THE NUMBER OF PRIME FACTORS IN h-FREE AND h-FULL POLYNOMIALS OVER FUNCTION FIELDS

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Abstract. We study the distribution of the number of prime divisors function Ω over polynomials of the function field $\mathbb{F}_q(T)$ when restricted to h-free polynomials and to h-full polynomials. We use an adaptation of the Selberg–Delange method to function fields due to Afshar and Porritt to compute the first and second moments in each case and show that the Erdős–Kac Theorem is satisfied.

1. Introduction

Let q be an odd prime power, and let $\mathbb{F}_q$ be the finite field with q elements. Given $f \in \mathbb{F}_q[T]$, let

$$f = \alpha P_1^{e_1} \cdots P_r^{e_r} \tag{1}$$

be its prime factorization, where each $P_j \in \mathbb{F}_q[T]$ is monic irreducible and $\alpha \in \mathbb{F}_q$.

Let $h \geq 2$ be an integer. We say that f is h-free if $e_j \leq h$ for $j = 1, \dots, r$ in (1). In particular, the case $h = 2$ yields the square-free polynomials. We denote by $\mathcal{S}_h$ the set of h-free monic polynomials. Analogously, we say that f is h-full if $e_j \geq h$ for $j = 1, \dots, r$ in (1). In particular, the case $h = 2$ yields the square-full polynomials. We denote by $\mathcal{N}_h$ the set of h-free monic polynomials.

The number of prime divisors function is defined by $\Omega(f) := e_1 + \cdots + e_r$. The distribution of this function, along with that of the number of distinct prime divisors $\omega(f) := r$, have been widely studied over the natural numbers. The average number of $\omega(n)$ is known to satisfy

$$\sum_{n \leq x} \omega(n) = \log \log x + O(1).$$

More precise expressions for this average were given by Sathe [Sat53], Selberg [Sel54], Delange [Del53, Del71], and Saidak [Sai02]. Hardy and Ramanujan [HR00] famously proved that $\omega(n)$ has normal order $\log \log n$, which means that $\omega(n) \sim \log \log n$ for almost all n. Turán [Tur34] proved that the variance of $\omega(n)$ for $n \leq x$ is $\ll x \log \log x$. Erdős and Kac [EK40] used the central limit theorem and Brun's sieve to famously prove that $(\omega(n) - \log \log n)/(\log \log n)^{1/2}$ for $n \geq 3$ has a normal distribution. This result was recovered by Delange [Del53, Del71] by the method of moments, and it was later extended by Halberstam [Hal55]. The method of moments was further simplified by Billingsley [Bil69] and by Granville and Soundararajan [GS07]. The function field version of Turán's and Erdős–Kac's theorems was proven by Liu [Liu04b, Liu04a] as consequences of much more general results. Rhoades [Rho09] considered an application of Granville–Soundararajan's method to the function field case. The function $\Omega(n)$ is closely related to $\omega(n)$ and satisfies similar properties including the Erdős–Kac Theorem.

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In this note we are interested in studying the distribution of $\omega(f)$ and $\Omega(f)$ as f goes over the h-free and the h-full polynomials of $\mathbb{F}_q[T]$. We focus on Ω because there is a very natural argument for obtaining the moments in the case of h-full polynomials, but we also state the analogous results for ω. The current work generalizes the results obtained in [JL22] for the first moments of $\Omega(n)$.

Let

$$B_1 = \gamma + \sum_P \left(\log\left(1-\frac{1}{|P|}\right)+\frac{1}{|P|}\right) \text{ and } B_2 = \gamma + \sum_P \left(\log\left(1-\frac{1}{|P|}\right)+\frac{1}{|P|-1}\right),$$

denote the function field analogues of the Mertens constants, where

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^n \frac{1}{k} - \log n\right) \approx 0.57721566\ldots$$

is the Euler–Mascheroni constant.

Above and for the rest of the article, $\sum_P$ and $\prod_P$ denote the sum and product over all monic irreducible polynomials in $\mathbb{F}_q[T]$. We let $\mathcal{M}_n$ denote the set of monic polynomials of $\mathbb{F}_q[T]$ of degree n.

We compute the first and second moments of Ω for h-free polynomials.

Theorem 1.1. *For any $\varepsilon > 0$ and as $n \to \infty$, the first moment of Ω over the h-free polynomials of degree n is given by*

$$\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n} \Omega(f) = \frac{q^n\log(n)}{\zeta_q(h)} + \frac{q^n}{\zeta_q(h)}\left(B_2 - \sum_P \frac{h}{|P|^h-1}\right) + O\left(\frac{q^n}{n^{1-\varepsilon}}\right), \tag{2}$$

while the second moment is given by

$$\begin{aligned}\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n} \Omega(f)^2 =& \frac{q^n(\log(n))^2}{\zeta_q(h)} + \frac{q^n(\log(n))}{\zeta_q(h)}\left[2\left(B_2 - \sum_P\frac{h}{|P|^h-1}\right)+1\right]\\ &+\frac{q^n}{\zeta_q(h)}\left[B_2^2 + B_2 - \zeta(2) + \sum_P\left(\frac{1}{(|P|-1)^2} - \frac{h^2}{|P|^h-1} - \frac{h^2}{(|P|^h-1)^2}\right)\right.\\ &\left.+\left(\sum_P \frac{h}{|P|^h-1}\right)^2 - 2B_2\sum_P\frac{h}{|P|^h-1}\right] + O\left(\frac{q^n}{n^{1-\varepsilon}}\right).\end{aligned} \tag{3}$$

We remark that the asymptotics of equation (2) over the integers was given in [JL22, Theorem 1]. The proof for Theorem (1.1) rests in a version of the Selberg–Delange method for function fields developed by Afshar and Porritt [AP19].

The above result allows us to compute the variance.

Theorem 1.2. *For any $\varepsilon > 0$ and as $n \to \infty$, the variance over Ω over the h-free polynomials of degree n is given by*

$$\mathrm{Var}_{h-\mathrm{free},n}(\Omega) = \log(n) + B_2 - \zeta(2) + \sum_P\left(\frac{1}{(|P|-1)^2} - \frac{h^2}{|P|^h-1} - \frac{h^2}{(|P|^h-1)^2}\right) + O\left(\frac{1}{n^{1-\varepsilon}}\right).$$

Finally, we obtain an analogue of Erdős–Kac theorem in this case.

Theorem 1.3. *As $n \to \infty$, $\Omega(f)$ with $f \in \mathcal{S}_h \cap \mathcal{M}_n$ approaches a normal distribution, namely, for $\alpha \leq \beta$,*

$$\left| \left\{ f \in \mathcal{S}_h \cap \mathcal{M}_n : \alpha \leq \frac{\Omega(f) - \log(n)}{\sqrt{\log(n)}} \leq \beta \right\} \right| \to \frac{1}{\sqrt{2\pi}} \int_\alpha^\beta e^{-t^2/2} dt.$$

This result is proven by computing the moments of $\frac{\Omega(f)-\log(n)}{\sqrt{\log(n)}}$ and using Stirling numbers of the second kind to understand their growth.

We also compute the first and second moments of Ω for h-full polynomials.

Theorem 1.4. *For any $\varepsilon > 0$ and as $n \to \infty$, the first moment of Ω over the h-full polynomials of degree n is given by*

$$\begin{aligned} \sum_{f \in \mathcal{N}_h \cap \mathcal{M}_n} \Omega(f) =& \frac{q^{\frac{n}{h}}}{h} \sum_{k=0}^{h-1} \xi_h^{kn} \prod_P \left(1 + (q^{\frac{1}{h}} \xi_h^k)^{-\deg(P)(h+1)} + \cdots + (q^{\frac{1}{h}} \xi_h^k)^{-\deg(P)(2h-1)}\right) \\ & \times \left[h \left(\log\left(\frac{n}{h}\right) + B_2 \right) \right. \\ & \left. + \sum_{j=1}^{h-1} (h+j) \sum_P \frac{(q^{\frac{1}{h}} \xi_h^k)^{-\deg(P)(h+j)}}{1 + (q^{\frac{1}{h}} \xi_h^k)^{-\deg(P)(h+1)} + \cdots + (q^{\frac{1}{h}} \xi_h^k)^{-\deg(P)(2h-1)}} \right] \\ & + O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}} \right), \end{aligned}$$

while the second moment is given by

$$\begin{aligned} \sum_{f \in \mathcal{N}_h \cap \mathcal{M}_n} \Omega(f)^2 =& \frac{q^{\frac{n}{h}}}{h} \sum_{k=0}^{h-1} \xi_h^{kn} \prod_P \left(1 + (q^{\frac{1}{h}} \xi_h^k)^{-\deg(P)(h+1)} + \cdots + (q^{\frac{1}{h}} \xi_h^k)^{-\deg(P)(2h-1)}\right) \\ & \times \left[h^2 \left(\log\left(\frac{n}{h}\right)^2 + (2B_2+1)\log\left(\frac{n}{h}\right) + B_2^2 + B_2 - \zeta(2) + \sum_P \frac{1}{(|P|-1)^2} \right) \right. \\ & + \sum_{j=1}^{h-1} (h+j) \left(2h \left(\log\left(\frac{n}{h}\right) + B_2 \right) + (h+j) \right. \\ & \left. + \sum_{\ell=1}^{h-1} (h+\ell) \sum_P \frac{(q^{\frac{1}{h}} \xi_h^k)^{-\deg(P)(h+\ell)}}{1 + (q^{\frac{1}{h}} \xi_h^k)^{-\deg(P)(h+1)} + \cdots + (q^{\frac{1}{h}} \xi_h^k)^{-\deg(P)(2h-1)}} \right) \\ & \times \sum_P \frac{(q^{\frac{1}{h}} \xi_h^k)^{-\deg(P)(h+j)}}{1 + (q^{\frac{1}{h}} \xi_h^k)^{-\deg(P)(h+1)} + \cdots + (q^{\frac{1}{h}} \xi_h^k)^{-\deg(P)(2h-1)}} \\ & \left. - \sum_{1 \leq j, \ell \leq h-1} (h+j)(h+\ell) \sum_P \frac{(q^{\frac{1}{h}} \xi_h^k)^{-\deg(P)(h+j)} (q^{\frac{1}{h}} \xi_h^k)^{-\deg(P)(h+\ell)}}{\left(1 + (q^{\frac{1}{h}} \xi_h^k)^{-\deg(P)(h+1)} + \cdots + (q^{\frac{1}{h}} \xi_h^k)^{-\deg(P)(2h-1)}\right)^2} \right] \\ & + O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}} \right). \end{aligned}$$

The above result allows us to compute the variance.

Theorem 1.5. *For any $\varepsilon > 0$ and as $n \to \infty$, the variance of Ω over the h-full polynomials is given by*

$$\begin{aligned}
\mathrm{Var}_{h-\mathrm{full},n}(\Omega) = \Bigg[& h^2\left(\log\left(\frac{n}{h}\right) + B_2 - \zeta(2) + \sum_P \frac{1}{(|P|-1)^2}\right) \\
&+ \left(\sum_{k=0}^{h-1} \xi_h^{kn} \prod_P \left(1 + (q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)} + \cdots + (q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right)\right)^{-1} \\
&\times \sum_{k=0}^{h-1} \xi_h^{kn} \prod_P \left(1 + (q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)} + \cdots + (q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right) \\
&\times \Bigg[\sum_{j=1}^{h-1}(h+j)\Bigg((h+j) + \sum_{\ell=1}^{h-1}(h+\ell)\sum_P \frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+\ell)}}{1 + (q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)} + \cdots + (q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}}\Bigg) \\
&\times \sum_P \frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+j)}}{1 + (q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)} + \cdots + (q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}} \\
&- \sum_{1\le j,\ell\le h-1}(h+j)(h+\ell)\sum_P \frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+j)}(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+\ell)}}{\left(1 + (q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)} + \cdots + (q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right)^2}\Bigg] \\
&- \Bigg[\left(\sum_{k=0}^{h-1} \xi_h^{kn} \prod_P \left(1 + (q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)} + \cdots + (q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right)\right)^{-1} \\
&\times \sum_{k=0}^{h-1} \xi_h^{kn} \prod_P \left(1 + (q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)} + \cdots + (q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right) \\
&\times \sum_{j=1}^{h-1}(h+j)\sum_P \frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+j)}}{1 + (q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)} + \cdots + (q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}}\Bigg]^2 + O\left(\frac{1}{n^{1-\varepsilon}}\right).
\end{aligned}$$

The above statements are proven by considering the factorization of each f to codify the h-full condition and by exploiting the fact that Ω is completely additive. As we will remark later, it is possible to give a proof that avoids using the completely additive property, that therefore can be applied to extend the results to ω. We have chosen to keep the method of proof that employs the completely additive property here because it yields clearer formulas for the first and second moments and the variant of Ω.

Finally, we also obtain an analogue of Erdős–Kac theorem in this case.

Theorem 1.6. *As $n \to \infty$, $\Omega(f)$ with $f \in \mathcal{N}_h \cap \mathcal{M}_n$ approaches a normal distribution, namely, for $\alpha \le \beta$,*

$$\left|\left\{ f \in \mathcal{N}_h \cap \mathcal{M}_n : \alpha \le \frac{\Omega(f) - h\log\left(\frac{n}{h}\right)}{h\sqrt{\log\left(\frac{n}{h}\right)}} \le \beta \right\}\right| \to \frac{1}{\sqrt{2\pi}}\int_\alpha^\beta e^{-t^2/2}dt.$$

This result is again proven by computing the moments of $\frac{\Omega(f)-h\log\left(\frac{n}{h}\right)}{h\sqrt{\log\left(\frac{n}{h}\right)}}$, using similar techniques as the ones for the h-free polynomial case.

Remark that we can recover the results over all monic polynomials with no restrictions if we let $h \to \infty$ in the formulas for h-free polynomials, or if we specialize at $h = 1$ in the formulas for h-full polynomials.

This article is organized as follows. Section 2 provides some notation and preliminary background. The distribution of Ω over the h-free polynomials is considered in detail in Section 3, while Section 4 states the corresponding results for ω. Similarly, the distribution of Ω over the h-full polynomials is treated in Section 5, including a discussion on the proof strategy for Theorem 1.4 in Section 5.3. Finally Section 6 states the corresponding results for ω.

2. Notation and preliminary statements

We denote by $\mathcal{M}$ the set of monic polynomials over $\mathbb{F}_q[T]$ and by $\mathcal{M}_n$ (respectively $\mathcal{M}_{\leq n}$) the subset of $\mathcal{M}$ containing the polynomials of degree n (respectively degree $\leq n$). Let $\mathcal{P}$ be the set of monic irreducible polynomials, and let $\mathcal{P}_n$, $\mathcal{P}_{\leq n}$ be defined analogously to the corresponding subsets of $\mathcal{M}$. The zeta function of $\mathbb{F}_q[T]$ is given by

$$\zeta_q(s) = \sum_{f \in \mathcal{M}} \frac{1}{|f|^s} = \prod_P \left(1 - \frac{1}{|P|^s}\right)^{-1}.$$

The above sum and product converge for $\mathrm{Re}(s) > 1$. However, one can also see that

$$\zeta_q(s) = \frac{1}{1 - q^{1-s}},$$

and this provides a meromorphic continuation for $\zeta_q(s)$ to the whole complex plane, with a single pole at $s = 1$. By making the change of variables $u = q^{-s}$ we can write

$$\mathcal{Z}_q(u) = \sum_{f \in \mathcal{M}} u^{\deg(f)} = \prod_P \left(1 - u^{\deg(P)}\right)^{-1},$$

which converges absolutely for $|u| < \frac{1}{q}$, and has a meromorphic continuation to the complex plane with a pole at $u = \frac{1}{q}$.

We recall Perron's formula over $\mathbb{F}_q[T]$ which we will use throughout this article.

Lemma 2.1 (Perron's Formula). *If the generating series $\mathcal{A}(u) = \sum_{f \in \mathcal{M}} a(f) u^{\deg(f)}$ is absolutely convergent in $|u| \leq r < 1$, then*

$$\sum_{f \in \mathcal{M}_n} a(f) = \frac{1}{2\pi i} \oint_{|u|=r} \frac{\mathcal{A}(u)}{u^n} \frac{du}{u},$$

where, in the usual notation, we take $\oint$ to signify the integral over the circle oriented counterclockwise.

The following result of Afshar and Porritt provides an extension of the Selberg–Delange method for function fields and will be crucial for obtaining the first and second moments in all cases.

Theorem 2.1. [AP19, Proposition 2.3] *Let $C(u, z) = \sum_{n \geq 0} C_z(n) u^n$ and $B(u, z) = \sum_{n \geq 0} B_z(n) u^n$ be power series with coefficients depending on z satisfying $C(u, z) = B(u, z)\mathcal{Z}_q(u)^z$. Suppose also that, uniformly for $|z| \leq A$,*

$$\sum_{n \geq 0} \frac{|B_z(n)|}{q^n} n^{2A+2} \ll_A 1.$$

Then, uniformly for $|z| \le A$ *and* $n \ge 1$*, we have*

$$C_z(n) = q^n \frac{n^{z-1}}{\Gamma(z)} B\left(1/q, z\right) + O_A\left(q^n n^{\mathrm{Re}(z)-2}\right).$$

We will need the following extension of the above theorem.

Theorem 2.2 (Generalized Theorem 2.1)**.** *Let* $C(u,z) = \sum_{n\ge 0} C_z(n) u^n$ *and* $B(u,z) = \sum_{n\ge 0} B_z(n) u^n$ *be power series with coefficients depending on* z *satisfying* $C(u,z) = B(u,z)\mathcal{Z}_q(u^h)^z$*, where* h *is a positive integer. Suppose also that, uniformly for* $|z| \le A$*,*

$$\sum_{a=0}^{\infty} \frac{|B_z(n)|}{q^{\frac{n}{h}}} n^{2A+2} \ll_A 1.$$

Then, uniformly for $|z| \le A$ *and* $n \ge 1$*, we have*

$$C_z(n) = \frac{q^{\frac{n}{h}} n^{z-1}}{h^z \Gamma(z)} \sum_{k=0}^{h-1} \xi_h^{kn} B\left((q^{\frac{1}{h}}\xi_h^k)^{-1}, z\right) + O_A\left(q^{\frac{n}{h}} n^{\mathrm{Re}(z)-2}\right),$$

where ξ_h *denotes a primitive* h*-root of unity in* $\mathbb{C}$*.*

Proof. Following Lemma 2.1 and Corollary 2.2 in [AP19], we define $D_z(n)$ for $n \ge 0$ via $\mathcal{Z}_q(u^h)^z = \sum_{n=0}^{\infty} D_z(n) u^n$. The binomial theorem gives

$$\mathcal{Z}_q(u^h)^z = (1 - qu^h)^{-z} = \sum_{n\ge 0} \binom{n+z-1}{n} q^n u^{hn},$$

and therefore,

$$D_z(n) = \frac{1}{h} \sum_{k=0}^{h-1} (q^{\frac{1}{h}}\xi_h^k)^n \binom{n/h + z - 1}{n/h}. \tag{4}$$

By [AP19, Corollary 2.2], $\frac{\Gamma(m+z)}{\Gamma(m+1)} = m^{z-1}(1 + O_A(\frac{1}{m}))$. Applying this to equation (4), we have

$$D_z(n) = \frac{1}{h} \sum_{k=0}^{h-1} (q^{\frac{1}{h}}\xi_h^k)^n \frac{\Gamma\left(\frac{n}{h} + z\right)}{\Gamma\left(\frac{n}{h} + 1\right)\Gamma(z)} = \frac{q^{\frac{n}{h}}}{h} \sum_{k=0}^{h-1} \xi_h^{kn} \frac{\left(\frac{n}{h}\right)^{z-1}}{\Gamma(z)} \left(1 + O_A\left(\frac{1}{n}\right)\right).$$

We now follow the proof of [AP19, Proposition 2.3]. First express the coefficients C_z as convolutions of B_z and D_z to obtain

$$\begin{aligned} C_z(n) &= \sum_{0\le a\le n} B_z(a) D_z(n-a) \\ &= \frac{q^{\frac{n}{h}}}{h^z} \sum_{0\le a<n} \frac{B_z(a)\sum_{k=0}^{h-1} \xi_h^{(n-a)k} (n-a)^{z-1}}{q^{\frac{a}{h}}\Gamma(z)} + B_z(n) + O_A\left(q^{\frac{n}{h}} \sum_{0\le a<n} \frac{|B_z(a)|}{q^{\frac{a}{h}}} (n-a)^{\mathrm{Re}(z)-2}\right). \end{aligned} \tag{5}$$

We split the first sum at $\frac{n}{2}$ and use that

$$(n-a)^{z-1} = \begin{cases} n^{z-1}(1 + O_A(a/n)), & \text{if } 0 \le a \le \frac{n}{2}, \\ O_A(n^{A-1}), & \text{if } \frac{n}{2} < a < n. \end{cases}$$

Then the first sum in (5) can be written as

$$\begin{aligned}
&\sum_{0\le a<n} \frac{B_z(a)\sum_{k=0}^{h-1}\xi_h^{(n-a)k}(n-a)^{z-1}}{q^{\frac{a}{h}}} \\
&= \sum_{0\le a\le \frac{n}{2}} \frac{B_z(a)\sum_{k=0}^{h-1}\xi_h^{(n-a)k}n^{z-1}}{q^{\frac{a}{h}}}(1+O_A(a/n)) + O_A\left(\sum_{\frac{n}{2}<a<n}\frac{B_z(a)n^{A-1}}{q^{\frac{a}{h}}}\right) \\
&= n^{z-1}\sum_{0\le a\le \frac{n}{2}}\sum_{k=0}^{h-1}\frac{B_z(a)\xi_h^{nk}}{(q^{\frac{1}{h}}\xi_h^k)^a} + O_A\left(n^{\mathrm{Re}(z)-2}\left(\sum_{0\le a\le\frac{n}{2}}\frac{|B_z(a)|a}{q^{\frac{a}{h}}} + \sum_{\frac{n}{2}<a<n}\frac{|B_z(a)|a^{2A+1}}{q^{\frac{a}{h}}}\right)\right),
\end{aligned}$$

where we have used that $n^{\mathrm{Re}(z)-2}a^{2A+1} \gg n^{-A-2}n^{2A+1} = n^{A-1}$ for $\frac{n}{2} < a < n$. Completing the sum over a, the above then equals

$$\begin{aligned}
&n^{z-1}\sum_{k=0}^{h-1}\xi_h^{kn}B\left((q^{\frac{1}{h}}\xi_h^k)^{-1},z\right) + O_A\left(n^{\mathrm{Re}(z)-1}\sum_{\frac{n}{2}<a}\frac{B_z(a)}{q^{\frac{a}{h}}} + n^{\mathrm{Re}(z)-2}\right) \\
&= n^{z-1}\sum_{k=0}^{h-1}\xi_h^{kn}B\left((q^{\frac{1}{h}}\xi_h^k)^{-1},z\right) + O_A\left(n^{\mathrm{Re}(z)-2}\sum_{\frac{n}{2}<a}\frac{B_z(a)n}{q^{\frac{a}{h}}} + n^{\mathrm{Re}(z)-2}\right) \\
&= n^{z-1}\sum_{k=0}^{h-1}\xi_h^{kn}B\left((q^{\frac{1}{h}}\xi_h^k)^{-1},z\right) + O_A\left(n^{\mathrm{Re}(z)-2}\right). \qquad (6)
\end{aligned}$$

For the other terms in (5), we get

$$B_z(n) \ll n^{\mathrm{Re}(z)-2}n^{A+2}|B_z(n)| \ll q^{\frac{n}{h}}n^{\mathrm{Re}(z)-2}\sum_{a=0}^{\infty}\frac{|B_z(a)|}{q^{\frac{a}{h}}}a^{A+2} \ll q^{\frac{n}{h}}n^{\mathrm{Re}(z)-2}, \qquad (7)$$

and

$$\begin{aligned}
&\sum_{0\le a<n}\frac{|B_z(a)|}{q^{\frac{a}{h}}}(n-a)^{\mathrm{Re}(z)-2} \\
&= \sum_{0\le a\le\frac{n}{2}}\frac{|B_z(a)|}{q^{\frac{a}{h}}}n^{\mathrm{Re}(z)-2}(1+O_A(a/n)) + O_A\left(\sum_{\frac{n}{2}a<n}\frac{|B_z(a)|}{q^{\frac{a}{h}}}n^{A-2}\right) \\
&\ll_A n^{\mathrm{Re}(z)-2}\sum_{0\le a\le\frac{n}{2}}\frac{|B_z(a)|}{q^{\frac{a}{h}}} + n^{\mathrm{Re}(z)-3}\sum_{0\le a\le\frac{n}{2}}\frac{|B_z(a)|a}{q^{\frac{a}{h}}} + n^{\mathrm{Re}(z)-3}\sum_{\frac{n}{2}<a<n}\frac{|B_z(a)|a^{2A+1}}{q^{\frac{a}{h}}} \\
&\ll_A n^{\mathrm{Re}(z)-2}. \qquad (8)
\end{aligned}$$

Combining (6) with (7) and (8) in (5), we get the result. □

3. The distribution of Ω over h-free polynomials

Recall that $h \ge 2$ is a fixed integer and that we denote by $\mathcal{S}_h$ the set of h-free monic polynomials. In order to prove Theorem 1.1, we start by considering the following Euler product (converging at

$\mathrm{Re}(s) > 1$), which gives the generating Dirichlet series for the h-free polynomials:

$$\frac{\mathcal{Z}_q(u)}{\mathcal{Z}_q(u^h)} = \prod_P \left(\frac{1-u^{h\deg(P)}}{1-u^{\deg(P)}}\right) = \prod_P \left(1+u^{\deg(P)}+\cdots+u^{(h-1)\deg(P)}\right) = \sum_{f\in\mathcal{S}_h} u^{\deg(f)}. \tag{9}$$

We incorporate the coefficient $\Omega(f)$ by introducing it as the exponent of an extra variable, which we will later differentiate. This preserves the additive structure of $\Omega(f)$.

$$\mathcal{C}(u,z) := \prod_P \left(1+zu^{\deg(P)}+\cdots+z^{h-1}u^{(h-1)\deg(P)}\right) = \sum_{f\in\mathcal{S}_h} z^{\Omega(f)}u^{\deg(f)}. \tag{10}$$

Extracting the singularity at $u=\frac{1}{q}$ leads us to consider

$$\begin{aligned}\mathcal{B}(u,z) :=& \mathcal{Z}_q(u)^{-z}\mathcal{C}(u,z)\\ =& \prod_P \left(1-u^{\deg(P)}\right)^z \left(\frac{1-z^hu^{h\deg(P)}}{1-zu^{\deg(P)}}\right).\end{aligned} \tag{11}$$

In order to estimate the derivative of $\mathcal{C}(u,z)$ we need the result of Afshar and Porritt, Theorem 2.1. We claim that for $\mathcal{B}(u,z)$ given by (11), if $\mathcal{B}(u,z)=\sum_{n\geq 0}\mathcal{B}_z(n)u^n$, then uniformly for $|z|\leq A$, we have $\sum_{n\geq 0}\frac{|\mathcal{B}_z(n)|}{q^n}n^{2A+2} \ll_A 1$. Indeed, we have the following result.

Lemma 3.1. *For $|z|\leq A$, $n\geq 2$ and $\sigma>\frac{1}{2}$,*

$$\sum_{0\leq a\leq n}\frac{|\mathcal{B}_z(a)|}{q^{\sigma a}} \leq c_{A,\sigma},$$

where $c_{A,\sigma}$ is a constant depending on A and σ.

Proof. Our proof follows very similar steps to those in the proof of [AP19, Proposition 2.5]. Let $b_z(f)$ be the function defined on the powers of monic irreducible polynomials P by

$$1+\sum_{k\geq 1} b_z(P^k)u^k = (1+zu+\cdots+z^{h-1}u^{h-1})(1-u)^z = \left(\frac{1-z^hu^h}{1-zu}\right)(1-u)^z, \tag{12}$$

and extended multiplicatively to all $f\in\mathcal{M}$.

Then $\mathcal{B}(u,z)=\sum_{f\in\mathcal{M}} b_z(f)u^{\deg(f)}$, and therefore, $\mathcal{B}_z(n)=\sum_{f\in\mathcal{M}_n} b_z(f)$. Expanding the right-hand side of (12), we see that $b_z(P)=0$. By Cauchy's formula integrating (12) over $|u|=\sqrt{\frac{2}{3}}$, we obtain

$$b_z(P^k) = \frac{1}{2\pi i}\oint_{|u|=\sqrt{\frac{2}{3}}}\left(\frac{1-z^hu^h}{1-zu}\right)(1-u)^z\frac{du}{u^{k+1}}.$$

Thus,

$$|b_z(P^k)| \leq \left(\frac{3}{2}\right)^{\frac{k}{2}} M_A,$$

for $k\geq 2$, where

$$M_A := \sup_{|z|\leq A, |u|\leq\sqrt{\frac{2}{3}}} \left|(1+zu+\cdots+z^{h-1}u^{h-1})(1-u)^z\right|$$

is a constant depending on A. Applying this bound, we have

$$\begin{aligned}\sum_{0\le a\le n}\frac{|\mathcal{B}_z(a)|}{q^{\sigma a}} &\le \sum_{f\in\mathcal{M}_{\le n}}\frac{|b_z(f)|}{q^{\sigma\deg(f)}}\\ &\le \prod_{P\in\mathcal{P}_{\le n}}\left(1+\sum_{k\ge 2}\frac{|b_z(P^k)|}{q^{k\sigma\deg(P)}}\right)\\ &\le \prod_{P\in\mathcal{P}_{\le n}}\left(1+M_A\sum_{k\ge 2}\left(\frac{\sqrt{3/2}}{q^{\sigma\deg(P)}}\right)^k\right)\\ &= \prod_{P\in\mathcal{P}_{\le n}}\left(1+\frac{M_A 3/2}{q^{\sigma\deg(P)}(q^{\sigma\deg(P)}-\sqrt{3/2})}\right).\end{aligned}$$

Taking the logarithm, noticing that $q^{\sigma\deg(P)}-\sqrt{3/2}\ge q^{\sigma\deg(P)}\frac{\sqrt{2}-1}{\sqrt{2}}$ for $q\ge 3$ and $\deg(P)\ge 1$, and applying the Prime Polynomial Theorem [Ros02, Theorem 2.2] that states that there are $\frac{q^n}{n}+O\left(\frac{q^{\frac{n}{2}}}{n}\right)$ monic irreducible polynomials of degree n, we have

$$\sum_{P\in\mathcal{P}_{\le n}}\log\left(1+\frac{M_A 3/2}{q^{\sigma\deg(P)}(q^{\sigma\deg(P)}-\sqrt{3/2})}\right)\ll M_A\sum_{1\le d\le n}\frac{q^{d(1-2\sigma)}}{d}\le\frac{M_A}{q^{2\sigma-1}-1}.$$

The statement follows by applying the exponential function. □

Since $a^{2A+2}<q^{a/3}$ as a approaches infinity, it follows from Lemma 3.1 that

$$\sum_{a\ge 0}\frac{|\mathcal{B}_z(a)|}{q^a}a^{2A+2}<\sum_{a\ge 0}\frac{|\mathcal{B}_z(a)|}{q^{\frac{2a}{3}}}\ll_A 1$$

uniformly for $|z|\le A$. Thus we can apply Theorem 2.1 to $\mathcal{B}(u,z)$.

Before proceeding to the proof of Theorem 1.1, we prove an auxiliary result that will allow us to differentiate inside an error term after applying Theorem 2.1.

Lemma 3.2. *Let $G_n(z), f(z)$ be complex functions, analytic at $z=1$ such that $G_n(\mathbb{R}), f(\mathbb{R})\subset\mathbb{R}$ and*

$$G_n(z)=f(z)n^{\mathrm{Re}(z)-1}+O\left(n^{\mathrm{Re}(z)-2}\right)$$

in a neighborhood of $z=1$. Then, for any small $\varepsilon>0$, we have

$$G_n'(1)=\frac{\partial}{\partial z}(f(z)n^{\mathrm{Re}(z)-1})\Big|_{z=1}+O\left(n^{-1+\varepsilon}\right), \tag{13}$$

$$G_n''(1)=\frac{\partial^2}{\partial z^2}(f(z)n^{\mathrm{Re}(z)-1})\Big|_{z=1}+O\left(n^{-1+\varepsilon}\right). \tag{14}$$

Proof. For ease of notation, we will assume that z is real and $z>1$ to compute the derivative. (This does not restrict the result, since the derivatives at $z=1$ exists by hypothesis.) Write $z=1+\ell$. Then for any $K>0$ there is an $M=M_K$ such that if $n\ge M$,

$$f(z)n^\ell-Kn^{\ell-1}\le G_n(z)\le f(z)n^\ell+Kn^{\ell-1}$$

in a neighborhood of $z=1$. Thus,

$$\left(f(z)n^\ell-Kn^{\ell-1}\right)-\left(f(1)+Kn^{-1}\right)\le G_n(z)-G_n(1)\le\left(f(z)n^\ell+Kn^{\ell-1}\right)-\left(f(1)-Kn^{-1}\right)$$

and

$$\left|\frac{f(z)n^{\ell}-f(1)}{\ell}-\frac{G_n(z)-G_n(1)}{\ell}\right|\leq\frac{K\left(n^{\ell}+1\right)}{\ell n}.$$

By enlarging M if necessary, we can assume that, for $n\geq M$,

$$\frac{K}{nf(z)},\frac{K}{nf(1)}\leq\ell n^{-1+\varepsilon}.$$

We have

$$Kn^{\ell-1}=f(z)n^{\ell}\left(\frac{K}{nf(z)}\right)\leq f(z)n^{\ell}\left(\ell n^{-1+\epsilon}\right)=\ell f(z)n^{\ell-1+\varepsilon},$$

and similarly,

$$Kn^{-1}\leq\ell f(1)n^{-1+\varepsilon}.$$

Thus

$$\left|\frac{f(z)n^{\ell}-f(1)}{\ell}-\frac{G_n(z)-G_n(1)}{\ell}\right|\leq\frac{f(z)n^{\ell}+f(1)}{n^{1-\varepsilon}}\ll\frac{1}{n^{1-\ell-\varepsilon}}.$$

Taking ℓ to be arbitrarily small in the above equation, we obtain

$$G_n'(1)=(f(z)n^{z-1})'|_{z=1}+O\left(\frac{1}{n^{1-\varepsilon}}\right),$$

as desired. This completes the proof of equation (13).

To prove (14) we proceed similarly. Writing $z_1=1+\ell$ and $z_2=z_1+m$ with $z_2>z_1>1$ for ease of notation, we have that for $K>0$ and $M=M_K$ and $n\geq M$,

$$\left|\frac{f(z_1)n^{\ell}-f(1)}{\ell}-\frac{G_n(z_1)-G(1)}{\ell}\right|\leq\frac{K\left(n^{\ell}+1\right)}{\ell n}$$

and

$$\left|\frac{f(z_2)n^{m+\ell}-f(z_1)n^{\ell}}{m}-\frac{G_n(z_2)-G_n(z_1)}{m}\right|\leq\frac{Kn^{\ell}\left(n^{m}+1\right)}{mn}.$$

By combining the above inequalities, we get

$$\begin{aligned}&\left|\frac{\frac{f(z_2)n^{m+\ell}-f(z_1)n^{\ell}}{m}-\frac{f(z_1)n^{\ell}-f(1)}{\ell}}{\ell+m}-\frac{\frac{G(z_2)-G(z_1)}{m}-\frac{G(z_1)-G(1)}{\ell}}{\ell+m}\right|\\&\leq\frac{Kn^{\ell}\left(n^{m}+1\right)}{m(\ell+m)n}+\frac{K(n^{\ell}+1)}{\ell(\ell+m)n}.\end{aligned}$$

As before, by enlarging M if necessary, we obtain

$$\frac{K}{nf(z_1)},\frac{K}{nf(z_2)},\frac{K}{nf(1)}\leq\frac{\ell m(\ell+m)}{n^{1-\varepsilon}}.$$

The above leads to

$$\begin{aligned}&\left|\frac{\frac{f(z_2)n^{m+\ell}-f(z_1)n^{\ell}}{m}-\frac{f(z_1)n^{\ell}-f(1)}{\ell}}{\ell+m}-\frac{\frac{G(z_2)-G(z_1)}{m}-\frac{G(z_1)-G(1)}{\ell}}{\ell+m}\right|\\&\leq\frac{f(z_2)n^{\ell+m}\ell+f(z_1)n^{\ell}\ell}{n^{1-\varepsilon}}+\frac{f(z_1)n^{\ell}m+f(1)m}{n^{1-\varepsilon}}\\&\ll\frac{1}{n^{1-\epsilon-\ell-m}}.\end{aligned}$$

Taking ℓ, m arbitrarily small, we obtain

$$G''(1) = (f(z)n^{z-1})''|_{z=1} + O\left(\frac{1}{n^{1-\varepsilon}}\right).$$

This concludes the proof of (14). ☐

We now have all the elements to proceed with the proof of Theorem 1.1.

Proof of Theorem 1.1. Starting from equation (10), we have

$$\sum_{f \in \mathcal{S}_h} z^{\Omega(f)} u^{\deg(f)} = \mathcal{C}(u,z) = \mathcal{B}(u,z)\mathcal{Z}_q(u)^z.$$

Applying Theorem 2.1, we obtain

$$\sum_{f \in \mathcal{S}_h \cap \mathcal{M}_n} z^{\Omega(f)} = q^n \frac{n^{z-1}}{\Gamma(z)} \mathcal{B}(1/q, z) + O_A\left(q^n n^{\mathrm{Re}(z)-2}\right). \tag{15}$$

Differentiating both sides of (15) with respect to z for z close to 1, and applying (13), we obtain

$$\sum_{f \in \mathcal{S}_h \cap \mathcal{M}_n} \Omega(f) z^{\Omega(f)-1} = \left(\frac{\mathcal{B}(1/q,z)}{\Gamma(z)}\right)' q^n n^{z-1} + \frac{\mathcal{B}(1/q,z)}{\Gamma(z)} q^n \log(n) n^{z-1} + O\left(\frac{q^n}{n^{1-\varepsilon}}\right). \tag{16}$$

Evaluating (16) at $z = 1$ gives

$$\sum_{f \in \mathcal{S}_h \cap \mathcal{M}_n} \Omega(f) = \frac{\frac{\partial}{\partial z}\mathcal{B}(1/q,1)\Gamma(1) - \mathcal{B}(1/q,1)\Gamma'(1)}{\Gamma(1)^2} q^n + \frac{\mathcal{B}(1/q,1)}{\Gamma(1)} q^n \log(n) + O\left(\frac{q^n}{n^{1-\varepsilon}}\right). \tag{17}$$

Notice that $\Gamma(1) = 1$ and $\Gamma'(1) = -\gamma$. We also have that

$$\mathcal{B}(1/q, 1) = \prod_P \left(1 - \frac{1}{|P|^h}\right) = \frac{1}{\zeta_q(h)}.$$

In addition, the logarithmic derivative gives

$$\frac{\frac{\partial}{\partial z}\mathcal{B}(1/q,z)}{\mathcal{B}(1/q,z)} = \sum_P \left(\log\left(1 - \frac{1}{|P|}\right) + \frac{1}{|P| - z} - \frac{hz^{h-1}}{|P|^h - z^h}\right).$$

Applying the above identities to (17), we obtain the proof of (2).

We now proceed to prove (3). Multiplying (16) by z, differentiating both sides with respect to z for z close to 1, and applying (14), we obtain

$$\begin{aligned}\sum_{f \in \mathcal{S}_h \cap \mathcal{M}_n} \Omega(f)^2 z^{\Omega(f)-1} = {} & \left(\frac{\mathcal{B}(1/q,z)}{\Gamma(z)}\right)'' q^n z n^{z-1} + \left(\frac{\mathcal{B}(1/q,z)}{\Gamma(z)}\right)' q^n (n^{z-1} + 2zn^{z-1}\log(n)) \\ & + \frac{\mathcal{B}(1/q,z)}{\Gamma(z)} q^n \log(n)(n^{z-1} + zn^{z-1}\log(n)) \\ & + O\left(\frac{q^n}{n^{1-\varepsilon}}\right).\end{aligned} \tag{18}$$

We remark that

$$\left(\frac{\mathcal{B}(1/q,z)}{\Gamma(z)}\right)''=\left(\frac{\frac{\partial}{\partial z}\mathcal{B}(1/q,z)\Gamma(z)-\mathcal{B}(1/q,z)\Gamma'(z)}{\Gamma(z)^2}\right)'$$
$$=\frac{\frac{\partial^2}{\partial z^2}\mathcal{B}(1/q,z)\Gamma(z)^2-2\frac{\partial}{\partial z}\mathcal{B}(1/q,z)\Gamma(z)\Gamma'(z)+\mathcal{B}(1/q,z)(2\Gamma'(z)^2-\Gamma(z)\Gamma''(z))}{\Gamma(z)^3}.$$

In addition, the second derivative of $\mathcal{B}(1/q,z)$ gives

$$\frac{\partial^2}{\partial z^2}\mathcal{B}(1/q,z)=\frac{\partial}{\partial z}\mathcal{B}(1/q,z)\sum_P\left(\log\left(1-\frac{1}{|P|}\right)+\frac{1}{|P|-z}-\frac{hz^{h-1}}{|P|^h-z^h}\right)$$
$$+\mathcal{B}(1/q,z)\sum_P\left(\frac{1}{(|P|-z)^2}-\frac{h(h-1)z^{h-2}}{|P|^h-z^h}-\frac{h^2z^{2h-2}}{(|P|^h-z^h)^2}\right).$$

Evaluating the above at $z=1$, we obtain

$$\frac{\partial^2}{\partial z^2}\mathcal{B}(1/q,1)=\frac{1}{\zeta_q(h)}\left(\sum_P\left(\log\left(1-\frac{1}{|P|}\right)+\frac{1}{|P|-1}-\frac{h}{|P|^h-1}\right)\right)^2$$
$$+\frac{1}{\zeta_q(h)}\sum_P\left(\frac{1}{(|P|-1)^2}-\frac{h(h-1)}{|P|^h-1}-\frac{h^2}{(|P|^h-1)^2}\right). \tag{19}$$

Evaluating (18) at $z=1$ gives

$$\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n}\Omega(f)^2=\frac{\frac{\partial^2}{\partial z^2}\mathcal{B}(1/q,1)\Gamma(1)^2-2\frac{\partial}{\partial z}\mathcal{B}(1/q,1)\Gamma(1)\Gamma'(1)+\mathcal{B}(1/q,1)(2\Gamma'(1)^2-\Gamma(1)\Gamma''(1))}{\Gamma(1)^3}q^n$$
$$+\frac{\frac{\partial}{\partial z}\mathcal{B}(1/q,1)\Gamma(1)-\mathcal{B}(1/q,1)\Gamma'(1)}{\Gamma(1)^2}q^n(1+2\log(n))$$
$$+\frac{\mathcal{B}(1/q,1)}{\Gamma(1)}q^n\log(n)(1+\log(n))+O\left(\frac{q^n}{n^{1-\varepsilon}}\right).$$

Replacing (19) in the above equation, and using the fact that $\Gamma''(1) = \gamma^2 + \zeta(2)$, we obtain

$$\begin{aligned}\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n} \Omega(f)^2 =&\frac{q^n(\log(n))^2}{\zeta_q(h)}\\ &+\frac{q^n(\log(n))}{\zeta_q(h)}\left(2\sum_P\left(\log\left(1-\frac{1}{|P|}\right)+\frac{1}{|P|-1}-\frac{h}{|P|^h-1}\right)+2\gamma+1\right)\\ &+\frac{q^n}{\zeta_q(h)}\left(\sum_P\left(\log\left(1-\frac{1}{|P|}\right)+\frac{1}{|P|-1}-\frac{h}{|P|^h-1}\right)\right)^2\\ &+\frac{q^n}{\zeta_q(h)}\sum_P\left(\frac{1}{(|P|-1)^2}-\frac{h(h-1)}{|P|^h-1}-\frac{h^2}{(|P|^h-1)^2}\right)\\ &+\frac{2\gamma+1}{\zeta_q(h)}q^n\sum_P\left(\log\left(1-\frac{1}{|P|}\right)+\frac{1}{|P|-1}-\frac{h}{|P|^h-1}\right)+\frac{\gamma^2+\gamma-\zeta(2)}{\zeta_q(h)}q^n\\ &+O\left(\frac{q^n}{n^{1-\varepsilon}}\right).\end{aligned}$$

□

We can compute the variance from Theorem 1.1. Before doing this, we need the following auxiliary result.

Lemma 3.3. *We have*

$$\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n} 1 = \frac{q^n}{\zeta_q(h)}.$$

Proof. Recall from (9) that

$$\sum_{f\in\mathcal{S}_h} u^{\deg(f)} = \frac{\mathcal{Z}_q(u)}{\mathcal{Z}_q(u^h)}.$$

By Perron's formula (Lemma 2.1), for $0<\delta<1/q$, we have

$$\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n} 1 =\frac{1}{2\pi i}\oint_{|u|=\delta}\frac{1-qu^h}{1-qu}\frac{du}{u^{n+1}} = -\mathrm{Res}_{u=1/q}\frac{1-qu^h}{u^{n+1}(1-qu)} = \frac{q^n}{\zeta_q(h)}.$$

□

Proof of Theorem 1.2. By combining the results of Theorem 1.1 with Lemma 3.3, we have

$$\begin{aligned}\mathbb{E}_{h,n}(\Omega^2)-(\mathbb{E}_{h,n}(\Omega))^2 =&\frac{1}{|\mathcal{S}_h\cap\mathcal{M}_n|}\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n}\Omega(f)^2-\left(\frac{1}{|\mathcal{S}_h\cap\mathcal{M}_n|}\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n}\Omega(f)\right)^2\\ =&\log(n)+\sum_P\left(\frac{1}{(|P|-1)^2}-\frac{h(h-1)}{|P|^h-1}-\frac{h^2}{(|P|^h-1)^2}\right)+(\gamma-\zeta(2))\\ &+\sum_P\left(\log\left(1-\frac{1}{|P|}\right)+\frac{1}{|P|-1}-\frac{h}{|P|^h-1}\right)+O\left(\frac{1}{n^{1-\varepsilon}}\right).\end{aligned}$$

□

By taking the limit as h goes to infinity, we recover results without the h-free condition.

Corollary 3.1. *As $h \to \infty$, we have*

$$\begin{aligned}\sum_{f\in\mathcal{M}_n}\Omega(f) =& q^n\log(n)+q^nB_2+O\left(\frac{q^n}{n^{1-\varepsilon}}\right),\\ \sum_{f\in\mathcal{M}_n}\Omega(f)^2 =& q^n(\log(n))^2+q^n(\log(n))\left(2B_2+1\right)\\ &+q^n\left(B_2^2+B_2-\zeta(2)+\sum_P\frac{1}{(|P|-1)^2}\right)+O\left(\frac{q^n}{n^{1-\varepsilon}}\right),\\ \mathrm{Var}_n(\Omega) =& \log(n)+B_2-\zeta(2)+\sum_P\frac{1}{(|P|-1)^2}+O\left(\frac{1}{n^{1-\varepsilon}}\right).\end{aligned}$$

3.1. **Higher moments of Ω and an Erdős–Kac result for h-free polynomials.** The goal of this section is to prove Theorem 1.3, namely the Erdős–Kac theorem for Ω over the h-free polynomials. To this end we will prove that as $n \to \infty$,

$$\frac{1}{|\mathcal{S}_h\cap\mathcal{M}_n|}\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n}\left(\frac{\Omega(f)-\log(n)}{\sqrt{\log(n)}}\right)^v \to C_v,$$

where

$$C_v=\begin{cases}\frac{v!}{2^{\frac{v}{2}}\left(\frac{v}{2}\right)!} & v \text{ even},\\ 0 & v\text{ odd}.\end{cases} \tag{20}$$

Notice that

$$\frac{1}{|\mathcal{S}_h\cap\mathcal{M}_n|}\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n}\left(\frac{\Omega(f)-\log(n)}{\sqrt{\log(n)}}\right)^v=\frac{1}{\log(n)^{\frac{v}{2}}}\sum_{\ell=0}^{v}\binom{v}{\ell}\mathbb{E}(\Omega^\ell)(-1)^{v-\ell}\log(n)^{v-\ell}.$$

Consider again the moment generating function (10)

$$\mathcal{C}(u,e^t):=\prod_P\left(1+e^tu^{\deg(P)}+\cdots+e^{(h-1)t}u^{(h-1)\deg(P)}\right)=\sum_{f\in\mathcal{S}_h\cap\mathcal{M}}e^{\Omega(f)t}u^{\deg(f)}.$$

After extracting the singularity at $u=\frac{1}{q}$, we obtain

$$\begin{aligned}\mathcal{B}(u,e^t):=&\mathcal{Z}_q(u)^{-e^t}\mathcal{C}(u,e^t)\\ =&\prod_P\left(1-u^{\deg(P)}\right)^{e^t}\left(\frac{1-e^{ht}u^{h\deg(P)}}{1-e^tu^{\deg(P)}}\right).\end{aligned}$$

Applying Theorem 2.1, we have

$$\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n}e^{\Omega(f)t}=q^nn^{e^t-1}\frac{\mathcal{B}(1/q,e^t)}{\Gamma(e^t)}+O_A\left(q^nn^{\mathrm{Re}(e^t)-2}\right).$$

By the property of the moment generating function, we have

$$\mathbb{E}\left(\Omega^\ell\right)=\mathbb{E}\left(e^{\Omega t}\right)^{(\ell)}|_{t=0}. \tag{21}$$

and therefore

$$\mathbb{E}\left(\Omega^\ell\right) = \zeta_q(h) \sum_{j=0}^{\ell} \binom{\ell}{j} (n^{e^t-1})^{(j)} \left(\frac{\mathcal{B}(1/q,e^t)}{\Gamma(e^t)}\right)^{(\ell-j)}\Bigg|_{t=0} + O\left(\frac{1}{n^{1-\varepsilon}}\right). \tag{22}$$

We claim that for $j \geq 1$,

$$(n^{e^t-1})^{(j)} = n^{e^t-1} \sum_{m=1}^{j} \left\{ {j \atop m} \right\} (e^t \log(n))^m = n^{e^t-1} T_j(e^t \log(n)), \tag{23}$$

where the $\left\{ {j \atop m} \right\}$ are the Stirling numbers of the second kind, and the T_j are the Touchard polynomials [Tou39]. Indeed, proceeding by induction, when $j = 1$, both sides of (23) become $n^{e^t-1} e^t \log(n)$. Now assume that (23) is true for j. Differentiating, we have

$$(n^{e^t-1})^{(j+1)} = n^{e^t-1} \sum_{m=1}^{j} \left\{ {j \atop m} \right\} m (e^t \log(n))^m + n^{e^t-1} \sum_{m=1}^{j} \left\{ {j \atop m} \right\} (e^t \log(n))^{m+1},$$

and the induction step follows from the fact that $m\left\{ {j \atop m} \right\} + \left\{ {j \atop m-1} \right\} = \left\{ {j+1 \atop m} \right\}$.

Combining (23) with (22), we have

$$\mathbb{E}(\Omega^\ell) = \zeta_q(h) \sum_{j=0}^{\ell} \binom{\ell}{j} T_j(\log(n)) \left(\frac{\mathcal{B}(1/q,e^t)}{\Gamma(e^t)}\right)^{(\ell-j)}\Bigg|_{t=0} + O\left(\frac{1}{n^{1-\varepsilon}}\right)$$

and

$$\begin{aligned} &\frac{1}{\log(n)^{\frac{v}{2}}} \sum_{\ell=0}^{v} \binom{v}{\ell} \mathbb{E}(\Omega^\ell)(-1)^{v-\ell} \log(n)^{v-\ell} \\ &= \frac{\zeta_q(h)}{\log(n)^{\frac{v}{2}}} \sum_{\ell=0}^{v} \binom{v}{\ell} \sum_{j=0}^{\ell} \binom{\ell}{j} T_j(\log(n)) \left(\frac{\mathcal{B}(1/q,e^t)}{\Gamma(e^t)}\right)^{(\ell-j)}\Bigg|_{t=0} (-1)^{v-\ell} \log(n)^{v-\ell} + O\left(\frac{1}{n^{1-\varepsilon}}\right). \end{aligned} \tag{24}$$

Consider the change of variables $u = v - \ell$, $k = v - \ell + j$. Then the main term in (24) becomes

$$\begin{aligned} &\frac{\zeta_q(h)}{\log(n)^{\frac{v}{2}}} \sum_{k=0}^{v} \left(\frac{\mathcal{B}(1/q,e^t)}{\Gamma(e^t)}\right)^{(v-k)}\Bigg|_{t=0} \sum_{u=0}^{k} \binom{v}{u}\binom{v-u}{k-u} T_{k-u}(\log(n))(-1)^u \log(n)^u \\ &= \frac{\zeta_q(h)}{\log(n)^{\frac{v}{2}}} \sum_{k=0}^{v} \binom{v}{k} \left(\frac{\mathcal{B}(1/q,e^t)}{\Gamma(e^t)}\right)^{(v-k)}\Bigg|_{t=0} \sum_{u=0}^{k} \binom{k}{u} T_{k-u}(\log(n))(-1)^u \log(n)^u. \end{aligned} \tag{25}$$

Since the generating function for the Touchard polynomials [Tou39] is

$$e^{x(e^t-1)} = \sum_{k=0}^{\infty} \frac{T_k(x)}{k!} t^k,$$

one can see that the generating function for the inner sum in (25) is given by

$$e^{x(e^t-1-t)} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{u=0}^{k} \binom{k}{u} T_{k-u}(x)(-1)^u x^u.$$

Notice that the coefficient of x^u in the power series of $e^{x(e^t-1-t)}$ is given by $\frac{(e^t-1-t)^u}{u!}$, whose lowest power of t is t^{2u}. Therefore, thinking of $e^{x(e^t-1-t)}$ as a power series in t, we have that the coefficient of t^v is a polynomial in x of degree at most $\lfloor \frac{v}{2} \rfloor$.

Now suppose that v is even. Then the coefficient of $x^{\frac{v}{2}}t^v$ in $e^{x(e^t-1-t)}$ is given by $\frac{1}{2^{\frac{v}{2}}\left(\frac{v}{2}\right)!}$. Back to the inner sum in (25), this gives a leading coefficient of $\frac{k!}{2^{\frac{k}{2}}\left(\frac{k}{2}\right)!}$ for $\log(n)^{\frac{k}{2}}$ when k is even. Incorporating this information in (25), we get, for v even,

$$\begin{aligned}
&\frac{1}{\log(n)^{\frac{v}{2}}}\sum_{\ell=0}^{v}\binom{v}{\ell}\mathbb{E}(\Omega(f)^\ell)(-1)^{v-\ell}\log(n)^{v-\ell}\\
&=\frac{\zeta_q(h)}{\log(n)^{\frac{v}{2}}}\frac{\mathcal{B}(1/q,1)}{\Gamma(1)}\sum_{u=0}^{v}\binom{v}{u}T_{v-u}(\log(n))(-1)^u\log(n)^u+O\left(\frac{1}{\log n}\right)\\
&=\frac{v!}{2^{\frac{v}{2}}\left(\frac{v}{2}\right)!}+O\left(\frac{1}{\log n}\right),
\end{aligned}$$

while for v odd we get

$$O\left(\frac{1}{\sqrt{\log n}}\right),$$

as desired.

4. The distribution of ω over h-free polynomials

In this short section we state, without proof, analogous results for ω, which can be proven using the same techniques as the results for Ω.

Theorem 4.1. *For any $\varepsilon > 0$ and as $n \to \infty$, the first moment of ω over the h-free polynomials of degree n is given by*

$$\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n}\omega(f)=\frac{q^n\log(n)}{\zeta_q(h)}+\frac{q^n}{\zeta_q(h)}\left(B_1-\sum_P\frac{|P|-1}{|P|(|P|^h-1)}\right)+O\left(\frac{q^n}{n^{1-\varepsilon}}\right),$$

while the second moment is given by

$$\begin{aligned}
\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n}\omega(f)^2=&\frac{q^n(\log(n))^2}{\zeta_q(h)}+\frac{q^n(\log(n))}{\zeta_q(h)}\left[2\left(B_1-\sum_P\frac{|P|-1}{|P|(|P|^h-1)}\right)+1\right]\\
&+\frac{q^n}{\zeta_q(h)}\left[B_1^2+B_1-\zeta(2)-\sum_P\left(\frac{|P|^{h-1}-1}{|P|^h-1}\right)^2+\left(\sum_P\frac{|P|-1}{|P|(|P|^h-1)}\right)^2\right.\\
&\left.-(2B_1+1)\sum_P\frac{|P|-1}{|P|(|P|^h-1)}\right]+O\left(\frac{q^n}{n^{1-\varepsilon}}\right).
\end{aligned}$$

Finally, the variance is given by

$$\mathrm{Var}_{h-\mathrm{free},n}(\omega)=\log(n)+B_1-\zeta(2)-\sum_P\left(\frac{|P|^{h-1}-1}{|P|^h-1}\right)^2-\sum_P\frac{|P|-1}{|P|(|P|^h-1)}+O\left(\frac{1}{n^{1-\varepsilon}}\right).$$

To prove the above results, one works with the Euler product

$$\begin{aligned}\mathcal{B}(u,z) &:= \prod_P \mathcal{Z}_q(u)^{-z}\left(1+z\left(u^{\deg(P)}+\cdots+u^{(h-1)\deg(P)}\right)\right)\\ &= \prod_P \left(1-u^{\deg(P)}\right)^z\left(1+z\frac{u^{\deg(P)}-u^{h\deg(P)}}{1-u^{\deg(P)}}\right).\end{aligned}$$

Moreover, working with $\mathcal{B}(u,e^t)$, we also obtain

Theorem 4.2. *As $n\to\infty$, $\omega(f)$ with $f\in\mathcal{S}_h\cap\mathcal{M}_n$ approaches a normal distribution, namely, for $\alpha\le\beta$,*

$$\left|\left\{f\in\mathcal{S}_h\cap\mathcal{M}_n : \alpha\le\frac{\omega(f)-\log(n)}{\sqrt{\log(n)}}\le\beta\right\}\right|\to\frac{1}{\sqrt{2\pi}}\int_\alpha^\beta e^{-t^2/2}dt.$$

5. The distribution of Ω over h-full polynomials

Recall that h is a fixed positive integer and that we denote by $\mathcal{N}_h$ the set of h-full monic polynomials. Let $f\in\mathcal{N}_h$. We proceed to regroup the primes in the factorization of f according to the congruence modulo h of their exponent. More precisely, we write

$$f=\underbrace{P_{1,0}^{ht_{1,0}}\cdots P_{\ell_0,0}^{ht_{\ell_0,0}}}_{\exp\equiv 0(\bmod h)}\underbrace{P_{1,1}^{ht_{1,1}+1}\cdots P_{\ell_1,1}^{ht_{\ell_1,1}+1}}_{\exp\equiv 1(\bmod h)}\cdots\underbrace{P_{1,h-1}^{ht_{1,h-1}+h-1}\cdots P_{\ell_{h-1},h-1}^{ht_{\ell_{h-1},h-1}+h-1}}_{\exp\equiv h-1(\bmod h)}.$$

By setting

$$\begin{aligned}m &= P_{1,0}^{t_{1,0}}\cdots P_{\ell_0,0}^{t_{\ell_0,0}}P_{1,1}^{t_{1,1}-1}\cdots P_{\ell_1,1}^{t_{\ell_1,1}-1}\cdots P_{1,h-1}^{t_{1,h-1}-1}\cdots P_{\ell_{h-1},h-1}^{t_{\ell_{h-1},h-1}-1},\\ r_j &= P_{1,j}\cdots P_{\ell_j,j},\qquad (j=1,\dots,h-1),\end{aligned}$$

we arrive at

$$f=m^h r_1^{h+1}\cdots r_{h-1}^{2h-1},$$

and we remark that the r_j are square-free and coprime in pairs.

5.1. The first moment.

After the above observation, the first moment over the h-full polynomials is given by

$$\sum_{f\in\mathcal{N}_h\cap\mathcal{M}_n}\Omega(f)=\sum_{m^h r_1^{h+1}\cdots r_{h-1}^{2h-1}\in\mathcal{M}_n}\mu^2(r_1\cdots r_{h-1})\Omega(m^h r_1^{h+1}\cdots r_{h-1}^{2h-1}).\tag{26}$$

Since Ω is completely additive and the r_j are square-free, we have that

$$\begin{aligned}\Omega(m^h r_1^{h+1}\cdots r_{h-1}^{2h-1}) &= h\Omega(m)+(h+1)\Omega(r_1)+\cdots+(2h-1)\Omega(r_{h-1})\\ &= h\Omega(m)+(h+1)\omega(r_1)+\cdots+(2h-1)\omega(r_{h-1}).\end{aligned}$$

Using this in (26), we obtain

$$\begin{aligned}\sum_{f\in\mathcal{N}_h\cap\mathcal{M}_n}\Omega(f) =& h\sum_{\substack{m,r_1,\dots,r_{h-1}\in\mathcal{M}\\ \deg(m^hr_1^{h+1}\cdots r_{h-1}^{2h-1})=n}}\mu^2(r_1\cdots r_{h-1})\Omega(m)\\ &+(h+1)\sum_{\substack{m,r_1,r_2,\dots,r_{h-1}\in\mathcal{M}\\ \deg(m^hr_2^{h+2}\cdots r_{h-1}^{2h-1})=n}}\mu^2(r_1\cdots r_{h-1})\omega(r_1)\\ &\vdots\\ &+(2h-1)\sum_{\substack{m,r_1,\dots,r_{h-1}\in\mathcal{M}\\ \deg(m^hr_1^{h+1}\cdots r_{h-2}^{2h-2})=n}}\mu^2(r_1\cdots r_{h-1})\omega(r_{h-1})\\ &= hT_0(n)+(h+1)T_1(n)+\cdots+(2h-1)T_{h-1}(n). \end{aligned} \tag{27}$$

Eventually we will see that $T_0(n)$ gives the main term, while the $T_j(n)$ give secondary terms.

In order to estimate the $T_j(n)$, as in the case of the h-free polynomials, we construct a generating function which we will later differentiate using Theorem 2.2.

5.1.1. *The sum over* $\Omega(m)$*.* We now consider a generating function for $T_0(n)$. Define

$$\begin{aligned}\mathcal{T}_0(u,z) =:& \sum_{m,r_1,\dots,r_{h-1}\in\mathcal{M}}\mu^2(r_1\cdots r_{h-1})z^{\Omega(m)}u^{\deg(m^hr_1^{h+1}\cdots r_{h-1}^{2h-1})}\\ =&\prod_P\left(1-zu^{\deg(P)h}\right)^{-1}\left(1+u^{\deg(P)(h+1)}+u^{\deg(P)(h+2)}+\cdots+u^{\deg(P)(2h-1)}\right).\end{aligned}$$

We remark that $\frac{\partial}{\partial z}\mathcal{T}_0(u,1)$ yields a generating series for T_0.

As usual we extract the singularities at $u^h=\frac{1}{q}$ and consider

$$\begin{aligned}\mathcal{B}_0(u,z) :=& \mathcal{Z}_q(u^h)^{-z}\mathcal{T}_0(u,z)\\ =&\prod_P\frac{\left(1-u^{\deg(P)h}\right)^z}{1-zu^{\deg(P)h}}\left(1+u^{\deg(P)(h+1)}+u^{\deg(P)(h+2)}+\cdots+u^{\deg(P)(2h-1)}\right).\end{aligned}\tag{28}$$

Our goal is to apply Theorem 2.2 to find the coefficients of $\mathcal{T}_0(u,z)$. Before doing that, we must verify that we can, indeed, apply the result.

Lemma 5.1. *Set $A<\frac{3}{2}$. For $|z|\le A$, $n\ge 2$, and $\sigma>\frac{1}{h+1}$,*

$$\sum_{0\le a\le n}\frac{|\mathcal{B}_{0,z}(a)|}{q^{\sigma a}}\le c_{A,\sigma},$$

where $c_{A,\sigma}$ is a constant depending on A and σ.

Proof. Our proof follows very similar steps to those in the proof of [AP19, Proposition 2.5] and Lemma 3.1. Let $b_z(f)$ be a multiplicative function on $\mathcal{M}$ defined for prime powers by

$$1+\sum_{k\ge1}b_z(P^k)u^k=\left(1+u^{h+1}+\cdots+u^{2h-1}\right)\frac{(1-u^h)^z}{1-zu^h}.$$

Then $\mathcal{B}_{0,z}(n)=\sum_{f\in\mathcal{M}_n} b_z(f)$. We see that $b_z(P^k)=0$ for $k\le h$. By Cauchy's formula integrating over $|u|=\left(\frac{2}{3}\right)^{\frac{1}{h}}$,

$$|b_z(P^k)|\le\left(\frac{3}{2}\right)^{\frac{k}{h}}M_A$$

for $k\ge h+1$, where

$$M_A:=\sup_{|z|\le A,|u|\le\left(\frac{2}{3}\right)^{\frac{1}{h}}}\left|\left(1+u^{h+1}+\cdots+u^{2h-1}\right)\frac{(1-u^h)^z}{1-zu^h}\right|$$

is a constant dependent on A. Therefore,

$$\begin{aligned}\sum_{0\le a\le n}\frac{\mathcal{B}_{0,z}(a)}{q^{a\sigma}}&\le\sum_{f\in\mathcal{M}_{\le n}}\frac{|b_z(f)|}{q^{\sigma\deg(f)}}\\&\le\prod_{P\in\mathcal{P}_{\le\frac{n}{h}}}\left(1+\sum_{k\ge h+1}\frac{|b_z(P^k)|}{q^{\sigma\deg(P^k)}}\right)\\&\le\prod_{P\in\mathcal{P}_{\le\frac{n}{h}}}\left(1+M_A\sum_{k\ge h+1}\left(\frac{(3/2)^{\frac{1}{h}}}{q^{\sigma\deg(P)}}\right)^k\right)\\&=\prod_{P\in\mathcal{P}_{\le\frac{n}{h}}}\left(1+\frac{M_A\,(3/2)^{\frac{h+1}{h}}}{q^{h\sigma\deg(P)}(q^{\sigma\deg(P)}-(3/2)^{\frac{1}{h}})}\right).\end{aligned}$$

Taking the logarithm, noticing that $q^{\sigma\deg(P)}-\left(\frac{3}{2}\right)^{\frac{1}{h}}\ge q^{\sigma\deg(P)}\left(1-\frac{(3/2)^{\frac{1}{h}}}{q^{\frac{1}{h+1}}}\right)$ for $\deg(P)\ge1$, and applying the Prime Polynomial Theorem [Ros02, Theorem 2.2], we have

$$\begin{aligned}\sum_{P\in\mathcal{P}_{\le\frac{n}{h}}}\log\left(1+\frac{M_A(3/2)^{\frac{h+1}{h}}}{q^{h\sigma\deg(P)}(q^{\sigma\deg(P)}-(3/2)^{\frac{1}{h}})}\right)&\ll M_A\sum_{1\le d\le\frac{n}{h}}\frac{q^{d(1-(h+1)\sigma)}}{d}\\&\le\frac{M_A}{q^{(h+1)\sigma-1}-1}.\end{aligned}$$

The statement follows by applying the exponential function. □

As in the h-free case, one can deduce from Lemma 5.1 that Theorem 2.2 can be applied to $\mathcal{B}_0$ in a neighborhood of $z=1$ to find T_0. Differentiating and evaluating $\mathcal{T}_0$ at $z=1$, we get

$$\begin{aligned}T_0(n)=&\sum_{\substack{m,r_1,\ldots,r_{h-1}\in\mathcal{M}\\\deg(m^hr_1^{h+1}\cdots r_{h-1}^{2h-1})=n}}\mu^2(r_1\cdots r_{h-1})\Omega(m)\\=&\frac{\partial}{\partial z}\left(\frac{q^{\frac{n}{h}}n^{z-1}}{h^z}\sum_{k=0}^{h-1}\frac{\xi_h^{kn}}{\Gamma(z)}\mathcal{B}_0\left((q^{\frac{1}{h}}\xi_h^k)^{-1},z\right)\right)\Bigg|_{z=1}+O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right)\\=&\frac{q^{\frac{n}{h}}}{h}\log\left(\frac{n}{h}\right)\sum_{k=0}^{h-1}\xi_h^{kn}\mathcal{B}_0\left((q^{\frac{1}{h}}\xi_h^k)^{-1},1\right)+\frac{q^{\frac{n}{h}}}{h}\sum_{k=0}^{h-1}\frac{\partial}{\partial z}\left(\frac{\xi_h^{kn}}{\Gamma(z)}\mathcal{B}_0\left((q^{\frac{1}{h}}\xi_h^k)^{-1},z\right)\right)\Bigg|_{z=1}+O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right),\end{aligned}$$

where we have applied (13) to compute the error term.

We remark that

$$\sum_{k=0}^{h-1} \xi_h^{kn} \mathcal{B}_0\left((q^{\frac{1}{h}}\xi_h^k)^{-1}, 1\right) = \sum_{k=0}^{h-1} \xi_h^{kn} \prod_P \left(1 + (q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)} + \cdots + (q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right), \tag{29}$$

while

$$\begin{aligned}\sum_{k=0}^{h-1} \frac{\partial}{\partial z}\left(\frac{\xi_h^{kn}}{\Gamma(z)} \mathcal{B}_0\left((q^{\frac{1}{h}}\xi_h^k)^{-1}, z\right)\right)\Big|_{z=1} &= \sum_{k=0}^{h-1} \xi_h^{kn} \frac{\frac{\partial}{\partial z}\mathcal{B}_0\left((q^{\frac{1}{h}}\xi_h^k)^{-1}, z\right)\Big|_{z=1}\Gamma(1) - \mathcal{B}_0\left((q^{\frac{1}{h}}\xi_h^k)^{-1}, 1\right)\Gamma'(1)}{\Gamma(1)^2}\\ &= \sum_{k=0}^{h-1} \xi_h^{kn}\left(\frac{\partial}{\partial z}\mathcal{B}_0\left((q^{\frac{1}{h}}\xi_h^k)^{-1}, z\right)\Big|_{z=1} + \mathcal{B}_0\left((q^{\frac{1}{h}}\xi_h^k)^{-1}, 1\right)\gamma\right). \end{aligned}\tag{30}$$

Now the logarithmic derivative gives

$$\frac{\frac{\partial}{\partial z}\mathcal{B}_0\left((q^{\frac{1}{h}}\xi_h^k)^{-1}, z\right)\Big|_{z=1}}{\mathcal{B}_0\left((q^{\frac{1}{h}}\xi_h^k)^{-1}, 1\right)} = \sum_P \left(\log\left(1 - \frac{1}{|P|}\right) + \frac{1}{|P|-1}\right) = B_2 - \gamma. \tag{31}$$

Combining all of the above, we obtain,

$$\begin{aligned}T_0(n) =& \frac{q^{\frac{n}{h}}}{h}\left(\log\left(\frac{n}{h}\right) + B_2\right) \sum_{k=0}^{h-1} \xi_h^{kn} \prod_P \left(1 + (q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)} + \cdots + (q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right)\\ &+ O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right). \end{aligned}\tag{32}$$

5.1.2. *The sums over $\omega(r_j)$.* We proceed similarly to calculate the $T_j(n)$ in (27). To this end, we consider the function

$$\begin{aligned}\mathcal{T}_j(u, z) =:& \sum_{m, r_1, \dots, r_{h-1} \in \mathcal{M}} \mu^2(r_1 \cdots r_{h-1}) z^{\omega(r_j)} u^{\deg(m^h r_1^{h+1} \cdots r_{h-1}^{2h-1})}\\ =& \prod_P \left(1 - u^{\deg(P)h}\right)^{-1} \left(1 + u^{\deg(P)(h+1)} + \cdots + z u^{\deg(P)(h+j)} + \cdots + u^{\deg(P)(2h-1)}\right)\\ =& \mathcal{Z}_q(u^h)\mathcal{B}_j(u, z), \end{aligned}\tag{33}$$

and we remark that $\frac{\partial}{\partial z}\mathcal{T}_j(u, 1)$ yields the generating series for T_j. Also remark that $\mathcal{B}_j(u, z)$ is absolutely convergent for $|z| \leq A$ (for any $A > 0$) and $|u| < q^{-\frac{1}{h+1}}$.

By applying Perron's formula (Lemma 2.1), we have

$$\sum_{\substack{m, r_1, \dots, r_{h-1} \in \mathcal{M}\\ \deg(m^h r_1^{h+1} \cdots r_{h-1}^{2h-1}) = n}} \mu^2(r_1 \cdots r_{h-1}) z^{\omega(r_j)} = \frac{1}{2\pi i} \oint_{|u|=\delta} \frac{\mathcal{B}_j(u, z)}{1 - qu^h} \frac{du}{u^{n+1}},$$

where $\delta > 0$ small. We move the above integral to the circle $|u| = q^{-\frac{\varepsilon}{n}-\frac{1}{h+1}}$ and we obtain the residues at the poles $u = (q^{\frac{1}{h}}\xi_h^j)^{-1}$. Thus we get

$$\begin{aligned}
&\sum_{\substack{m,r_1,\dots,r_{h-1}\in\mathcal{M}\\ \deg(m^h r_1^{h+1}\cdots r_{h-1}^{2h-1})=n}} \mu^2(r_1\cdots r_{h-1})z^{\omega(r_j)}\\
&= -\sum_{k=0}^{h-1} \operatorname{Res}_{u=(q^{\frac{1}{h}}\xi_h^k)^{-1}} \frac{\mathcal{B}_j(u,z)}{u^{n+1}(1-qu^h)} + \frac{1}{2\pi i}\oint_{|u|=q^{-\frac{\varepsilon}{n}-\frac{1}{h+1}}} \frac{\mathcal{B}_j(u,z)}{1-qu^h}\frac{du}{u^{n+1}}\\
&= \sum_{k=0}^{h-1} (q^{\frac{1}{h}}\xi_h^k)^{n+1}\mathcal{B}_j\left((q^{\frac{1}{h}}\xi_h^k)^{-1}, z\right)\frac{1}{q^{\frac{1}{h}}\xi_h^k\prod_{m\neq k}\left(1-\xi_h^{m-k}\right)} + O_z(1)O\left(q^{\frac{n}{h+1}+\varepsilon}\right)\\
&= \frac{q^{\frac{n}{h}}}{h}\sum_{k=0}^{h-1}\xi_h^{kn}\mathcal{B}_j\left((q^{\frac{1}{h}}\xi_h^k)^{-1}, z\right) + O_z(1)O\left(q^{\frac{n}{h+1}+\varepsilon}\right).
\end{aligned}$$

Now we have

$$T_j(n) = \sum_{\substack{m,r_1,\dots,r_{h-1}\in\mathcal{M}\\ \deg(m^h r_1^{h+1}\cdots r_{h-1}^{2h-1})=n}} \mu^2(r_1\cdots r_{h-1})\omega(r_j) = \frac{q^{\frac{n}{h}}}{h}\sum_{k=0}^{h-1}\xi_h^{kn}\frac{\partial}{\partial z}\mathcal{B}_j\left((q^{\frac{1}{h}}\xi_h^k)^{-1}, z\right)\Big|_{z=1} + O\left(q^{\frac{n}{h+1}+\varepsilon}\right).$$

Computing the logarithmic derivative,

$$\frac{\frac{\partial}{\partial z}\mathcal{B}_j\left((q^{\frac{1}{h}}\xi_h^k)^{-1}, z\right)\Big|_{z=1}}{\mathcal{B}_j\left((q^{\frac{1}{h}}\xi_h^k)^{-1}, 1\right)} = \sum_P \frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+j)}}{1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}}.$$

In sum, we get

$$\begin{aligned}
T_j(n) =& \frac{q^{\frac{n}{h}}}{h}\sum_{k=0}^{h-1}\xi_h^{kn}\prod_P\left(1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right)\\
&\times\sum_P \frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+j)}}{1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}} + O\left(q^{\frac{n}{h+1}+\varepsilon}\right).
\end{aligned} \tag{34}$$

5.1.3. *Conclusion of the first moment computation.* Finally, by combining (32) and (34) with (27), we obtain

$$\begin{aligned}
&\sum_{f\in\mathcal{N}_h\cap\mathcal{M}_n}\Omega(f)\\
&= q^{\frac{n}{h}}\left(\log\left(\frac{n}{h}\right)+B_2\right)\sum_{k=0}^{h-1}\xi_h^{kn}\prod_P\left(1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right)\\
&+\frac{q^{\frac{n}{h}}}{h}\sum_{j=1}^{h-1}(h+j)\sum_{k=0}^{h-1}\xi_h^{kn}\prod_P\left(1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right)\\
&\times\sum_P\frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+j)}}{1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}} + O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right).
\end{aligned}$$

5.2. **The second moment.** Recall that we have written

$$f=m^hr_1^{h+1}\cdots r_{h-1}^{2h-1},$$

where the r_j are square-free and coprime in pairs. As in the first moment case, we have

$$\Omega(m^hr_1^{h+1}\cdots r_{h-1}^{2h-1})^2 = h^2\Omega(m)^2+2\sum_{j=1}^{h-1}h(h+j)\Omega(m)\omega(r_j)+\sum_{\ell=1}^{h-1}\sum_{j=1}^{h-1}(h+j)(h+\ell)\omega(r_j)\omega(r_\ell).$$

Using this, we obtain

$$\begin{aligned}\sum_{f\in\mathcal{N}_h\cap\mathcal{M}_n}\Omega(f)^2 = & h^2\sum_{\substack{m,r_1,\dots,r_{h-1}\in\mathcal{M}\\ \deg(m^hr_1^{h+1}\cdots r_{h-1}^{2h-1})=n}}\mu^2(r_1\cdots r_{h-1})\Omega(m)^2\\ &+2\sum_{j=1}^{h-1}h(h+j)\sum_{\substack{m,r_1,\dots,r_{h-1}\in\mathcal{M}\\ \deg(m^hr_1^{h+1}\cdots r_{h-1}^{2h-1})=n}}\mu^2(r_1\cdots r_{h-1})\Omega(m)\omega(r_j)\\ &+\sum_{j=1}^{h-1}(h+j)^2\sum_{\substack{m,r_1,\dots,r_{h-1}\in\mathcal{M}\\ \deg(m^hr_1^{h+1}\cdots r_{h-1}^{2h-1})=n}}\mu^2(r_1\cdots r_{h-1})\omega(r_j)^2\\ &+\sum_{\substack{1\le j,\ell\le h-1\\ j\neq\ell}}(h+j)(h+\ell)\sum_{\substack{m,r_1,\dots,r_{h-1}\in\mathcal{M}\\ \deg(m^hr_1^{h+1}\cdots r_{h-1}^{2h-1})=n}}\mu^2(r_1\cdots r_{h-1})\omega(r_j)\omega(r_\ell)\\ =&h^2T_{0,0}(n)+2\sum_{j=1}^{h-1}h(h+j)T_{0,j}(n)+\sum_{j=1}^{h-1}(h+j)^2T_{j,j}(n)+\sum_{\substack{1\le j,\ell\le h-1\\ j\neq\ell}}(h+j)(h+\ell)T_{j,\ell}(n).\end{aligned} \tag{35}$$

In this section we show how to obtain the formula for the second moment. We will see that the main term comes from $T_{0,0}(n)$.

5.2.1. *The sum over* $\Omega^2(m)$. For the sum over $\Omega^2(m)$, we apply Theorem 2.2 to $\mathcal{T}_0$ and $\mathcal{B}_0$ and obtain the following.

$$\begin{aligned}T_{0,0}(n)=&\sum_{\substack{m,r_1,\dots,r_{h-1}\in\mathcal{M}\\ \deg(m^hr_1^{h+1}\cdots r_{h-1}^{2h-1})=n}}\mu^2(r_1\cdots r_{h-1})\Omega(m)^2\\ =&\frac{\partial}{\partial z}\left(z\frac{\partial}{\partial z}\left(\frac{q^{\frac{n}{h}}n^{z-1}}{h^z}\sum_{k=0}^{h-1}\frac{\xi_h^{kn}}{\Gamma(z)}\mathcal{B}_0\left((q^{\frac1h}\xi_h^k)^{-1},z\right)\right)\right)\Bigg|_{z=1}+O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right)\\ =&\frac{q^{\frac{n}{h}}}{h}\left(\log\left(\frac{n}{h}\right)^2+\log\left(\frac{n}{h}\right)\right)\sum_{k=0}^{h-1}\xi_h^{kn}\mathcal{B}_0\left((q^{\frac1h}\xi_h^k)^{-1},1\right)\\ &+\frac{q^{\frac{n}{h}}}{h}\left(2\log\left(\frac{n}{h}\right)+1\right)\sum_{k=0}^{h-1}\frac{\partial}{\partial z}\left(\frac{\xi_h^{kn}}{\Gamma(z)}\mathcal{B}_0\left((q^{\frac1h}\xi_h^k)^{-1},z\right)\right)\Bigg|_{z=1}\\ &+\frac{q^{\frac{n}{h}}}{h}\sum_{k=0}^{h-1}\frac{\partial^2}{\partial z^2}\left(\frac{\xi_h^{kn}}{\Gamma(z)}\mathcal{B}_0\left((q^{\frac1h}\xi_h^k)^{-1},z\right)\right)\Bigg|_{z=1}+O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right),\end{aligned} \tag{36}$$

where we have applied (14) to compute the error term.

We now proceed to compute the second derivative of $\mathcal{B}_0$.

$$\begin{aligned}
&\sum_{k=0}^{h-1}\frac{\partial^2}{\partial z^2}\left(\frac{\xi_h^{kn}}{\Gamma(z)}\mathcal{B}_0\left((q^{\frac{1}{h}}\xi_h^k)^{-1},z\right)\right)\bigg|_{z=1}\\
=&\sum_{k=0}^{h-1}\xi_h^{kn}\left(\frac{\frac{\partial^2}{\partial z^2}\mathcal{B}_0\left((q^{\frac{1}{h}}\xi_h^k)^{-1},z\right)\Big|_{z=1}\Gamma(1)-2\,\frac{\partial}{\partial z}\mathcal{B}_0\left((q^{\frac{1}{h}}\xi_h^k)^{-1},z\right)\Big|_{z=1}\Gamma'(1)}{\Gamma(1)^2}\right.\\
&\left.+\frac{\mathcal{B}_0\left((q^{\frac{1}{h}}\xi_h^k)^{-1},1\right)(2\Gamma'(1)^2-\Gamma(1)\Gamma''(1))}{\Gamma(1)^3}\right)\\
=&\sum_{k=0}^{h-1}\xi_h^{kn}\left(\frac{\partial^2}{\partial z^2}\mathcal{B}_0\left((q^{\frac{1}{h}}\xi_h^k)^{-1},z\right)\bigg|_{z=1}+2\gamma\,\frac{\partial}{\partial z}\mathcal{B}_0\left((q^{\frac{1}{h}}\xi_h^k)^{-1},z\right)\bigg|_{z=1}\right.\\
&\left.+\mathcal{B}_0\left((q^{\frac{1}{h}}\xi_h^k)^{-1},1\right)(\gamma^2-\zeta(2)\right),
\end{aligned}$$

where we have used the fact that $\Gamma''(1)=\gamma^2+\zeta(2)$. A computation via the logarithmic derivative finally yields

$$\begin{aligned}
\frac{\frac{\partial^2}{\partial z^2}\mathcal{B}_0\left((q^{\frac{1}{h}}\xi_h^k)^{-1},z\right)\Big|_{z=1}}{\frac{\partial}{\partial z}\mathcal{B}_0\left((q^{\frac{1}{h}}\xi_h^k)^{-1},z\right)\Big|_{z=1}}=&\frac{\left(\sum_P\left(\log\left(1-\frac{1}{|P|}\right)+\frac{1}{|P|-1}\right)\right)^2+\sum_P\frac{1}{(|P|-1)^2}}{\sum_P\left(\log\left(1-\frac{1}{|P|}\right)+\frac{1}{|P|-1}\right)}\\
=&\frac{(B_2-\gamma)^2+\sum_P\frac{1}{(|P|-1)^2}}{B_2-\gamma}.
\end{aligned}$$

Finally, the result follows from applying the above as well as (29), (30), and (31) to (36):

$$\begin{aligned}
T_{0,0}(n)=&\frac{q^{\frac{n}{h}}}{h}\left(\log\left(\frac{n}{h}\right)^2+(2B_2+1)\log\left(\frac{n}{h}\right)+B_2^2+B_2-\zeta(2)+\sum_P\frac{1}{(|P|-1)^2}\right)\\
&\times\sum_{k=0}^{h-1}\xi_h^{kn}\prod_P\left(1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right)\\
&+O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right).
\end{aligned}$$

5.2.2. *The sums over* $\Omega(m)\omega(r_j)$. We proceed to compute these terms by considering the generating function

$$\begin{aligned}
\mathcal{T}_{0,j}(u,w,z)=:&\sum_{m,r_1,\ldots,r_{h-1}\in\mathcal{M}}\mu^2(r_1\cdots r_{h-1})w^{\omega(r_j)}z^{\Omega(m)}u^{\deg(m^hr_1^{h+1}\cdots r_{h-1}^{2h-1})}\\
=&\prod_P\left(1-zu^{\deg(P)h}\right)^{-1}\left(1+u^{\deg(P)(h+1)}+\cdots+wu^{\deg(P)(h+j)}+\cdots+u^{\deg(P)(2h-1)}\right).
\end{aligned}$$

We remark that $\frac{\partial^2}{\partial w\partial z}\mathcal{T}_{0,j}(u,1,1)$ yields a generating series for the $T_{0,j}(n)$.

Consider

$$\begin{aligned}\mathcal{B}_{0,j}(u,w,z) :=&\mathcal{Z}_q(u^h)^{-z}\mathcal{T}_{0,j}(u,w,z)\\ =&\prod_P \frac{\left(1-u^{\deg(P)h}\right)^z}{1-zu^{\deg(P)h}}\left(1+u^{\deg(P)(h+1)}+\cdots+wu^{\deg(P)(h+j)}+\cdots+u^{\deg(P)(2h-1)}\right).\end{aligned}$$

Fix w in a neighborhood of 1, Theorem 2.2 applies to $\mathcal{T}_{0,j}(u,w,z)$. Indeed, we have the following result, whose proof is almost identical to the proof of Lemma 5.1, and thus we will not repeat here.

Lemma 5.2. *Set $A < \frac{3}{2}$. For $|z| \le A$, $|w-1| \le \delta$, $n \ge 2$, and $\sigma > \frac{1}{h+1}$,*

$$\sum_{0\le a\le n} \frac{|\mathcal{B}_{0,j,w,z}(a)|}{q^{\sigma a}} \le c_{A,\sigma,\delta},$$

where $c_{A,\sigma,\delta}$ is a constant depending on A, σ and δ.

By applying Theorem 2.2, we obtain

$$\mathcal{T}_{0,j,z}(n,w) = \frac{q^{\frac{n}{h}}n^{z-1}}{h^z\Gamma(z)}\sum_{k=0}^{h-1}\xi_h^{kn}\mathcal{B}_{0,j}\left((q^{\frac{1}{h}}\xi_h^k)^{-1},w,z\right) + O_w(1)O_A\left(q^{\frac{n}{h}}n^{\mathrm{Re}(z)-2}\right),$$

where the error term is obtained by considering the factors involving w separated from the factors involving z.

By a similar calculation to the one presented in the $T_0(n)$ term of the first moment yields

$$\begin{aligned}T_{0,j}(n) =& \sum_{\substack{m,r_1,\dots,r_{h-1}\in\mathcal{M}\\ \deg(m^hr_1^{h+1}\cdots r_{h-1}^{2h-1})=n}} \mu^2(r_1\cdots r_{h-1})\omega(r_j)\Omega(m)\\
=& \frac{\partial^2}{\partial w\partial z}\left(\frac{q^{\frac{n}{h}}n^{z-1}}{h^z\Gamma(z)}\sum_{k=0}^{h-1}\xi_h^{kn}\mathcal{B}_{0,j}\left((q^{\frac{1}{h}}\xi_h^k)^{-1},w,z\right)\right)\Bigg|_{w=1,z=1} + O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right)\\
=& \frac{q^{\frac{n}{h}}}{h}\log\left(\frac{n}{h}\right)\sum_{k=0}^{h-1}\xi_h^{kn}\frac{\partial}{\partial w}\mathcal{B}_{0,j}\left((q^{\frac{1}{h}}\xi_h^k)^{-1},w,1\right)\Big|_{w=1}\\
&+\frac{q^{\frac{n}{h}}}{h}\sum_{k=0}^{h-1}\xi_h^{kn}\frac{\frac{\partial^2}{\partial z\partial w}\mathcal{B}_{0,j}\left((q^{\frac{1}{h}}\xi_h^k)^{-1},w,z\right)\Big|_{w=1,z=1}\Gamma(1) - \frac{\partial}{\partial w}\mathcal{B}_{0,j}\left((q^{\frac{1}{h}}\xi_h^k)^{-1},w,1\right)\Big|_{w=1}\Gamma'(1)}{\Gamma(1)^2}\\
&+O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right).\end{aligned}$$

Notice that

$$\begin{aligned}\frac{\partial}{\partial w}\mathcal{B}_{0,j}\left((q^{\frac{1}{h}}\xi_h^k)^{-1},w,z\right)\Big|_{w=1} =& \prod_P \frac{\left(1-(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)h}\right)^z}{1-z(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)h}}\left(1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right)\\
&\times\sum_P \frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+j)}}{1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}}\end{aligned}$$

and, by a similar computation to (31),

$$\frac{\partial^2}{\partial z\partial w}\,\mathcal{B}_{0,j}\left((q^{\frac{1}{h}}\xi_h^k)^{-1},w,z\right)\Big|_{w=1,z=1} = (B_2-\gamma)\prod_P\left(1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right)$$
$$\times\sum_P\frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+j)}}{1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}}.$$

Finally we get

$$\begin{aligned}T_{0,j}(n) = {} & \frac{q^{\frac{n}{h}}}{h}\left(\log\left(\frac{n}{h}\right)+B_2\right)\\ & \times\sum_{k=0}^{h-1}\xi_h^{kn}\prod_P\left(1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right)\\ & \times\sum_P\frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+j)}}{1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}}\\ & + O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right).\end{aligned}$$

5.2.3. *The sums over $\omega(r_j)\omega(r_\ell)$ for $j\neq\ell$.* We proceed to calculate these terms by considering the generating function

$$\begin{aligned}&\mathcal{T}_{j,l}(u,z,w)\\ &:=\sum_{m,r_1,\dots,r_{h-1}\in\mathcal{M}}\mu^2(r_1\cdots r_{h-1})w^{\omega(r_j)}z^{\omega(r_\ell)}u^{\deg(m^h r_1^{h+1}\cdots r_{h-1}^{2h-1})}\\ &=\prod_P(1-u^{h\deg(P)})^{-1}(1+u^{\deg(P)(h+1)}+\cdots+wu^{\deg(P)(h+j)}+\cdots+zu^{\deg(P)(h+\ell)}+\cdots+u^{\deg(P)(2h-1)})\\ &=\mathcal{Z}_q(u^h)\mathcal{B}_{j,\ell}(u,w,z).\end{aligned}$$

Remark that $\mathcal{B}_{j,\ell}(u,w,z)$ is absolutely convergent for $|w|,|z|\le A$ (for any $A>0$) and $|u|<q^{-\frac{1}{h+1}}$.

By Perron's formula (Lemma 2.1), we have

$$\sum_{\substack{m,r_1,\dots,r_{h-1}\in\mathcal{M}\\ \deg(m^h r_1^{h+1}\cdots r_{h-1}^{2h-1})=n}}\mu^2(r_1\cdots r_{h-1})w^{\omega(r_j)}z^{\omega(r_\ell)}=\frac{1}{2\pi i}\oint_{|u|=\delta}\frac{\mathcal{B}_{j,\ell}(u,w,z)}{1-qu^h}\frac{du}{u^{n+1}},$$

with $\delta>0$ small. We move the above integral to the circle $|u|=q^{-\frac{\varepsilon}{n}-\frac{1}{h+1}}$ and we obtain the residues at the poles $u=(q^{\frac{1}{h}}\xi_h^k)^{-1}$.

Thus we get

$$\begin{aligned}
T_{j,\ell}(n) =& \sum_{\substack{m,r_1,\dots,r_{h-1}\in\mathcal{M}\\ \deg(m^h r_1^{h+1}\cdots r_{h-1}^{2h-1})=n}} \mu^2(r_1\cdots r_{h-1}) w^{\omega(r_j)} z^{\omega(r_\ell)}\\
=& -\sum_{k=0}^{h-1} \mathrm{Res}_{u=(q^{\frac{1}{h}}\xi_h^k)^{-1}} \frac{\mathcal{B}_{j,\ell}(u,w,z)}{u^{n+1}(1-qu^h)} + \frac{1}{2\pi i}\oint_{|u|=q^{-\frac{\varepsilon}{n}-\frac{1}{h+1}}} \frac{\mathcal{B}_{j,\ell}(u,w,z)}{1-qu^h}\frac{du}{u^{n+1}}\\
=& \sum_{k=0}^{h-1}(q^{\frac{1}{h}}\xi_h^k)^{n+1}\mathcal{B}_{j,\ell}\left((q^{\frac{1}{h}}\xi_h^k)^{-1},w,z\right)\frac{1}{q^{\frac{1}{h}}\xi_h^k\prod_{m\neq k}\left(1-\xi_h^{m-k}\right)} + O_{w,z}(1)O\left(q^{\frac{n}{h+1}+\varepsilon}\right)\\
=& \frac{q^{\frac{n}{h}}}{h}\sum_{k=0}^{h-1}\xi_h^{kn}\mathcal{B}_{j,\ell}\left((q^{\frac{1}{h}}\xi_h^k)^{-1},w,z\right) + O_{w,z}(1)O\left(q^{\frac{n}{h+1}+\varepsilon}\right).
\end{aligned}$$

Now we have

$$\begin{aligned}
T_{j,\ell}(n) =& \sum_{\substack{m,r_1,\dots,r_{h-1}\in\mathcal{M}\\ \deg(m^h r_1^{h+1}\cdots r_{h-1}^{2h-1})=n}} \mu^2(r_1\cdots r_{h-1})\omega(r_j)\omega(r_\ell)\\
=& \frac{q^{\frac{n}{h}}}{h}\sum_{k=0}^{h-1}\xi_h^{kn}\,\frac{\partial^2}{\partial w\partial z}\mathcal{B}_\ell\left((q^{\frac{1}{h}}\xi_h^k)^{-1},w,z\right)\Big|_{w=1,z=1} + O\left(q^{\frac{n}{h+1}+\varepsilon}\right).
\end{aligned}$$

Using the usual techniques with the logarithmic derivative, we arrive at

$$\begin{aligned}
\frac{\partial^2}{\partial z\partial w}\mathcal{B}_{j,\ell}\left((q^{\frac{1}{h}}\xi_h^k)^{-1},w,z\right)\Big|_{w=1,z=1} =& \prod_P\left(1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right)\\
&\times\left(\sum_P\frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+j)}}{1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}}\right.\\
&\times\sum_P\frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+\ell)}}{1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}}\\
&\left.-\sum_P\frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+j)}(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+\ell)}}{\left(1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right)^2}\right).
\end{aligned}$$

In sum, we get

$$\begin{aligned}T_{j,\ell}(n) =&\frac{q^{\frac{n}{h}}}{h}\sum_{k=0}^{h-1}\xi_h^{kn}\prod_P\left(1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right)\\ &\times\left(\sum_P\frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+j)}}{1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}}\right.\\ &\times\sum_P\frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+\ell)}}{1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}}\\ &\left.-\sum_P\frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+j)}(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+\ell)}}{\left(1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right)^2}\right)+O\left(q^{\frac{n}{h+1}+\varepsilon}\right).\end{aligned}$$

5.2.4. *The sums over $\omega^2(r_j)$.* We remark that the sum over $\omega^2(r_j)$ is given by $\frac{\partial^2}{\partial z^2}\mathcal{T}_j(u,1)$, where $\mathcal{T}_j(u,z)$ was defined in (33). Indeed, we have

$$\begin{aligned}T_{j,j}(n) =&\sum_{\substack{m,r_1,\ldots,r_{h-1}\in\mathcal{M}\\ \deg(m^hr_1^{h+1}\cdots r_{h-1}^{2h-1})=n}}\mu^2(r_1\cdots r_{h-1})\omega^2(r_j)\\ =&\frac{q^{\frac{n}{h}}}{h}\sum_{k=0}^{h-1}\xi_h^{kn}\frac{\partial}{\partial z}\left(z\frac{\partial}{\partial z}\mathcal{B}_j\left((q^{\frac{1}{h}}\xi_h^k)^{-1},z\right)\right)\Big|_{z=1}+O\left(q^{\frac{n}{h+1}+\varepsilon}\right)\\ =&\frac{q^{\frac{n}{h}}}{h}\sum_{k=0}^{h-1}\xi_h^{kn}\,\frac{\partial^2}{\partial z^2}\mathcal{B}_j\left((q^{\frac{1}{h}}\xi_h^k)^{-1},z\right)\Big|_{z=1}+\frac{q^{\frac{n}{h}}}{h}\sum_{k=0}^{h-1}\xi_h^{kn}\,\frac{\partial}{\partial z}\mathcal{B}_j\left((q^{\frac{1}{h}}\xi_h^k)^{-1},z\right)\Big|_{z=1}+O\left(q^{\frac{n}{h+1}+\varepsilon}\right).\end{aligned}$$

By following a similar computation as in the other cases, we get

$$\begin{aligned}T_{j,j}(n) =&\frac{q^{\frac{n}{h}}}{h}\sum_{k=0}^{h-1}\xi_h^{kn}\prod_P\left(1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right)\\ &\times\left(\sum_P\frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+j)}}{1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}}\right.\\ &+\left(\sum_P\frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+j)}}{1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}}\right)^2\\ &\left.-\sum_P\left(\frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+j)}}{1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}}\right)^2\right)+O\left(q^{\frac{n}{h+1}+\varepsilon}\right).\end{aligned}$$

5.2.5. *Conclusion of the second moment computation.* By combining the results from the previous subsections into (35), we finally obtain

$$\frac{q^{\frac{n}{h}}}{h}\sum_{k=0}^{h-1}\xi_h^{kn}\prod_P\left(1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right)$$
$$\times\Bigg[h^2\left(\log\left(\frac{n}{h}\right)^2+(2B_2+1)\log\left(\frac{n}{h}\right)+B_2^2+B_2-\zeta(2)+\sum_P\frac{1}{(|P|-1)^2}\right)$$
$$+\sum_{j=1}^{h-1}(h+j)\Bigg(2h\left(\log\left(\frac{n}{h}\right)+B_2\right)+(h+j)$$
$$+\sum_{\ell=1}^{h-1}(h+\ell)\sum_P\frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+\ell)}}{1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}}\Bigg)$$
$$\times\sum_P\frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+j)}}{1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}}$$
$$-\sum_{1\leq j,\ell\leq h-1}(h+j)(h+\ell)\sum_P\frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+j)}(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+\ell)}}{\left(1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right)^2}\Bigg]+O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right).$$

We can compute the variance from Theorem 1.4. Before proceeding, we need the following auxiliary result.

Lemma 5.3. *We have*

$$\sum_{f\in\mathcal{N}_h\cap\mathcal{M}_n}1=\frac{q^{\frac{n}{h}}}{h}\sum_{k=0}^{h-1}\xi_h^{kn}\prod_P\left(1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right)+O\left(q^{\frac{n}{h+1}+\varepsilon}\right).$$

Proof. We obtain the generating function for the h-full polynomials by evaluating $z=1$ in (28)

$$\begin{aligned}\sum_{f\in\mathcal{N}_h}u^{\deg(f)}=&\sum_{m,r_1,\dots,r_{h-1}\in\mathcal{M}}\mu^2(r_1\cdots r_{h-1})u^{\deg(m^hr_1^{h+1}\cdots r_{h-1}^{2h-1})}\\ =&\prod_P\left(1-u^{\deg(P)h}\right)^{-1}\left(1+u^{\deg(P)(h+1)}+\cdots+u^{\deg(P)(2h-1)}\right)\\ =&\mathcal{Z}_q(u^h)\mathcal{B}_0(u,1),\end{aligned}$$

where we recall that $\mathcal{B}_0(u,1)$ converges absolutely for $|u|<q^{-\frac{1}{h+1}}$.

We apply Perron's formula (Lemma 2.1), first integrating over $\delta > 0$ small, and then moving the integral to the circle $|u| = q^{-\frac{\varepsilon}{n}-\frac{1}{h+1}}$ and collecting the residues at $u = (q^{\frac{1}{h}}\xi_h j)^{-1}$. This gives

$$\begin{aligned}
\sum_{f\in\mathcal{N}_h\cap\mathcal{M}_n} 1 =& \frac{1}{2\pi i}\oint_{|u|=\delta} \frac{\mathcal{B}_0(u,1)}{1-qu^h}\frac{du}{u^{n+1}} \\
=& -\sum_{k=0}^{h-1} \mathrm{Res}_{u=(q^{\frac{1}{h}}\xi_h^k)^{-1}} \frac{\mathcal{B}_0(u,1)}{u^{n+1}(1-qu^h)} + \frac{1}{2\pi i}\oint_{|u|=q^{-\frac{\varepsilon}{n}-\frac{1}{h+1}}} \frac{\mathcal{B}_0(u,1)}{1-qu^h}\frac{du}{u^{n+1}} \\
=& \sum_{k=0}^{h-1} (q^{\frac{1}{h}}\xi_h^k)^{n+1}\mathcal{B}_0\left((q^{\frac{1}{h}}\xi_h^k)^{-1},1\right)\frac{1}{q^{\frac{1}{h}}\xi_h^k\prod_{m\neq k}\left(1-\xi_h^{m-k}\right)} + O\left(q^{\frac{n}{h+1}+\varepsilon}\right) \\
=& \frac{q^{\frac{n}{h}}}{h}\sum_{k=0}^{h-1}\xi_h^{kn}\prod_P\left(1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right) + O\left(q^{\frac{n}{h+1}+\varepsilon}\right).
\end{aligned}$$

□

Proof of Theorem 1.5. By combining the results of Theorem 1.4 and Lemma 5.3, we have

$$\begin{aligned}
&\mathbb{E}_{h,n}(\Omega^2) - (\mathbb{E}_{h,n}(\Omega))^2 \\
=& \Bigg[h^2\left(\log\left(\frac{n}{h}\right) + B_2 - \zeta(2) + \sum_P \frac{1}{(|P|-1)^2}\right) \\
&+ \left(\sum_{k=0}^{h-1}\xi_h^{kn}\prod_P\left(1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right)\right)^{-1} \\
&\times\sum_{k=0}^{h-1}\xi_h^{kn}\prod_P\left(1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right) \\
&\times\Bigg[\sum_{j=1}^{h-1}(h+j)\Bigg((h+j)+\sum_{\ell=1}^{h-1}(h+\ell)\sum_P\frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+\ell)}}{1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}}\Bigg) \\
&\times\sum_P\frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+j)}}{1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}} \\
&-\sum_{1\le j,\ell\le h-1}(h+j)(h+\ell)\sum_P\frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+j)}(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+\ell)}}{\left(1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right)^2}\Bigg] \\
&-\Bigg[\left(\sum_{k=0}^{h-1}\xi_h^{kn}\prod_P\left(1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right)\right)^{-1} \\
&\times\sum_{k=0}^{h-1}\xi_h^{kn}\prod_P\left(1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}\right) \\
&\times\sum_{j=1}^{h-1}(h+j)\sum_P\frac{(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+j)}}{1+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(h+1)}+\cdots+(q^{\frac{1}{h}}\xi_h^k)^{-\deg(P)(2h-1)}}\Bigg]^2 + O\left(\frac{1}{n^{1-\varepsilon}}\right).
\end{aligned}$$

□

Remark. *By setting $h = 1$ in Theorems 1.4 and 1.5, we can recover once again the result of Corollary 3.1.*

5.3. **A discussion on the proof strategy for Theorem 1.4.** The starting point in our proof of Theorem 1.4 was to consider the decomposition

$$f = m^h r_1^{h+1} \cdots r_{h-1}^{2h-1}. \tag{37}$$

Then we used crucially that Ω is completely multiplicative. Another possible strategy would have been to start from the generating series

$$\mathcal{C}(u,z) := \prod_P \left(1 + z^h u^{h\deg(P)} + z^{h+1} u^{(h+1)\deg(P)} + \cdots\right) = \sum_{f \in \mathcal{N}_h} z^{\Omega(f)} u^{\deg(f)}$$

and to apply an extension of Theorem 2.2 to

$$\begin{aligned}\mathcal{B}(u,z) :=& \mathcal{Z}_q(u^h)^{-z^h} \mathcal{C}(u,z)\\ =& \prod_P \left(1 - u^{h\deg(P)}\right)^{z^h} \left(1 + \frac{z^h u^{h\deg(P)}}{1 - z u^{\deg(P)}}\right).\end{aligned} \tag{38}$$

In fact, the extension of Theorem 2.2 that we need is the following.

Theorem 5.1 (Generalized Theorem 2.1)**.** *Let $C(u,z) = \sum_{n\geq 0} C_z(n) u^n$ and $B(u,z) = \sum_{n \geq 0} B_z(n) u^n$ be power series with coefficients depending on z satisfying $C(u,z) = B(u,z)\mathcal{Z}_q(u^h)^{z^h}$. Suppose also that, uniformly for $|z^h| \leq A$,*

$$\sum_{a=0}^{\infty} \frac{|B_z(n)|}{q^{\frac{n}{h}}} n^{2A+2} \ll_A 1.$$

Then, uniformly for $|z^h| \leq A$ and $n \geq 1$, we have

$$C_z(n) = \frac{q^{\frac{n}{h}} n^{z^h - 1}}{h^{z^h} \Gamma(z^h)} \sum_{k=0}^{h-1} \xi_h^{kn} B\left((q^{\frac{1}{h}} \xi_h^k)^{-1}, z\right) + O_A\left(q^{\frac{n}{h}} n^{\mathrm{Re}(z^h) - 2}\right).$$

This method leads to equivalent results. We have opted to follow the decomposition (37) to prove Theorem 1.4 because the proof is more clear, it shows the origin of the main term, and it leads to cleaner and more symmetric formulas for the first and second moment and the variance.

A crucial point that makes working with (38) particularly cumbersome has to do with the mixture of z and z^h, and their different behavior under differentiation. This does not make the analysis of the Erdős–Kac theorem more complicated, and therefore we will use Theorem 5.1 in Section 5.4 for that.

It should be remarked however, that a proof that depends on the decomposition (37) can not be applied to ω. Nevertheless, in the case of ω, the direct generating series is simpler and does not involve the mixture of z and z^h. We state the results obtained in this way for ω in Section 6.

5.4. **Higher moments of Ω and an Erdős–Kac result for h-full polynomials.** The goal of this section is to prove Theorem 1.6, which is the Erdős–Kac theorem for Ω over the h-full polynomials. To this end we will prove that

$$\frac{1}{|\mathcal{N}_h \cap \mathcal{M}_n|} \sum_{f \in \mathcal{N}_h \cap \mathcal{M}_n} \left(\frac{\Omega(f) - h\log\left(\frac{n}{h}\right)}{h\sqrt{\log\left(\frac{n}{h}\right)}} \right)^v \to C_v,$$

where the C_v's are given by (20).

As before, we will consider

$$\frac{1}{|\mathcal{N}_h \cap \mathcal{M}_n|} \sum_{f \in \mathcal{N}_h \cap \mathcal{M}_n} \left(\frac{\Omega(f) - h \log\left(\frac{n}{h}\right)}{h\sqrt{\log\left(\frac{n}{h}\right)}} \right)^v = \frac{1}{\left(h^2 \log\left(\frac{n}{h}\right)\right)^{\frac{v}{2}}} \sum_{\ell=0}^{v} \binom{v}{\ell} \mathbb{E}(\Omega^\ell)(-1)^{v-\ell} \left(h \log\left(\frac{n}{h}\right)\right)^{v-\ell}.$$

By substituting $z = e^t$ into $\mathcal{B}(u, z)$ given by (38), we have

$$\begin{aligned} \mathcal{B}(u, e^t) :=& \mathcal{Z}_q(u^h)^{-e^{ht}} \mathcal{C}(u, e^t) \\ =& \prod_P \left(1 - u^{\deg(P)h}\right)^{e^{ht}} \left(1 + \frac{e^{ht} u^{h \deg(P)}}{1 - e^t u^{\deg(P)}}\right). \end{aligned}$$

Theorem 5.1 applies to $\mathcal{B}(u, e^t)$ at $t = 0$, and therefore,

$$\sum_{f \in \mathcal{N}_h \cap \mathcal{M}_n} e^{\Omega(f)t} = \frac{q^{\frac{n}{h}} n^{e^{ht}-1}}{h^{e^{ht}}} \sum_{s=0}^{h-1} \frac{\xi_h^{sn}}{\Gamma(e^{ht})} \mathcal{B}\left((q^{\frac{1}{h}} \xi_h^s)^{-1}, e^t\right) + O_{A,\delta}\left(q^{\frac{n}{h}} n^{\mathrm{Re}(e^t)-2}\right).$$

By the property of the moment generating function (21) combined with (23), we have

$$\begin{aligned} \mathbb{E}(\Omega^\ell) = & \left(\sum_{s=0}^{h-1} \xi_h^{sn} \mathcal{B}\left((q^{\frac{1}{h}} \xi_h^s)^{-1}, 1\right)\right)^{-1} \sum_{j=0}^{\ell} \binom{\ell}{j} h^j T_j \left(\log\left(\frac{n}{h}\right)\right) \sum_{s=0}^{h-1} \xi_h^{sn} \left(\frac{\mathcal{B}\left((q^{\frac{1}{h}} \xi_h^s)^{-1}, e^t\right)}{\Gamma(e^{ht})}\right)^{(\ell-j)} \Bigg|_{t=0} \\ & + O\left(\frac{1}{n^{1-\varepsilon}}\right) \end{aligned}$$

and

$$\begin{aligned} & \frac{\left(\sum_{s=0}^{h-1} \xi_h^{sn} \mathcal{B}\left((q^{\frac{1}{h}} \xi_h^s)^{-1}, 1\right)\right)^{-1}}{h^v \log\left(\frac{n}{h}\right)^{\frac{v}{2}}} \sum_{\ell=0}^{v} \binom{v}{\ell} \mathbb{E}(\Omega^\ell)(-1)^{v-\ell} \left(h \log\left(\frac{n}{h}\right)\right)^{v-\ell} \\ & = \frac{\left(\sum_{s=0}^{h-1} \xi_h^{sn} \mathcal{B}\left((q^{\frac{1}{h}} \xi_h^s)^{-1}, 1\right)\right)^{-1}}{h^v \log\left(\frac{n}{h}\right)^{\frac{v}{2}}} \sum_{\ell=0}^{v} \binom{v}{\ell} \sum_{j=0}^{\ell} \binom{\ell}{j} h^j T_j \left(\log\left(\frac{n}{h}\right)\right) \sum_{s=0}^{h-1} \xi_h^{sn} \left(\frac{\mathcal{B}\left((q^{\frac{1}{h}} \xi_h^s)^{-1}, e^t\right)}{\Gamma(e^{ht})}\right)^{(\ell-j)} \Bigg|_{t=0} \\ & \times (-1)^{v-\ell} \left(h \log\left(\frac{n}{h}\right)\right)^{v-\ell} + O\left(\frac{1}{n^{1-\varepsilon}}\right). \end{aligned} \tag{39}$$

Consider the change of variables $u = v - \ell$, $k = v - \ell + j$. Then the main term in (39) becomes

$$\frac{\left(\sum_{s=0}^{h-1} \xi_h^{sn}\mathcal{B}\left((q^{\frac{1}{h}}\xi_h^s)^{-1},1\right)\right)^{-1}}{h^v \log\left(\frac{n}{h}\right)^{\frac{v}{2}}} \sum_{k=0}^{v}\sum_{s=0}^{h-1} \xi_h^{sn} \left(\frac{\mathcal{B}\left((q^{\frac{1}{h}}\xi_h^s)^{-1},e^t\right)}{\Gamma(e^{ht})}\right)^{(v-k)}\Bigg|_{t=0}$$
$$\times \sum_{u=0}^{k}\binom{v}{u}\binom{v-u}{k-u} h^{k-u}T_{k-u}\left(\log\left(\frac{n}{h}\right)\right)(-1)^u\left(h\log\left(\frac{n}{h}\right)\right)^u$$
$$= \frac{\left(\sum_{s=0}^{h-1} \xi_h^{sn}\mathcal{B}\left((q^{\frac{1}{h}}\xi_h^s)^{-1},1\right)\right)^{-1}}{h^v \log\left(\frac{n}{h}\right)^{\frac{v}{2}}} \sum_{k=0}^{v}\binom{v}{k}\sum_{s=0}^{h-1} \xi_h^{sn} \left(\frac{\mathcal{B}\left((q^{\frac{1}{h}}\xi_h^s)^{-1},e^t\right)}{\Gamma(e^{ht})}\right)^{(v-k)}\Bigg|_{t=0}$$
$$\times \sum_{u=0}^{k}\binom{k}{u} h^{k-u}T_{k-u}\left(\log\left(\frac{n}{h}\right)\right)(-1)^u\left(h\log\left(\frac{n}{h}\right)\right)^u.$$

Similarly to the h-free case we get, for v even, that the main term should come from setting $k = v$, which leads to

$$\frac{1}{h^v \log\left(\frac{n}{h}\right)^{\frac{v}{2}}}\sum_{\ell=0}^{v}\binom{v}{\ell}\mathbb{E}(\Omega(f)^\ell)(-1)^{v-\ell}\left(h\log\left(\frac{n}{h}\right)\right)^{v-\ell}$$
$$= \frac{\left(\sum_{s=0}^{h-1} \xi_h^{sn}\mathcal{B}\left((q^{\frac{1}{h}}\xi_h^s)^{-1},1\right)\right)^{-1}}{h^v \log\left(\frac{n}{h}\right)^{\frac{v}{2}}} \sum_{s=0}^{h-1} \xi_h^{sn} \left(\frac{\mathcal{B}\left((q^{\frac{1}{h}}\xi_h^s)^{-1},1\right)}{\Gamma(1)}\right)$$
$$\times \sum_{u=0}^{v}\binom{v}{u} h^{v-u}T_{v-u}\left(\log\left(\frac{n}{h}\right)\right)(-1)^u\left(h\log\left(\frac{n}{h}\right)\right)^u + O\left(\frac{1}{\log n}\right)$$
$$= \frac{v!}{2^{\frac{v}{2}}\left(\frac{v}{2}\right)!} + O\left(\frac{1}{\log n}\right),$$

while for v odd we get

$$O\left(\frac{1}{\sqrt{\log n}}\right).$$

6. The distribution of ω over h-full polynomials

In this short section we state, without proof, analogous results for ω. The proofs are quite direct from the corresponding generating function.

Theorem 6.1. *For any $\varepsilon > 0$ and as $n \to \infty$, the first moment of ω over the h-full polynomials of degree n is given by*

$$\sum_{f\in\mathcal{N}_h\cap\mathcal{M}_n}\omega(f) = \frac{q^{\frac{n}{h}}}{h}\sum_{k=0}^{h-1}\xi_h^{kn}\prod_P\left(1-\frac{1}{|P|}\right)\left(1+\frac{1}{|P|-|P|^{1-\frac{1}{h}}\xi_h^{-\deg(P)k}}\right)$$
$$\times\left(\log\left(\frac{n}{h}\right)+B_1+\sum_P\left(\frac{1}{|P|-|P|^{1-\frac{1}{h}}\xi_h^{-\deg(P)k}+1}-\frac{1}{|P|}\right)\right)+O\left(\frac{q^n}{n^{1-\varepsilon}}\right).$$

while the second moment is given by

$$\begin{aligned}\sum_{f\in\mathcal{N}_h\cap\mathcal{M}_n}\omega(f)^2 =&\frac{q^{\frac{n}{h}}}{h}\sum_{k=0}^{h-1}\xi_h^{kn}\prod_P\left(1-\frac{1}{|P|}\right)\left(1+\frac{1}{|P|-|P|^{1-\frac{1}{h}}\xi_h^{-\deg(P)k}}\right)\\ &\times\left[\log\left(\frac{n}{h}\right)^2+\log\left(\frac{n}{h}\right)\left[2B_1+1+2\sum_P\left(\frac{1}{|P|-|P|^{1-\frac{1}{h}}\xi_h^{-\deg(P)k}+1}-\frac{1}{|P|}\right)\right]\right.\\ &+B_1^2+B_1-\zeta(2)-\sum_P\frac{1}{\left(|P|-|P|^{1-\frac{1}{h}}\xi_h^{-\deg(P)k}+1\right)^2}\\ &+\left(\sum_P\left(\frac{1}{|P|-|P|^{1-\frac{1}{h}}\xi_h^{-\deg(P)k}+1}-\frac{1}{|P|}\right)\right)^2\\ &\left.+(2B_1+1)\sum_P\left(\frac{1}{|P|-|P|^{1-\frac{1}{h}}\xi_h^{-\deg(P)k}+1}-\frac{1}{|P|}\right)\right]+O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right).\end{aligned}$$

Finally, the variance is given by

$$\begin{aligned}&\mathrm{Var}_{h-\mathrm{full},n}(\omega)=\log\left(\frac{n}{h}\right)+B_1-\zeta(2)+\left(\sum_{k=0}^{h-1}\xi_h^{kn}\prod_P\left(1-\frac{1}{|P|}\right)\left(1+\frac{1}{|P|-|P|^{1-\frac{1}{h}}\xi_h^{-\deg(P)k}}\right)\right)^{-1}\\ &\times\sum_{k=0}^{h-1}\xi_h^{kn}\prod_P\left(1-\frac{1}{|P|}\right)\left(1+\frac{1}{|P|-|P|^{1-\frac{1}{h}}\xi_h^{-\deg(P)k}}\right)\left[-\sum_P\frac{1}{\left(|P|-|P|^{1-\frac{1}{h}}\xi_h^{-\deg(P)k}+1\right)^2}\right.\\ &\left.+\left(\sum_P\left(\frac{1}{|P|-|P|^{1-\frac{1}{h}}\xi_h^{-\deg(P)k}+1}-\frac{1}{|P|}\right)\right)^2+\sum_P\left(\frac{1}{|P|-|P|^{1-\frac{1}{h}}\xi_h^{-\deg(P)k}+1}-\frac{1}{|P|}\right)\right]\\ &-\left[\left(\sum_{k=0}^{h-1}\xi_h^{kn}\prod_P\left(1-\frac{1}{|P|}\right)\left(1+\frac{1}{|P|-|P|^{1-\frac{1}{h}}\xi_h^{-\deg(P)k}}\right)\right)^{-1}\right.\\ &\left.\times\sum_{k=0}^{h-1}\xi_h^{kn}\prod_P\left(1-\frac{1}{|P|}\right)\left(1+\frac{1}{|P|-|P|^{1-\frac{1}{h}}\xi_h^{-\deg(P)k}}\right)\sum_P\left(\frac{1}{|P|-|P|^{1-\frac{1}{h}}\xi_h^{-\deg(P)k}+1}-\frac{1}{|P|}\right)\right]^2\\ &+O\left(\frac{1}{n^{1-\varepsilon}}\right).\end{aligned}$$

To prove the above results, one works with the Euler product

$$\begin{aligned}\mathcal{B}(u,z):=&\prod_P\mathcal{Z}_q(u^h)^{-z}\left(1+z\left(u^{h\deg(P)}+u^{(h+1)\deg(P)}+\cdots\right)\right)\\ =&\prod_P\left(1-u^{h\deg(P)}\right)^z\left(1+\frac{zu^{h\deg(P)}}{1-u^{\deg(P)}}\right).\end{aligned}$$

Moreover, working with $\mathcal{B}(u,e^t)$, we also obtain

Theorem 6.2. *As $n \to \infty$, $\omega(f)$ with $f \in \mathcal{N}_h \cap \mathcal{M}_n$ approaches a normal distribution, namely, for $\alpha \leq \beta$,*

$$\left| \left\{ f \in \mathcal{N}_h \cap \mathcal{M}_n : \alpha \leq \frac{\omega(f) - \log(n)}{\sqrt{\log(n)}} \leq \beta \right\} \right| \to \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-t^2/2} dt.$$

References

[AP19] Ardavan Afshar and Sam Porritt, *The function field Sathe-Selberg formula in arithmetic progressions and 'short intervals'*, Acta Arith. **187** (2019), no. 2, 101–124. MR 3897499

[Bil69] Patrick Billingsley, *On the central limit theorem for the prime divisor functions*, Amer. Math. Monthly **76** (1969), 132–139. MR 242222

[Del53] Hubert Delange, *Sur le nombre des diviseurs premiers de n*, C. R. Acad. Sci. Paris **237** (1953), 542–544. MR 57282

[Del71] ———, *Sur des formules de Atle Selberg*, Acta Arith. **19** (1971), 105–146. (errata insert). MR 289432

[EK40] P. Erdös and M. Kac, *The Gaussian law of errors in the theory of additive number theoretic functions*, Amer. J. Math. **62** (1940), 738–742. MR 2374

[GS07] Andrew Granville and K. Soundararajan, *Sieving and the Erdös-Kac theorem*, Equidistribution in number theory, an introduction, NATO Sci. Ser. II Math. Phys. Chem., vol. 237, Springer, Dordrecht, 2007, pp. 15–27. MR 2290492

[Hal55] H. Halberstam, *On the distribution of additive number-theoretic functions*, J. London Math. Soc. **30** (1955), 43–53. MR 66406

[HR00] G. H. Hardy and S. Ramanujan, *The normal number of prime factors of a number n [Quart. J. Math.* **48** *(1917), 76–92]*, Collected papers of Srinivasa Ramanujan, AMS Chelsea Publ., Providence, RI, 2000, pp. 262–275. MR 2280878

[JL22] Rafael Jakimczuk and Matilde Lalín, *The number of prime factors on average in certain integer sequences*, J. Integer Seq. **25** (2022), no. 2, Art. 22.2.3, 15. MR 4390042

[Liu04a] Yu-Ru Liu, *A generalization of the Erdös-Kac theorem and its applications*, Canad. Math. Bull. **47** (2004), no. 4, 589–606. MR 2099756

[Liu04b] ———, *A generalization of the Turán theorem and its applications*, Canad. Math. Bull. **47** (2004), no. 4, 573–588. MR 2099755

[Rho09] Robert C. Rhoades, *Statistics of prime divisors in function fields*, Int. J. Number Theory **5** (2009), no. 1, 141–152. MR 2499026

[Ros02] Michael Rosen, *Number theory in function fields*, Graduate Texts in Mathematics, vol. 210, Springer-Verlag, New York, 2002. MR 1876657

[Sai02] Filip Saidak, *An elementary proof of a theorem of Delange*, C. R. Math. Acad. Sci. Soc. R. Can. **24** (2002), no. 4, 144–151. MR 1940553

[Sat53] L. G. Sathe, *On a problem of Hardy on the distribution of integers having a given number of prime factors. I-IV*, J. Indian Math. Soc. (N.S.) **17,18** (1953), 63–82, 83–141, 27–42, 43–81. MR 56625

[Sel54] Atle Selberg, *Note on a paper by L. G. Sathe*, J. Indian Math. Soc. (N.S.) **18** (1954), 83–87. MR 67143

[Tou39] Jacques Touchard, *Sur les cycles des substitutions*, Acta Math. **70** (1939), no. 1, 243–297. MR 1555449

[Tur34] Paul Turán, *On a Theorem of Hardy and Ramanujan*, J. London Math. Soc. **9** (1934), no. 4, 274–276. MR 1574877

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