k$ be the *dual projective space* parameterizing *hyperplanes* in affine $(n+1)$-space. Concretely, we have $\mathbb{P}^{n*}_k \cong \mathbb{P}^n_k$, with a point $[a_0 : \cdots : a_n] \in \mathbb{P}^n_k$ corresponding to the hyperplane $a_0X_0 + \cdots + a_nX_n = 0$. The map $W(k) \to \mathbb{P}^{n*}_k$ defined by $P \mapsto T_P(W)$ can be promoted to a regular map $\mathcal{G}_W \colon W \to \mathbb{P}^{n*}_k$; using the identification $\mathbb{P}^{n*}_k \cong \mathbb{P}^n_k$, it is given by the homogeneous polynomials $[\partial H/\partial X_0 : \ldots : \partial H/\partial X_n]$. We call $\mathcal{G}_W$ the *Gauss map*; its image W^* is the *dual variety of* W. Algebraically, the Gauss map is defined by the k-algebra homomorphism $g_W \colon k[Y_0, \ldots, Y_n] \to k[X_0, \ldots, X_n]/(H)$ sending Y_i to $\partial H/\partial X_i$, and W^* is defined by $\ker(g_W)$.

More generally, if W is singular then W^* is defined to be the closure of the image of the nonsingular locus under the Gauss map.

The following proposition summarizes the main facts about projective duality for smooth hypersurfaces in arbitrary characteristic. All results are extracted from [Kle86], which is an excellent reference.

Proposition 11.2. *Let $n \geq 2$, let $W \subset \mathbb{P}^n_k$ be a smooth hypersurface defined by $H \in k[X_0, \ldots, X_n]$, and let $W^* \subset \mathbb{P}^{n*}_k$ be the dual variety. Suppose that W is not a hyperplane, i.e. that* $\deg(H) > 1$.

1. *The dual W^* is a geometrically integral hypersurface.*
2. *The Gauss map $\mathcal{G}_W \colon W \to W^*$ is generically finite.*
3. *If W^* is defined by a homogeneous polynomial $H^* \in k[X_0, \ldots, X_n]$, then*

   $$[k(W) : k(W^*)] \deg(H^*) = \deg(H)(\deg(H) - 1)^{n-1},$$

   where $k(\ )$ denotes the field of rational functions.
4. *If the field extension $k(W)/k(W^*)$ is separable (e.g. if* $\operatorname{char}(k) = 0$*), then $k(W) = k(W^*)$.*
5. *(Reciprocity) If $k(W) = k(W^*)$ then $(W^*)^* = W$.*

Proof. The image of a (geometrically) integral variety under a regular map is again (geometrically) integral. This is the geometric version of the following algebraic fact: if $f \colon A \to B$ is a homomorphism from a non-zero unitary commutative ring A to an integral domain B, then $\ker(f)$ is prime. If $\dim(W^*) = n - 1$ then $\mathcal{G}_W$ is generically finite, as it is then a dominant morphism of varieties of the same dimension. The fact that $\dim(W^*) = n - 1$, along with the degree formula in (3), follow from [Kle86, Proposition II.2(iv) and Proposition II.9]. Assertions (4) and (5) follow from [Kle86, Proposition II.15a]. $\square$

Remark 11.3. Proposition 11.2(4) does not assert that $k(W)$ is a purely inseparable extension of $k(W^*)$. Indeed, there is no restriction on the separable degree of $k(W)/k(W^*)$ (provided that $k(W)/k(W^*)$ is not separable): see [Kle86, §II.3].

Remark 11.4. With the notation in Proposition 11.2, suppose that $(W^*)^* = W$. Let $d = \deg(H)$ and $d^* = \deg(H^*)$, so that $d^* = d(d-1)^{n-1}$. If W^* is also smooth then $d = d^*(d^* - 1)^{n-1}$, which is true only if $d = 2$. Hence W^* is not smooth if $(W^*)^* = W$ and W is not a quadric. See [Kle86, Corollary II.10] for more details.

Now we apply the above considerations when k is a *global* field (of any characteristic). If S is a finite set of finite places of k then we let $\mathcal{O}_S$ denote the ring of S-integers. A finite place $\mathfrak{p}$ not contained in S corresponds to a maximal ideal of $\mathcal{O}_S$; we denote the residue field by $\kappa(\mathfrak{p}) = \mathcal{O}_S/\mathfrak{p}$. We wish to prove a version of Proposition 11.2 that holds for all finite places outside of some S depending only on H. We will do so by "spreading out from the generic fiber": we will consider varieties (or schemes) defined over $\operatorname{Spec}(\mathcal{O}_S)$, and take the closure of W in $\mathbb{P}^n_{\mathcal{O}_S}$. If $\operatorname{char}(k) = p > 0$ then k is the function field of a smooth, projective, geometrically integral curve C defined over a finite field $\mathbb{F}_q$, and S may be identified with a finite set of (closed) points of C; in this case, $\operatorname{Spec}(\mathcal{O}_S)$ is simply the variety $C \setminus S$. In characteristic zero, we are forced to use some scheme theory, as $\operatorname{Spec}(\mathcal{O}_S)$ is not a variety over a field; in both cases, our proofs are written in the language of elementary commutative algebra.

Proposition 11.5. *Let k be a global field, let $n \geq 2$, and let $W \subset \mathbb{P}^n_k$ be a smooth hypersurface defined by $H \in k[X_0, \ldots, X_n]$ of degree at least 2. Let $W^* \subset \mathbb{P}^{n*}_k$ be the dual variety, and let $H^* \in k[Y_0, \ldots, Y_n]$ be a homogeneous polynomial defining W^*. There exists a finite set S of finite places of k, depending only on H and H^*, such that the following hold.*

1. *The polynomials H and H^* have coefficients in $\mathcal{O}_S$.*
2. *For all finite places $\mathfrak{p}$ not in S, the polynomial $H \pmod{\mathfrak{p}} \in \kappa(\mathfrak{p})[X_0, \ldots, X_n]$ is nonzero (thus $\deg(H \bmod \mathfrak{p}) = \deg(H)$ since H is homogeneous), and the hypersurface $W_\mathfrak{p} \subset \mathbb{P}^n_{\kappa(\mathfrak{p})}$ defined by $H \pmod{\mathfrak{p}}$ is smooth.*
3. *For all finite places $\mathfrak{p}$ not in S, the dual variety $(W_\mathfrak{p})^* \subset \mathbb{P}^{n*}_{\kappa(\mathfrak{p})}$ is defined by $H^* \pmod{\mathfrak{p}}$.*

Proof. The first assertion is true once S contains all places with respect to which some coefficient of H or H^* has negative valuation. The reduction $H \pmod{\mathfrak{p}}$ is nonzero as long as $\mathfrak{p}$ is not one of the finite set of places whose valuation is strictly positive on all coefficients of H; we include such places in S as well. Finally, we enlarge S to assume that $\mathcal{O}_S$ is a unique factorization domain. Note that H is a *primitive* polynomial over $\mathcal{O}_S$ by construction: its coefficients have no common factors because we included those in S. Similarly, by enlarging S if necessary, we may assume that $H^* \pmod{\mathfrak{p}}$ is nonzero for all $\mathfrak{p} \notin S$, so that H^* is primitive over $\mathcal{O}_S$.

Consider the closed subscheme $\overline{W} \subset \mathbb{P}^n_{\mathcal{O}_S}$ defined by H. (If $\operatorname{char}(k) = p > 0$ then this is a subvariety of $\mathbb{P}^n_{\mathcal{O}_S} = \mathbb{P}^n_{\mathbb{F}_q} \times (C \setminus S)$.) Let I be the (homogeneous) ideal of $\mathcal{O}_S[X_0, \ldots, X_n]$ defined by H and $\partial H/\partial X_0, \ldots, \partial H/\partial X_n$. Since W is smooth, the extended ideal $Ik[X_0, \ldots, X_n]$ (the ideal of $k[X_0, \ldots, X_n]$ generated by the image of I) contains $(X_0^m, \ldots, X_n^m)$ for some $m > 0$. This means that each X_i^m is a linear combination of H and the $\partial H/\partial X_j$ with coefficients in k. Enlarging S to contain all places with negative valuation on some coefficient of one of these linear combinations, we may assume $(X_0^m, \ldots, X_n^m) \subset I$. Then for $\mathfrak{p} \notin S$ we have $(X_0^m, \ldots, X_n^m) \subset (H, \partial H/\partial X_0, \ldots, \partial H/\partial X_n) \pmod{\mathfrak{p}}$, so $W_\mathfrak{p}$ is smooth.

(Geometrically, the generic fiber of $\overline{W} \to \operatorname{Spec}(\mathcal{O}_S)$ is the hypersurface W, and the fiber over a place $\mathfrak{p} \in \operatorname{Spec}(\mathcal{O}_S)$ is $W_\mathfrak{p}$. Lemma 11.6 below shows that $\overline{W}$ is the closure of W in $\mathbb{P}^n_{\mathcal{O}_S}$. The singular locus of $\overline{W}$ is a closed subscheme not intersecting the generic fiber of $\overline{W} \to \operatorname{Spec}(\mathcal{O}_S)$, so its image in $\operatorname{Spec}(\mathcal{O}_S)$ is a finite set of closed points. Deleting these points allows us to assume $\overline{W} \to \operatorname{Spec}(\mathcal{O}_S)$ is smooth.)

Now consider the morphism (regular map) $\mathcal{G}\colon \overline{W} \to \mathbb{P}^n_{\mathcal{O}_S}$ defined by the homogeneous polynomials $[\partial H/\partial X_0 : \ldots : \partial H/\partial X_n]$. As before, this is well-defined because H and its partial derivatives have no common zeros. Let $\overline{W}^*$ denote the image of $\mathcal{G}$. Algebraically, the morphism $\mathcal{G}$ corresponds to the $\mathcal{O}_S$-algebra map $g\colon \mathcal{O}_S[Y_0, \ldots, Y_n] \to \mathcal{O}_S[X_0, \ldots, X_n]/(H)$ sending Y_i to $\partial H/\partial X_i$, and $\overline{W}^*$ is defined by $J = \ker(g)$. The Gauss map $\mathcal{G}_W$ corresponds to $g_W = g \otimes_{\mathcal{O}_S} k\colon k[Y_0, \ldots, Y_n] \to k[X_0, \ldots, X_n]/(H)$, and $\mathcal{G}_{W_\mathfrak{p}}$ is defined by $g_\mathfrak{p} = g \pmod{\mathfrak{p}}\colon k(\mathfrak{p})[Y_0, \ldots, Y_n] \to k(\mathfrak{p})[X_0, \ldots, X_n]/(H \pmod{\mathfrak{p}})$ for $\mathfrak{p} \notin S$. Hence the dual W^* is defined by the ideal $\ker(g_W) = Jk[Y_0, \ldots, Y_n]$, and the dual $(W_\mathfrak{p})^*$ is defined by $\ker(g_\mathfrak{p}) = J \pmod{\mathfrak{p}}$. But $\ker(g_W)$ is generated by H^*, and we are assuming H^* to be primitive, so $\ker(g_W) \cap \mathcal{O}_S[Y_0, \ldots, Y_n] = (H^*)$ by Lemma 11.6. On the other hand, we have $\ker(g_W) \cap \mathcal{O}_S[Y_0, \ldots, Y_n] = J$ by [AM16, Proposition 3.11(iv)], since J is prime and $\ker(g_W) = Jk[Y_0, \ldots, Y_n]$ is not the unit ideal. Hence $(W_\mathfrak{p})^*$ is defined by $J \pmod{\mathfrak{p}} = (H^* \pmod{\mathfrak{p}})$, as desired.

(Geometrically, the restriction of $\mathcal{G}$ to the generic fiber of $\overline{W} \to \mathrm{Spec}(\mathcal{O}_S)$ is the Gauss map $\mathcal{G}_W$, and the restriction to the fiber over $\mathfrak{p}$ is $\mathcal{G}_{W_\mathfrak{p}}$. Since H^* is irreducible, it defines an integral hypersurface X in $\mathbb{P}^n_{\mathcal{O}_S}$, which is thus the closure of its generic fiber. But $\overline{W}^*$ is also irreducible, and $\overline{W}^*$ and X both have generic fiber W.) □

We used the following lemma in the above proof.

Lemma 11.6. *Let R be a unique factorization domain with fraction field k, let $H \in R[X_1, \ldots, X_n]$ be a primitive polynomial of positive degree, let I be the ideal of $R[X_1, \ldots, X_n]$ generated by H, and let J be the ideal of $k[X_0, \ldots, X_n]$ generated by H. Then $J \cap R[X_1, \ldots, X_n] = I$.*

Proof. Since H is primitive, it is a prime element of $R[X_1, \ldots, X_n]$, so I is prime. Since H has positive degree, the ideal J is not the unit ideal. Now use [AM16, Proposition 3.11(iv)]. □

Joseph Rabinoff: Department of Mathematics, Duke University, 120 Science Drive, Durham, North Carolina 27708, USA. *Email:* `jdr@math.duke.edu`

References

[AM16] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, economy ed., Addison-Wesley Series in Mathematics, Westview Press, Boulder, CO, 2016, For the 1969 original see [MR0242802]. MR 3525784

[Bon21] D. Bonolis, *A polynomial sieve and sums of Deligne type*, Int. Math. Res. Not. IMRN (2021), no. 2, 1096–1137. MR 4201962

[Bra15] J. Brandes, *Sums and differences of power-free numbers*, Acta Arith. **169** (2015), no. 2, 169–180. MR 3359952

[Bro09] N. Broberg, *Rational points on finite covers of $\mathbb{P}^1$ and $\mathbb{P}^2$*, J. Number Theory **101** (2009), 195–207. MR 1979659

[Bro15] T. D. Browning, *The polynomial sieve and equal sums of like polynomials*, Int. Math. Res. Not. IMRN (2015), no. 7, 1987–2019. MR 3335239

[BV15] T. D. Browning and P. Vishe, *Rational points on cubic hypersurfaces over $\mathbb{F}_q(t)$*, Geom. Funct. Anal. **25** (2015), no. 3, 671–732. MR 3361770

[CD08] A. C. Cojocaru and C. David, *Frobenius fields for Drinfeld modules of rank 2*, Compos. Math. **144** (2008), no. 4, 827–848. MR 2441247

[Coh81] S. D. Cohen, *The distribution of Galois groups and Hilbert's irreducibility theorem*, Proc. London Math. Soc. (3) **43** (1981), no. 2, 227–250. MR 628276

[Del74] P. Deligne, *La conjecture de Weil I*, Inst. Hautes Études Sc. Publ. Math. No. **43** (1974), 273–307. MR 340258

[Har77] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157

[HB84] D. R. Heath-Brown, *The square sieve and consecutive square-free numbers*, Math. Ann. **266** (1984), 251–259. MR 730168

[HB02] ———, *The density of rational points on curves and surfaces*, Ann. of Math. **155** (2002), 553–595. MR 1906595

[HBP12] D. R. Heath-Brown and L. B. Pierce, *Counting rational points on smooth cyclic covers*, J. Number Theory **132** (2012), no. 8, 1741–1757. MR 2922342

[Hoo78] C. Hooley, *On the representations of a number as the sum of four cubes. I*, Proc. London Math. Soc. (3) **36** (1978), no. 1, 117–140. MR 506025

[Hsu96] Chih-Nung Hsu, *A large sieve inequality for rational function fields*, J. Number Theory **58** (1996), no. 2, 267–287. MR 1393616

[IK04] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, vol. 53, Amer. Math. Soc. Colloquium Publications, Providence RI, 2004. MR 2061214

[IR90] K. Ireland and M. Rosen, *A classical introduction to modern number theory*, second ed., Graduate Texts in Mathematics, vol. 84, Springer-Verlag, New York, 1990. MR 1070716

[Jan96] Gerald J. Janusz, *Algebraic number fields*, second ed., Graduate Studies in Mathematics, vol. 7, American Mathematical Society, Providence, RI, 1996. MR 1362545

[Kat02] N. M. Katz, *Estimates for nonsingular multiplicative character sums*, Int. Math. Res. Not. **2002** (2002), 333–349. MR 1883179

[Kat07] ———, *Estimates for nonsingular mixed character sums*, Int. Math. Res. Not. **2007** (2007), 19 pp. MR 2359542

[Kle86] S. L. Kleiman, *Tangency and duality*, Proceedings of the 1984 Vancouver conference in algebraic geometry, CMS Conf. Proc., vol. 6, Amer. Math. Soc., Providence, RI, 1986, pp. 163–225. MR 846021

[LW54] Serge Lang and André Weil, *Number of points of varieties in finite fields*, Amer. J. Math. **76** (1954), 819–827. MR 65218

[Mun09] R. Munshi, *Density of rational points on cyclic covers of* $\mathbb{P}^n$, Journal de Théorie des Nombres de Bordeaux **21** (2009), 335–341. MR 2541429

[Pie06] L. B. Pierce, *A bound for the 3-part of class numbers of quadratic fields by means of the square sieve*, Forum Math. **18** (2006), 677–698. MR 2254390

[RL05] A. Rojas-León, *Estimates for singular multiplicative character sums*, Int. Math. Res. Not. (2005), no. 20, 1221–1234. MR 2144086

[Ros02] M. Rosen, *Number theory in function fields*, Graduate Texts in Mathematics, vol. 210, Springer-Verlag, New York, 2002. MR 1876657

[Ser97] J.-P. Serre, *Lectures on the Mordell-Weil theorem*, third ed., Aspects of Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 1997, Translated from the French and edited by Martin Brown from notes by Michel Waldschmidt, With a foreword by Brown and Serre. MR 1757192

(Alina Bucur)

- Department of Mathematics, University of California at San Diego, 9500 Gilman Dr 0112, La Jolla, California 92093, USA

Email address, Alina Bucur: `alina@math.ucsd.edu`

(Alina Carmen Cojocaru)

- Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, 851 S Morgan St, 322 SEO, Chicago, 60607, IL, USA;
- Institute of Mathematics “Simion Stoilow” of the Romanian Academy, 21 Calea Grivitei St, Bucharest, 010702, Sector 1, Romania

Email address, Alina Carmen Cojocaru: `cojocaru@uic.edu`

(Matilde N. Lalín)

- Département de Mathématiques et de Statistique, Université de Montréal, CP 6128, Succ. Centre-Ville, Montreal, Quebec H3C 3J7, Canada

Email address, Matilde N. Lalín: `mlalin@gmail.com`

(Lillian B. Pierce)

- Department of Mathematics, Duke University, 120 Science Drive, Durham, North Carolina 27708, USA;

Email address, Lillian B. Pierce: `pierce@math.duke.edu`