SUMS OF $\omega(n)$ AND $\Omega(n)$ OVER THE k-FREE PARTS AND k-FULL PARTS OF SOME PARTICULAR SEQUENCES

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Abstract. The k-free part of a positive integer n is the product of the prime powers dividing n that have exponent less than k in the factorization, while the k-full part of n is the product of the prime powers that have exponent at least k. We consider sums of the prime factor counting functions ω and Ω going over the k-free parts and k-full parts of some particular number sequences.

1. Introduction

For a positive integer with prime factorization

$$n = q_1^{s_1} \cdots q_r^{s_r}, \tag{1}$$

where the q_j are the prime factors and the $s_j \geq 1$ are their respective exponents, the prime factor counting functions are defined by $\omega(n) = r$ and $\Omega(n) = s_1 + \cdots + s_r$.

For $k \geq 1$, and n as above, let

$$L_k(n) = \prod_{\substack{1\leq j\leq r\\ s_j<k}} q_j^{s_j} \qquad \text{and} \qquad U_k(n) = \prod_{\substack{1\leq j\leq r\\ k\leq s_j}} q_j^{s_j}.$$

We say that $L_k(n)$ is the k-free part of n and that $U_k(n)$ is the k-full part of n. By convention, $L_1(n) = 1$, while naturally $U_1(n) = n$. Similarly, when $k > \max_j s_j$, we have $L_k(n) = n$ and $U_k(n) = 1$. Remark that $n = L_k(n)U_k(n)$ for any k and that $L_k(n)$ and $U_k(n)$ are coprime.

The aim of this article is to consider sums of ω and Ω composed with U_k and L_k evaluated in certain sequences of positive integer numbers.

To begin, we consider the evaluation in the whole sequence of positive integer numbers.

Theorem 1. *Let $k \geq 1$ be an integer. We have that*

$$\sum_{n\leq x} \omega(U_k(n)) = \left(\sum_p \frac{1}{p^k}\right) x + O_k\left(\frac{x^{\frac{1}{k}}}{\log x}\right), \tag{2}$$

and

$$\sum_{n\leq x} \Omega(U_k(n)) = \left(\sum_p \frac{1-k+kp}{p^{k+1}-p^k}\right) x + O_k\left(\frac{x^{\frac{1}{k}}}{\log x}\right), \tag{3}$$

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where the sums over p indicate that the sums are taken over all prime numbers.

For the rest of this article we will continue to use the convention that sums and products over p indicate over all the primes, unless stated otherwise.

Corollary 2. *Let $k \geq 1$ be an integer. We have that*

$$\sum_{n\leq x} \omega(L_k(n)) = x \log\log x + \left(B_1 - \sum_p \frac{1}{p^k}\right) x + O\left(\frac{x}{\log x}\right), \tag{4}$$

where B_1 is the Mertens constant is given by

$$B_1 = \gamma + \sum_p \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right), \tag{5}$$

and $\gamma = 0.57721\ldots$ is the Euler–Mascheroni constant.

We have that

$$\sum_{n\leq x} \Omega(L_k(n)) = x \log\log x + \left(B_2 - \sum_p \frac{1-k+kp}{p^{k+1}-p^k}\right) x + O\left(\frac{x}{\log x}\right), \tag{6}$$

where

$$B_2 = B_1 + \sum_p \frac{1}{p(p-1)}. \tag{7}$$

Let $h \geq 1$ be an integer. A positive integer n is said to be h-free if all its prime factors have exponents less than h. In other words, if n has prime factorization (1), then $s_j \leq h-1$ for all j. In particular, n is square-free if all $s_j = 1$. We denote by $\mathcal{S}_h$ the set of h-free positive integers.

We have the following result.

Theorem 3. *Let $h > k > 1$ be integers. Then we have*

$$\sum_{\substack{n\in\mathcal{S}_h\\ n\leq x}} \Omega(U_k(n)) = \frac{1}{\zeta(h)} D_{\Omega,k,h} x + O_h\left(x^{\frac{2k-1}{k^2}} \log\log x\right), \tag{8}$$

$$\sum_{\substack{n\in\mathcal{S}_h\\ n\leq x}} \omega(U_k(n)) = \frac{1}{\zeta(h)} D_{\omega,k,h} x + O_h\left(x^{\frac{2k-1}{k^2}} \log\log x\right). \tag{9}$$

where

$$D_{\Omega,k,h} = \sum_p \frac{h-1-(k-1)p^{h-k} - hp + kp^{h-k+1}}{(p-1)(p^h-1)}, \tag{10}$$

and

$$D_{\omega,k,h} = \sum_p \frac{p^{h-k}-1}{p^h-1}. \tag{11}$$

Corollary 4. *Let $h > k > 1$ be integers. Then we have*

$$\sum_{\substack{n\in\mathcal{S}_h\\ n\le x}} \Omega(L_k(n)) = \frac{1}{\zeta(h)} x\log\log x + O(x), \tag{12}$$

$$\sum_{\substack{n\in\mathcal{S}_h\\ n\le x}} \omega(L_k(n)) = \frac{1}{\zeta(h)} x\log\log x + O(x). \tag{13}$$

Let $h \ge 1$ be an integer. A positive integer n is said to be h-full if all its prime factors have exponents greater or equal than h. In other words, if n has prime factorization (1), then $s_j \ge h$ for all j. (This definition is trivial for $h = 1$.) We denote by $\mathcal{N}_h$ the set of h-full positive integers.

We prove the following estimates.

Theorem 5. *Let $k > h > 0$ be integers. Then we have*

$$\sum_{\substack{n\in\mathcal{N}_h\\ n\le x}} \Omega(U_k(n)) = \gamma_{0,h} E_{\Omega,k,h} x^{\frac{1}{h}} + O\left(x^{\frac{1}{h} - \left(\frac{k}{h}-1\right)\frac{1}{k+2h(h+1)}+\varepsilon} \log\log x\right), \tag{14}$$

$$\sum_{\substack{n\in\mathcal{N}_h\\ n\le x}} \omega(U_k(n)) = \gamma_{0,h} E_{\omega,k,h} x^{\frac{1}{h}} + O\left(x^{\frac{1}{h} - \left(\frac{k}{h}-1\right)\frac{1}{k+2h(h+1)}+\varepsilon} \log\log x\right), \tag{15}$$

where

$$\gamma_{0,h} = \prod_p \left(1 + \frac{p - p^{\frac{1}{h}}}{p^2\left(p^{\frac{1}{h}} - 1\right)}\right), \tag{16}$$

$$E_{\Omega,k,h} = \sum_p \frac{kp^{\frac{1}{h}} - k + 1}{p^{\frac{k-h-1}{h}}\left(p^{\frac{1}{h}} - 1\right)\left(p^{1+\frac{1}{h}} + p^{\frac{1}{h}} - p\right)}, \tag{17}$$

and

$$E_{\omega,k,h} = \sum_p \frac{1}{p^{\frac{k-h-1}{h}}\left(p^{1+\frac{1}{h}} + p^{\frac{1}{h}} - p\right)}. \tag{18}$$

Corollary 6. *Let $k > h > 0$ be integers. The following formulas hold*

$$\sum_{\substack{n\in\mathcal{N}_h\\ n\le x}} \Omega(L_k(n)) = h\gamma_{0,h} x^{\frac{1}{h}} \log\log x + \gamma_{0,h}\left(C_{\Omega,h} - E_{\omega,k,h}\right) x^{\frac{1}{h}} + O_h\left(\frac{x^{\frac{1}{h}}}{\sqrt{\log x}}\right), \tag{19}$$

$$\sum_{\substack{n\in\mathcal{N}_h\\ n\le x}} \omega(L_k(n)) = \gamma_{0,h} x^{\frac{1}{h}} \log\log x + \gamma_{0,h}\left(C_{\omega,h} - E_{\omega,k,h}\right) x^{\frac{1}{h}} + O_h\left(\frac{x^{\frac{1}{h}}}{\sqrt{\log x}}\right), \tag{20}$$

where

$$C_{\Omega,h} = h(B_2 - \log h) + \sum_p \frac{(h+1)p^{1+\frac{1}{h}} - hp - 2hp^{\frac{2}{h}} + (2h-1)p^{\frac{1}{h}}}{(p-1)\left(p^{\frac{1}{h}} - 1\right)\left(p^{1+\frac{1}{h}} + p^{\frac{1}{h}} - p\right)} \tag{21}$$

and

$$C_{\omega,h} = B_1 - \log h + \sum_p \frac{p - p^{\frac{1}{h}}}{(p-1)\left(p^{1+\frac{1}{h}} + p^{\frac{1}{h}} - p\right)}. \tag{22}$$

This article is organized as follows. Section 2 includes the proof of Theorem 1 and Corollary 2 by elementary counting, as well as a corollary considering the sum going over h-powers. Theorem 3 is proven in Section 3. This is achieved by counting first the h-free integers that are coprime to certain fixed number. Corollary 4 is obtained as a consequence of known results for the count over all h-free numbers. Finally, Section 4 contains a proof of Theorem 5, which follows from counting integers that are simultaneously h-free and k-full, while Corollary 6 is obtained as a consequence of known results for the count over all h-full numbers.

2. Sums over integers

In this section we prove Theorem 1. We start by recalling the following results involving sums of primes.

Lemma 7. [AE77, Lemma 1.2] *If $s > 1$,*

$$\sum_{p \geq x} \frac{1}{p^s} = \frac{1}{(s-1)x^{s-1}\log x} + O\left(\frac{1}{x^{s-1}\log^2 x}\right).$$

Lemma 8. [AE77, Lemma 1.4] *If $r, s \geq 0$,*

$$\sum_{p \leq x} \frac{p^s}{\log^r p} = \frac{x^{s+1}}{(s+1)\log^{r+1} x} + O\left(\frac{x^{s+1}}{\log^{r+2} x}\right).$$

Proof of Theorem 1. We consider formula (2). Notice that summing over all the numbers of the form $\omega(U_k(n))$ is equivalent to counting the number of powers $p^\ell \leq x$ such that $\ell \geq k$, and each power must be counted with multiplicity equal to the number of $n \leq x$ such that $p^\ell \mid n$. But this is equivalent to counting the multiples of p^k that are less or equal to x. In other words, we have

$$\sum_{n \leq x} \omega(U_k(n)) = \sum_{p^k \leq x} \left\lfloor \frac{x}{p^k} \right\rfloor = \sum_{p \leq x^{\frac{1}{k}}} \frac{x}{p^k} - \sum_{p \leq x^{\frac{1}{k}}} \left\{ \frac{x}{p^k} \right\}.$$

Applying the Prime Number Theorem as well as Lemma 7, we have

$$\begin{aligned}\sum_{n \leq x} \omega(U_k(n)) =& x\sum_p \frac{1}{p^k} - x \sum_{p > x^{\frac{1}{k}}} \frac{1}{p^k} + O_k\left(\frac{x^{\frac{1}{k}}}{\log x}\right)\\ =& x\sum_p \frac{1}{p^k} + O_k\left(\frac{x^{\frac{1}{k}}}{\log x}\right).\end{aligned}$$

Equation (3) is proven similarly. Summing over all the numbers of the form $\Omega(U_k(n))$ is equivalent to counting the number of powers $p^\ell \leq x$ such that $\ell \geq k$, and each power must be counted with multiplicity equal to the number of $n \le x$ such that $p^\ell \mid n$ but $p^{\ell+1} \nmid n$, multiplied by ℓ. Set $t = \lfloor \log_p x \rfloor$. We have

$$\begin{aligned}\sum_{n\le x}\Omega(U_k(n)) =& \sum_{p^k\le x}\sum_{\ell=k}^{t}\ell\left(\left\lfloor\frac{x}{p^\ell}\right\rfloor-\left\lfloor\frac{x}{p^{\ell+1}}\right\rfloor\right)\\ =&\sum_{p^k\le x}\left(k\left\lfloor\frac{x}{p^k}\right\rfloor+\left\lfloor\frac{x}{p^{k+1}}\right\rfloor+\cdots+\left\lfloor\frac{x}{p^t}\right\rfloor\right)\\ =&x\sum_{p^k\le x}\left(\frac{k}{p^k}+\frac{1}{p^{k+1}}+\cdots+\frac{1}{p^t}\right)\\ &-\sum_{p^k\le x}\left(k\left\{\frac{x}{p^k}\right\}+\left\{\frac{x}{p^{k+1}}\right\}+\cdots+\left\{\frac{x}{p^t}\right\}\right)\\ =&x\sum_{p^k\le x}\frac{\frac{1}{p^{k+1}}-\frac{1}{p^{t+1}}}{1-\frac{1}{p}}+\frac{k}{p^k}+O\left(\sum_{p\le x^{\frac{1}{k}}}t\right)\\ =&x\sum_{p^k\le x}\frac{\frac{1-k}{p^{k+1}}-\frac{1}{p^{t+1}}+\frac{k}{p^k}}{1-\frac{1}{p}}+O\left(\log x\sum_{p\le x^{\frac{1}{k}}}\frac{1}{\log p}\right).\end{aligned}$$

Now we use the Prime Number Theorem to estimate

$$x\sum_{p^k\le x}\frac{1}{p^{t+1}(1-\frac{1}{p})}\ll x\sum_{p\le x^{\frac{1}{k}}}\frac{1}{x}\ll_k\frac{x^{\frac{1}{k}}}{\log x}.$$

By applying the above estimate as well as Lemmas 7 and 8 (with $r=1, s=0$), we obtain

$$\begin{aligned}\sum_{n\le x}\Omega(U_k(n)) =&x\sum_{p}\frac{\frac{1-k}{p^{k+1}}+\frac{k}{p^k}}{1-\frac{1}{p}}-x\sum_{p>x^{\frac{1}{k}}}\frac{\frac{1-k}{p^{k+1}}+\frac{k}{p^k}}{1-\frac{1}{p}}+O_k\left(\frac{x^{\frac{1}{k}}}{\log x}\right)\\ =&x\sum_{p}\frac{1-k+kp}{p^{k+1}-p^k}+O_k\left(\frac{x^{\frac{1}{k}}}{\log x}\right).\end{aligned}$$

This conclude the proof of Theorem 1. □

Proof of Corollary 2. To prove (4) and (6) we use the well-known identities [HW08, Theorem 430] and [Fin19, Section 1.4.4]) for $x \ge 2$:

(23)

$$\sum_{n\le x}\omega(n)=x\log\log x+B_1x+O\left(\frac{x}{\log x}\right),\qquad \sum_{n\le x}\Omega(n)=x\log\log x+B_2x+O\left(\frac{x}{\log x}\right),$$

where B_1 and B_2 are given by (5) and (7) respectively.

Notice that $\Omega(n)=\Omega(L_k(n))+\Omega(U_k(n))$ and, since $L_k(n)$ and $U_k(n)$ are coprime, $\omega(n) = \omega(L_k(n))+\omega(U_k(n))$ as well. Combining equations (2) and (3) with (23), we get (4) and (6). □

A perfect power is a number of the form n^h, where $h \geq 2$ and n are positive integers. We can immediately deduce the following result from Theorem 1.

Corollary 9. *Let $k \geq 2$ be an integer. The following formulas hold*

$$\begin{aligned}\sum_{n^h \leq x} \Omega(U_k(n^h)) &= h\left(\sum_p \frac{1-k+kp}{p^{k+1}-p^k}\right) x^{\frac{1}{h}} + O_{k,h}\left(\frac{x^{\frac{1}{hk}}}{\log x}\right),\\ \sum_{n^h \leq x} \omega(U_k(n^h)) &= \left(\sum_p \frac{1}{p^k}\right) x^{\frac{1}{h}} + O_{k,h}\left(\frac{x^{\frac{1}{hk}}}{\log x}\right).\end{aligned}$$

In addition, the following formulas hold

$$\begin{aligned}\sum_{n^h \leq x} \Omega(L_k(n^h)) &= hx^{\frac{1}{h}} \log\log x + h\left(B_2 - \log h - \sum_p \frac{1-k+kp}{p^{k+1}-p^k}\right) x^{\frac{1}{h}} + O_h\left(\frac{x^{\frac{1}{h}}}{\log x}\right),\\ \sum_{n^h \leq x} \omega(L_k(n^h)) &= x^{\frac{1}{h}} \log\log x + \left(B_1 - \log h - \sum_p \frac{1}{p^k}\right) x^{\frac{1}{h}} + O_h\left(\frac{x^{\frac{1}{h}}}{\log x}\right).\end{aligned}$$

Let $\omega_k(n)$ be the number of primes with exponent k in the prime factorization of n.

Corollary 10. *Let $k \geq 1$ be an integer. We have the asymptotic formula*

$$\sum_{n \leq x} \omega_k(n) = \left(\sum_p \frac{p-1}{p^{k+1}}\right) x + O_k\left(\frac{x^{\frac{1}{k}}}{\log x}\right).$$

This recovers a result of Elma and Liu [EL21], who also studied the second moment of ω_k.

Proof. By equation (2), we have

$$\sum_{n \leq x} \omega_k(n) = \sum_{n \leq x} \omega(U_k(n)) - \omega(U_{k+1}(n)) = x\left(\sum_p \frac{1}{p^k} - \sum_p \frac{1}{p^{k+1}}\right) + O_k\left(\frac{x^{\frac{1}{k}}}{\log x}\right),$$

and the result follows. □

Remark 11. *It is interesting to consider the quotient of the sums appearing in formulas* (2) *and* (3)*. We get*

$$\frac{\sum_{n \leq x} \Omega(U_k(n))}{\sum_{n \leq x} \omega(U_k(n))} \to \frac{\sum_p \frac{1-k+kp}{p^{k+1}-p^k}}{\sum_p \frac{1}{p^k}}. \tag{24}$$

Since we have that

$$\frac{k}{p^k} = \frac{k(p-1)}{p^k(p-1)} < \frac{kp-(k-1)}{p^k(p-1)} \leq \frac{(k+1)(p-1)}{p^k(p-1)} = \frac{k+1}{p^k},$$

and the second inequality is strict for $p > 2$, we conclude that the limit (24) *belongs to the interval* $(k, k+1)$.

Remark 12. *The constants appearing in formulas* (2) *and* (3) *can also be expressed as*

$$\sum_p \frac{1}{p^k} = \frac{1}{\zeta(k)} \sum_{U \in \mathcal{N}_k} \prod_{q|U} \left(\frac{q^k - q^{k-1}}{q^k - 1}\right) \frac{\omega(U)}{U} \tag{25}$$

and

$$\sum_{p} \frac{1-k+kp}{p^{k+1}-p^k} = \frac{1}{\zeta(k)} \sum_{U\in\mathcal{N}_k} \prod_{p|n} \left(\frac{q^k-q^{k-1}}{q^k-1}\right) \frac{\Omega(U)}{U}. \tag{26}$$

This can be seen by working with the generating functions, in a method that will be employed to find the constants in Theorems 3 and 5. In fact, (25) *and* (26) *can be obtained from* $D_{\omega,k,h}$ *and* (35) *as well as* $D_{\Omega,k,h}$ *and* (34) *by letting* $h \to \infty$ *and therefore removing the condition* h*-free.*

3. Sums over h-free numbers

In this section we prove Theorem 3. We start with the following estimate for the number of k-free positive integers that are not divisible by some fixed primes.

Lemma 13. *Let* $q_1, \ldots, q_r$ *be prime numbers, and let* $\mathfrak{Q}_{k,q_1\cdots q_r}(x)$ *be the number of* k*-free positive integers not exceeding* x *such that they are relatively prime to* $q_1 \cdots q_r$*. The following formula holds*

$$\mathfrak{Q}_{k,q_1\cdots q_r}(x) = \frac{1}{\zeta(k)} \prod_{j=1}^{r} \frac{\left(1-\frac{1}{q_j}\right)}{\left(1-\frac{1}{q_j^k}\right)} x + O_k\left(2^r x^{\frac{1}{k}}\right).$$

We remark that the above formula generalizes the classical estimate giving

$$Q_k(x) = \frac{x}{\zeta(k)} + O\left(x^{\frac{1}{k}}\right),$$

where $Q_k(x)$ is the number of k-free numbers not exceeding x.

Proof. Consider the modified Möbius function defined as

$$\mu_{q_1\cdots q_r}(d) = \begin{cases} \mu(d) & (d, q_1 \cdots q_r) = 1, \\ 0 & \text{otherwise.} \end{cases}.$$

By Möbius inversion, we have

$$\mathfrak{Q}_{k,q_1\cdots q_r}(x) = \sum_{\substack{n\in\mathcal{S}_k \\ n\leq x \\ (n,q_1\cdots q_r)=1}} 1 = \sum_{\substack{n\leq x \\ (n,q_1\cdots q_r)=1}} \sum_{\substack{d^k|n \\ (d,q_1\cdots q_r)=1}} \mu(d) = \sum_{\substack{n\leq x \\ (n,q_1\cdots q_r)=1}} \sum_{d^k|n} \mu_{q_1\cdots q_r}(d).$$

Writing $n = d^k e$, we have

$$\mathfrak{Q}_{k,q_1\cdots q_r}(x) = \sum_{d^k\leq x} \mu_{q_1\cdots q_r}(d) \sum_{\substack{e\leq x/d^k \\ (e,q_1\cdots q_r)=1}} 1.$$

Estimating the inner sum with inclusion-exclusion, we obtain

$$\begin{aligned}\mathfrak{Q}_{k,q_1\cdots q_r}(x) &= \sum_{d^k\le x}\mu_{q_1\cdots q_r}(d)\left(\left\lfloor\frac{x}{d^k}\right\rfloor - \left\lfloor\frac{x}{q_i d^k}\right\rfloor + \left\lfloor\frac{x}{q_i q_j d^k}\right\rfloor + \cdots\right)\\ &= \sum_{d^k\le x}\mu_{q_1\cdots q_r}(d)\frac{x}{d^k}\prod_{j=1}^{r}\left(1-\frac{1}{q_j}\right)\\ &\quad + O\left(\sum_{d^k\le x}\mu_{q_1\cdots q_r}(d)\left(\left\{\frac{x}{d^k}\right\} - \left\{\frac{x}{q_i d^k}\right\} + \left\{\frac{x}{q_i q_j d^k}\right\} + \cdots\right)\right)\\ &= \sum_{d^k\le x}\mu_{q_1\cdots q_r}(d)\frac{x}{d^k}\prod_{j=1}^{r}\left(1-\frac{1}{q_j}\right) + O\left(2^r\sum_{d^k\le x}1\right).\end{aligned}$$

After using the full sum to estimate, the above becomes,

$$\begin{aligned}&\sum_{d}\mu_{q_1\cdots q_r}(d)\frac{x}{d^k}\prod_{j=1}^{r}\left(1-\frac{1}{q_j}\right) - \sum_{d^k>x}\mu_{q_1\cdots q_r}(d)\frac{x}{d^k}\prod_{j=1}^{r}\left(1-\frac{1}{q_j}\right) + O\left(2^r x^{\frac{1}{k}}\right)\\ =&x\prod_{p\ne q_j}\left(1-\frac{1}{p^k}\right)\prod_{j=1}^{r}\left(1-\frac{1}{q_j}\right) + O\left(x\sum_{d^k>x}\frac{1}{d^k}\right) + O\left(2^r x^{\frac{1}{k}}\right).\end{aligned}$$

Estimating the first big-O term by approximating with an integral, we obtain $O_k(x^{\frac{1}{k}})$, and this yields

$$\begin{aligned}\mathfrak{Q}_{k,q_1\cdots q_r}(x) =&x\prod_{p}\left(1-\frac{1}{p^k}\right)\prod_{j=1}^{r}\frac{\left(1-\frac{1}{q_j}\right)}{\left(1-\frac{1}{q_j^k}\right)} + O_k\left(2^r x^{\frac{1}{k}}\right)\\ =&\frac{1}{\zeta(k)}\prod_{j=1}^{r}\frac{\left(1-\frac{1}{q_j}\right)}{\left(1-\frac{1}{q_j^k}\right)}x + O_k\left(2^r x^{\frac{1}{k}}\right).\end{aligned}$$

□

We now state some results involving sums of prime factor counting functions over h-full numbers that will be needed for the proof of Theorem 3 and Corollary 4.

Theorem 14. *Let $h\ge 1$ be an integer. We have*

$$\sum_{\substack{n\in\mathcal{N}_h\\ n\le x}}\Omega(n) = h\gamma_{0,h}x^{\frac{1}{h}}\log\log x + \gamma_{0,h}C_{\Omega,h}x^{\frac{1}{h}} + O_h\left(\frac{x^{\frac{1}{h}}}{\sqrt{\log x}}\right), \tag{27}$$

where $\gamma_{0,h}$ is given by (16), *$C_{\Omega,h}$ is given by* (21), *and*

$$\sum_{\substack{n\in\mathcal{N}_h\\ n\le x}}\omega(n) = \gamma_{0,h}x^{\frac{1}{h}}\log\log x + \gamma_{0,h}C_{\omega,h}x^{\frac{1}{h}} + O_h\left(\frac{x^{\frac{1}{h}}}{\sqrt{\log x}}\right), \tag{28}$$

where $C_{\omega,h}$ is given by (22).

Proof. Equation (27) was proven in [JL, Theorem 2]. Equation (28) can be proven similarly. □

Lemma 15. *Let $\alpha \in \mathbb{R}$. Then, we have*

$$\sum_{\substack{n\in\mathcal{N}_h\\ x<n\leq y}} \Omega(n)n^{\alpha} = O_h\left(y^{\frac{1}{h}+\alpha}\log\log y\right) + O_h\left(x^{\frac{1}{h}+\alpha}\log\log x\right) \tag{29}$$

and

$$\sum_{\substack{n\in\mathcal{N}_h\\ x<n\leq y}} \omega(n)n^{\alpha} = O_h\left(y^{\frac{1}{h}+\alpha}\log\log y\right) + O_h\left(x^{\frac{1}{h}+\alpha}\log\log x\right). \tag{30}$$

Proof. Denote

$$\mathscr{N}_h(x) = \sum_{\substack{n\in\mathcal{N}_h\\ n\leq x}} \Omega(n),$$

and remark that the asymptotics for $\mathscr{N}_h(x)$ is given by (27).

By Abel's summation formula,

$$\begin{aligned}
\sum_{\substack{n\in\mathcal{N}_h\\ x<n\leq y}} \Omega(n)n^{\alpha} =& \mathscr{N}_h(y)y^{\alpha} - \mathscr{N}_h(x)x^{\alpha} - \alpha\int_x^y \mathscr{N}_h(t)t^{\alpha-1}dt\\
=& h\gamma_{0,h}y^{\frac{1}{h}+\alpha}\log\log y - h\gamma_{0,h}x^{\frac{1}{h}+\alpha}\log\log x\\
&+ O\left(y^{\frac{1}{h}+\alpha}\right) + O\left(x^{\frac{1}{h}+\alpha}\right) + O\left(\int_x^y t^{\alpha-\frac{h-1}{h}}\log\log t dt\right)\\
=& O_h\left(y^{\frac{1}{h}+\alpha}\log\log y\right) + O_h\left(x^{\frac{1}{h}+\alpha}\log\log x\right).
\end{aligned}$$

Estimate (30) is proven in the same way, using (28) instead. □

Proof of Theorem 3. We prove equations (8) and (10). Fix $0 < B \leq x$ (to be determined later) and suppose that $U = U_k(n)$ is such that $U \leq B$. We start by counting all the possible values of $L = L_k(n)$ satisfying $L \leq x/U$. By Lemma 13, the number of possible values of L is given by

$$\mathfrak{Q}_{k,q_1\cdots q_r}\left(\frac{x}{U}\right) = \frac{1}{\zeta(k)}\prod_{j=1}^{r}\left(\frac{q_j^k - q_j^{k-1}}{q_j^k - 1}\right)\frac{x}{U} + O\left(2^r\frac{x^{\frac{1}{k}}}{U^{\frac{1}{k}}}\right), \tag{31}$$

where $q_1, \dots, q_r$ are the primes in the factorization of U. Thus we have

$$\sum_{\substack{n\in\mathcal{S}_h\\ n\le x}} \Omega(U_k(n)) = \sum_{\substack{n\in\mathcal{S}_h\\ n\le x\\ U_k(n)\le B}} \Omega(U_k(n)) + \sum_{\substack{n\in\mathcal{S}_h\\ n\le x\\ B<U_k(n)\le x}} \Omega(U_k(n))$$

$$= \frac{x}{\zeta(k)} \sum_{\substack{U\in\mathcal{N}_k\cap\mathcal{S}_h\\ U\le B}} \prod_{q|U} \left(\frac{q^k - q^{k-1}}{q^k-1}\right) \frac{\Omega(U)}{U} + O\left(\sum_{\substack{U\in\mathcal{N}_k\cap\mathcal{S}_h\\ U\le B}} \Omega(U) 2^{\omega(U)} \frac{x^{\frac{1}{k}}}{U^{\frac{1}{k}}} \right)$$

$$+ \sum_{\substack{n\in\mathcal{S}_h\\ n\le x\\ B<U_k(n)\le x}} \Omega(U_k(n)).$$

Notice that for $U \in \mathcal{N}_k$, we have $2^{\omega(U)} \le q_1 \cdots q_r \le U^{\frac{1}{k}}$. Using this to bound the error term gives

$$\sum_{\substack{n\in\mathcal{S}_h\\ n\le x}} \Omega(U_k(n)) = \frac{x}{\zeta(k)} \sum_{U\in\mathcal{N}_k\cap\mathcal{S}_h} \prod_{q|U} \left(\frac{q^k - q^{k-1}}{q^k-1}\right) \frac{\Omega(U)}{U} + O\left(x^{\frac{1}{k}} \sum_{\substack{U\in\mathcal{N}_k\cap\mathcal{S}_h\\ U\le B}} \Omega(U) \right)$$

$$+ \sum_{\substack{n\in\mathcal{S}_h\\ n\le x\\ B<U_k(n)\le x}} \Omega(U_k(n)) - \frac{x}{\zeta(k)} \sum_{\substack{U\in\mathcal{N}_k\cap\mathcal{S}_h\\ B<U}} \prod_{q|U} \left(\frac{q^k - q^{k-1}}{q^k-1}\right) \frac{\Omega(U)}{U}. \tag{32}$$

We have the following estimate

$$\sum_{\substack{n\in\mathcal{S}_h\\ n\le x\\ B<U_k(n)\le x}} \Omega(U_k(n)) \le \sum_{\substack{U\in\mathcal{N}_k\cap\mathcal{S}_h\\ B<U\le x}} \left\lfloor \frac{x}{U} \right\rfloor \Omega(U) \le \sum_{\substack{U\in\mathcal{N}_k\\ U\le x}} \frac{x}{U}\Omega(U). \tag{33}$$

Applying Lemma 15 to (32) and (33), we have

$$\sum_{\substack{n\in\mathcal{S}_h\\ n\le x}} \Omega(U_k(n)) = \frac{x}{\zeta(k)} \sum_{U\in\mathcal{N}_k\cap\mathcal{S}_h} \prod_{q|U} \left(\frac{q^k - q^{k-1}}{q^k-1}\right) \frac{\Omega(U)}{U} + O_h\left(x^{\frac{1}{k}} B^{\frac{1}{k}} \log\log B \right)$$

$$+ O_h\left(x^{\frac{1}{k}} \log\log x \right) + O_h\left(x B^{\frac{1}{k}-1} \log\log B \right).$$

Let $B = x^{1-\frac{1}{k}}$. We get

$$\sum_{\substack{n\in\mathcal{S}_h\\ n\le x}} \Omega(U_k(n)) = \frac{x}{\zeta(k)} \sum_{U\in\mathcal{N}_k\cap\mathcal{S}_h} \prod_{q|U} \left(\frac{q^k - q^{k-1}}{q^k-1}\right) \frac{\Omega(U)}{U} + O_h\left(x^{\frac{2k-1}{k^2}} \log\log x \right).$$

We now proceed to find a closed expression for

$$\frac{1}{\zeta(k)} \sum_{U\in\mathcal{N}_k\cap\mathcal{S}_h} \prod_{q|U} \left(\frac{q^k - q^{k-1}}{q^k-1}\right) \frac{\Omega(U)}{U}. \tag{34}$$

We consider a generating function given by

$$\begin{aligned}\mathcal{D}_{\Omega,k,h}(z) &= \sum_{n\in\mathcal{N}_k\cap\mathcal{S}_h} \frac{z^{\Omega(n)}}{n}\prod_{q|n}\frac{q^k-q^{k-1}}{q^k-1}\\ &=\prod_p\left(1+\left(\frac{p^k-p^{k-1}}{p^k-1}\right)\frac{z^k}{p^k}\left(1+\frac{z}{p}+\cdots+\frac{z^{h-k-1}}{p^{h-k-1}}\right)\right)\\ &=\prod_p\left(1+\left(\frac{p^k-p^{k-1}}{p^k-1}\right)\frac{\frac{z^h}{p^h}-\frac{z^k}{p^k}}{\frac{z}{p}-1}\right),\end{aligned}$$

which is absolutely convergent over compact sets.

We will recover our term of interest from considering $\mathcal{D}'_{\Omega,k,h}(1)$. In order to find this term, we consider the logarithmic derivative of $\mathcal{D}_{\Omega,k,h}(z)$:

$$\frac{\partial}{\partial z}\frac{\mathcal{D}'_{\Omega,k,h}(z)}{\mathcal{D}_{\Omega,k,h}(z)} = \sum_p \frac{\left(\frac{p^k-p^{k-1}}{p^k-1}\right)\left((h-1)\frac{z^h}{p^{h+1}}-(k-1)\frac{z^k}{p^{k+1}}-h\frac{z^{h-1}}{p^h}+k\frac{z^{k-1}}{p^k}\right)}{\left(\frac{z}{p}-1\right)^2\left(1+\left(\frac{p^k-p^{k-1}}{p^k-1}\right)\frac{\frac{z^h}{p^h}-\frac{z^k}{p^k}}{\frac{z}{p}-1}\right)}.$$

Evaluating at $z=1$, we obtain,

$$\left.\frac{\partial}{\partial z}\frac{\mathcal{D}'_{\Omega,k,h}(z)}{\mathcal{D}_{\Omega,k,h}(z)}\right|_{z=1} = \sum_p \frac{\left(\frac{p^k}{p^k-1}\right)\left(\frac{h-1}{p^{h+1}}-\frac{k-1}{p^{k+1}}-\frac{h}{p^h}+\frac{k}{p^k}\right)}{\left(1-\frac{1}{p}\right)\left(1-\frac{p^k}{p^k-1}\left(\frac{1}{p^h}-\frac{1}{p^k}\right)\right)}.$$

Multiplying the above by $\mathcal{D}_{\Omega,k,h}(1)$ and by the coefficient $\frac{1}{\zeta(k)}=\prod_p\left(1-\frac{1}{p^k}\right)$ provides the coefficient for the main term of (8):

$$\begin{aligned}\frac{\mathcal{D}'_{\Omega,k,h}(1)}{\zeta(k)} &= \frac{1}{\zeta(k)}\sum_p \frac{\left(\frac{p^k}{p^k-1}\right)\left(\frac{h-1}{p^{h+1}}-\frac{k-1}{p^{k+1}}-\frac{h}{p^h}+\frac{k}{p^k}\right)}{\left(1-\frac{1}{p}\right)\left(1-\frac{p^k}{p^k-1}\left(\frac{1}{p^h}-\frac{1}{p^k}\right)\right)}\prod_p\left(1-\left(\frac{p^k}{p^k-1}\right)\left(\frac{1}{p^h}-\frac{1}{p^k}\right)\right)\\ &=\sum_p\frac{\frac{h-1}{p^{h+1}}-\frac{k-1}{p^{k+1}}-\frac{h}{p^h}+\frac{k}{p^k}}{\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^h}\right)}\prod_p\left(1-\frac{1}{p^h}\right)\\ &=\frac{1}{\zeta(h)}\sum_p\frac{h-1-(k-1)p^{h-k}-hp+kp^{h-k+1}}{(p-1)\left(p^h-1\right)}.\end{aligned}$$

Equations (9) and (11) are proven analogously. Here the difference is that we must consider instead

$$\frac{1}{\zeta(k)}\sum_{U\in\mathcal{N}_k\cap\mathcal{S}_h}\prod_{q|U}\left(\frac{q^k-q^{k-1}}{q^k-1}\right)\frac{\omega(U)}{U}. \tag{35}$$

In this case the generating function is given by

$$\begin{aligned}\mathcal{D}_{\omega,k,h}(z) =& \sum_{n\in\mathcal{N}_k\cap\mathcal{S}_h} \frac{z^{\omega(n)}}{n}\prod_{q|n}\frac{q^k-q^{k-1}}{q^k-1}\\ =&\prod_p\left(1+\left(\frac{p^k-p^{k-1}}{p^k-1}\right)\frac{z}{p^k}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{h-k-1}}\right)\right)\\ =&\prod_p\left(1+\left(\frac{p^k-p^{k-1}}{p^k-1}\right)\frac{z\left(\frac{1}{p^h}-\frac{1}{p^k}\right)}{\frac{1}{p}-1}\right),\end{aligned}$$

which is absolutely convergent.

In order to find $\mathcal{D}'_{\omega,k,h}(1)$, we consider the logarithmic derivative:

$$\frac{\partial}{\partial z}\frac{\mathcal{D}'_{\omega,k,h}(z)}{\mathcal{D}_{\omega,k,h}(z)} = \sum_p \frac{\left(\frac{p^k-p^{k-1}}{p^k-1}\right)\frac{\left(\frac{1}{p^h}-\frac{1}{p^k}\right)}{\frac{1}{p}-1}}{1+\left(\frac{p^k-p^{k-1}}{p^k-1}\right)\frac{z\left(\frac{1}{p^h}-\frac{1}{p^k}\right)}{\frac{1}{p}-1}}.$$

Therefore,

$$\left.\frac{\partial}{\partial z}\frac{\mathcal{D}'_{\omega,k,h}(z)}{\mathcal{D}_{\omega,k,h}(z)}\right|_{z=1} = \sum_p \frac{p^{h-k}-1}{p^h-1}.$$

Multiplying the above by $\mathcal{D}_{\omega,k,h}(1)$ and by the coefficient $\frac{1}{\zeta(k)} = \prod_p\left(1-\frac{1}{p^k}\right)$ yields the coefficient for the main term of (9):

$$\begin{aligned}\frac{\mathcal{D}'_{\omega,k,h}(1)}{\zeta(k)} =& \frac{1}{\zeta(k)}\sum_p\frac{p^{h-k}-1}{p^h-1}\prod_p\left(1-\left(\frac{p^k}{p^k-1}\right)\left(\frac{1}{p^h}-\frac{1}{p^k}\right)\right)\\ =&\sum_p\frac{p^{h-k}-1}{p^h-1}\prod_p\left(1-\frac{1}{p^h}\right)\\ =&\frac{1}{\zeta(h)}\sum_p\frac{p^{h-k}-1}{p^h-1}.\end{aligned}$$

This concludes the proof of Theorem 3. □

Theorem 16. *The following asymptotic formula hold*

$$\sum_{\substack{n\in\mathcal{S}_h\\ n\le x}}\Omega(n) = \frac{1}{\zeta(h)}x\log\log x + O(x), \tag{36}$$

and

$$\sum_{\substack{n\in\mathcal{S}_h\\ n\le x}}\omega(n) = \frac{1}{\zeta(h)}x\log\log x + O(x). \tag{37}$$

Proof. Equation (36) was proven in [JL, Theorem 1]. Equation (37) can be proven similarly. □

Proof of Corollary 4. Since $n = L_k(n)U_k(n)$, we have $\Omega(n) = \Omega(L_k(n)) + \Omega(U_k(n))$, and similarly with ω (since $L_k(n)$ and $U_k(n)$ are coprime). Combining equations (8) and (36), we immediately obtain equation (12). Equation (13) follows by combining equations (9) and (37).

□

4. Sums over h-full numbers

In this section we prove Theorem 5. Before proceeding to the proof, we need the following generalization of Lemma 13.

Lemma 17. *Let $q_1, \dots, q_r$ be prime numbers and let $k > h$ be integers. Set $\mathfrak{Q}_{k,h,q_1\cdots q_r}(x)$ to be the number of k-free, h-full positive integers not exceeding x such that they are relatively prime to $q_1 \cdots q_r$. The following formula holds*

$$\mathfrak{Q}_{k,h,q_1\cdots q_r}(x) = \prod_{j=1}^{r}\left(1+\frac{\frac{1}{q_j}-\frac{1}{q_j^{\frac{k}{h}}}}{1-\frac{1}{q_j^{\frac{1}{h}}}}\right)^{-1}\prod_p\left(1-\frac{1}{p}\right)\left(1+\frac{\frac{1}{p}-\frac{1}{p^{\frac{k}{h}}}}{1-\frac{1}{p^{\frac{1}{h}}}}\right)x^{\frac{1}{h}} + O\left(2^r x^{\frac{2h+1}{2h(h+1)}+\varepsilon}\right),$$

where $\varepsilon > 0$ is arbitrarily small.

Proof. Consider the generating function

$$\begin{aligned}\sum_{\substack{n\in\mathcal{N}_h\cap\mathcal{S}_k\\(n,q_1\cdots q_r)=1}}\frac{1}{n^s} &= \prod_{p\neq q_j}\left(1+\frac{1}{p^{sh}}+\cdots+\frac{1}{p^{s(k-1)}}\right)\\ &= \prod_{p\neq q_j}\left(1+\frac{\frac{1}{p^{sh}}-\frac{1}{p^{sk}}}{1-\frac{1}{p^s}}\right)\\ &= \prod_{j=1}^{r}\left(1+\frac{1}{q_j^{sh}}\right)^{-1}\prod_p\left(1+\frac{1}{p^{sh}}\right)\prod_{p\neq q_j}\left(1+\frac{\frac{1}{p^{s(h+1)}}-\frac{1}{p^{sk}}}{\left(1-\frac{1}{p^s}\right)\left(1+\frac{1}{p^{sh}}\right)}\right)\\ &= \prod_{j=1}^{r}\left(1+\frac{1}{q_j^{sh}}\right)^{-1}\frac{\zeta(sh)}{\zeta(2sh)}\mathcal{H}_{q_1\cdots q_r}(s).\end{aligned}$$

Notice that for $\mathrm{Re}(s) > \frac{1}{h+1}$,

$$|\mathcal{H}_{q_1\cdots q_r}(s)| \le \prod_{p\neq q_j}\left(1+\left|\frac{\frac{1}{p^{s(h+1)}}-\frac{1}{p^{sk}}}{\left(1-\frac{1}{p^s}\right)\left(1+\frac{1}{p^{sh}}\right)}\right|\right) \le \prod_p\left(1+\left|\frac{\frac{1}{p^{s(h+1)}}-\frac{1}{p^{sk}}}{\left(1-\frac{1}{p^s}\right)\left(1+\frac{1}{p^{sh}}\right)}\right|\right), \tag{38}$$

which is convergent for $\mathrm{Re}(s) \ge \frac{1}{h+1}+\varepsilon$, and therefore $\mathcal{H}_{q_1\cdots q_r}(s)$ is convergent for $\mathrm{Re}(s) > \frac{1}{h+1}$. Now we use Perron's formula ([MV07, Section 5.1], [Mur08, Section 4.4], more precisely, Problems 4.4.15-4.4.17). Take $\sigma_0 = \frac{1}{h} + \varepsilon$. As $T \to \infty$,

$$\begin{aligned}\mathfrak{Q}_{k,h,q_1\cdots q_r}(x) = &\sum_{\substack{n\in\mathcal{N}_h\cap\mathcal{S}_k\\ n\le x\\ (n,q_1\cdots q_r)=1}} 1\\ =&\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\prod_{j=1}^{r}\left(1+\frac{1}{q_j^{sh}}\right)^{-1}\frac{\zeta(sh)}{\zeta(2sh)}\mathcal{H}_{q_1\cdots q_r}(s)\frac{x^s}{s}ds + O\left(\frac{x^{\sigma_0+\varepsilon}}{T}\right).\end{aligned}$$

To compute this integral we consider the rectangle of vertical sides $[\sigma_0 - iT, \sigma_0 + iT]$ and $[\sigma_1 - iT, \sigma_1 + iT]$ and horizontal sides $[\sigma_0 \pm iT, \sigma_1 \pm iT]$. The integral over the sides is equal the residue from the pole at $s = \frac{1}{h}$, which can be computed as follows:

$$\begin{aligned}&\prod_{j=1}^{r}\left(1+\frac{1}{q_j}\right)^{-1}\frac{h}{\zeta(2)}\mathcal{H}_{q_1\cdots q_r}\left(\frac{1}{h}\right)x^{\frac{1}{h}}\mathrm{Res}_{s=\frac{1}{h}}\zeta(sh)\\ =&\prod_{j=1}^{r}\left(1+\frac{\frac{1}{q_j}-\frac{1}{q_j^{\frac{k}{h}}}}{1-\frac{1}{q_j^{\frac{1}{h}}}}\right)^{-1}\frac{1}{\zeta(2)}\prod_p\left(1+\frac{\frac{1}{p^{\frac{1}{h}+1}}-\frac{1}{p^{\frac{k}{h}}}}{\left(1-\frac{1}{p^{\frac{1}{h}}}\right)\left(1+\frac{1}{p}\right)}\right)x^{\frac{1}{h}}.\end{aligned}$$

Since we are interested in the integral over the segment $[\sigma_0 - iT, \sigma_0 + iT]$, we proceed to bound the integral at the vertical segment $[\sigma_1 - iT, \sigma_1 + iT]$ and at the horizontal lines $[\sigma_0 \pm iT, \sigma_1 \pm iT]$. First we note that inequality (38) gives a uniform bound for $\mathcal{H}_{q_1\cdots q_r}(s)$ which is independent of the choice of $q_1, \ldots, q_r$. Next notice that we have, over the same segments,

$$\left|1+\frac{1}{q^{sh}}\right|^{-1} \le \frac{1}{1-\frac{1}{q^{\mathrm{Re}(s)h}}} \le \frac{1}{1-\frac{1}{q^{\frac{h}{h+1}}}} \le \frac{1}{1-\frac{1}{q^{\frac{1}{2}}}},$$

and the above bound is less or equal than 2 when $q \neq 2, 3$, and for $q = 2, 3$ it is bounded by 4 and 3 respectively. Thus, we have the following bound over the vertical segment $[\sigma_1 - iT, \sigma_1 + iT]$ and at the horizontal lines $[\sigma_0 \pm iT, \sigma_1 \pm iT]$:

$$\left|\prod_{j=1}^{r}\left(1+\frac{1}{q_j^{sh}}\right)^{-1}\right| < 12\cdot 2^r.$$

Since $\zeta(\sigma \pm iT) = O\left(T^{\frac{1}{2}}\right)$ uniformly for $\varepsilon \le \sigma \le 1$ as $T \to \infty$ (see for example, [Ivi85, Theorem 1.9]), the horizontal integrals on $[\sigma_0 \pm iT, \sigma_1 \pm iT]$ contribute $O\left(2^r\frac{x^{\sigma_0}T^{-\frac{1}{2}}}{\log x}\right)$.

The vertical line $[\sigma_1 - iT, \sigma_1 + iT]$ contributes to $O\left(2^r x^{\sigma_1}T^{\frac{1}{2}}\right)$.

Finally, taking $T = x^{\frac{1}{h(h+1)}}$ gives a final estimate of

$$\mathfrak{Q}_{k,h,q_1\cdots q_r}(x) = \prod_{j=1}^{r}\left(1+\frac{\frac{1}{q_j}-\frac{1}{q_j^{\frac{k}{h}}}}{1-\frac{1}{q_j^{\frac{1}{h}}}}\right)^{-1}\frac{1}{\zeta(2)}\prod_{p}\left(1+\frac{\frac{1}{p^{\frac{1}{h}+1}}-\frac{1}{p^{\frac{k}{h}}}}{\left(1-\frac{1}{p^{\frac{1}{h}}}\right)\left(1+\frac{1}{p}\right)}\right)x^{\frac{1}{h}}$$
$$+O\left(2^r x^{\frac{2h+1}{2h(h+1)}+\varepsilon}\right).$$

□

Remark that the main term in Lemma 17 reduces to the main term in Lemma 13 when $h=1$. However, the error term has size $O\left(2^r x^{\frac{3}{4}+\varepsilon}\right)$ and is worse. The reason for this is that we are we are only considering the pole at $s=\frac{1}{h}$ in Perron's formula. To eliminate the dependence on h we would need to remove all the poles up to $\frac{1}{k}$.

Another interesting case is when $k\to\infty$ and $r=0$. This counts the h-full numbers not exceeding x and recovers the formula

$$\gamma_{0,h}x^{\frac{1}{h}}+O\left(x^{\frac{2h+1}{2h(h+1)}+\varepsilon}\right).$$

This is a much weaker version of the result of Ivić and Shiu [IS82], who estimate this number to be

$$\gamma_{0,h}x^{\frac{1}{h}}+\gamma_{1,h}x^{\frac{1}{h+1}}+\cdots+\gamma_{h-1,h}x^{\frac{1}{2h-1}}+\Delta_h(x),$$

where $\gamma_{0,h},\gamma_{1,h}\ldots,\gamma_{h-1,h}$ are certain computable constants and $\Delta_h(x)\ll x^{\rho}$ for ρ small.

Proof of Theorem 5. First, we proceed to prove equations (14) and (17). Fix $0<B\le x$ (to be determined later) and suppose that $U=U_k(n)$ is such that $U\le B$. We start by counting all the possible $L=L_k(n)$ satisfying $L\le x/U$. Since L must be both k-free and h-full, Lemma 17 implies that the number of possible values of L is given by

$$\mathfrak{Q}_{k,h,q_1\cdots q_r}\left(\frac{x}{U}\right) = \prod_{j=1}^{r}\left(1+\frac{\frac{1}{q_j}-\frac{1}{q_j^{\frac{k}{h}}}}{1-\frac{1}{q_j^{\frac{1}{h}}}}\right)^{-1}\prod_{p}\left(1-\frac{1}{p}\right)\left(1+\frac{\frac{1}{p}-\frac{1}{p^{\frac{k}{h}}}}{1-\frac{1}{p^{\frac{1}{h}}}}\right)\frac{x^{\frac{1}{h}}}{U^{\frac{1}{h}}}$$
$$+O\left(2^r\frac{x^{\frac{2h+1}{2h(h+1)}+\varepsilon}}{U^{\frac{2h+1}{2h(h+1)}+\varepsilon}}\right), \tag{39}$$

where $q_1,\ldots,q_r$ are the primes in the factorization of U.

To make the proof easier to follow, we define

$$f(k,h) := \prod_{p}\left(1-\frac{1}{p}\right)\left(1+\frac{\frac{1}{p}-\frac{1}{p^{\frac{k}{h}}}}{1-\frac{1}{p^{\frac{1}{h}}}}\right).$$

Thus we have

$$\begin{aligned}\sum_{\substack{n\in\mathcal{N}_h\\ n\leq x}}\Omega(U_k(n)) =& \sum_{\substack{n\in\mathcal{N}_h\\ n\leq x\\ U_k(n)\leq B}}\Omega(U_k(n)) + \sum_{\substack{n\in\mathcal{N}_h\\ n\leq x\\ B<U_k(n)\leq x}}\Omega(U_k(n))\\ =&f(k,h)x^{\frac{1}{h}}\sum_{\substack{U\in\mathcal{N}_k\\ U\leq B}}\prod_{q\mid U}\left(1+\frac{\frac{1}{q}-\frac{1}{q^{\frac{k}{h}}}}{1-\frac{1}{q^{\frac{1}{h}}}}\right)^{-1}\frac{\Omega(U)}{U^{\frac{1}{h}}}\\ &+O\left(\sum_{\substack{U\in\mathcal{N}_k\\ U\leq B}}2^{\omega(U)}\Omega(U)\frac{x^{\frac{2h+1}{2h(h+1)}+\varepsilon}}{U^{\frac{2h+1}{2h(h+1)}+\varepsilon}}\right)+\sum_{\substack{n\in\mathcal{N}_h\\ n\leq x\\ B<U_k(n)\leq x}}\Omega(U_k(n)).\end{aligned}$$

Notice that for $U\in\mathcal{N}_k$, we have $2^{\omega(U)}\leq q_1\cdots q_r\leq U^{\frac{1}{k}}$. Using this to bound the error term above gives

$$\begin{aligned}\sum_{\substack{n\in\mathcal{N}_h\\ n\leq x}}\Omega(U_k(n)) =&f(k,h)x^{\frac{1}{h}}\sum_{U\in\mathcal{N}_k}\prod_{q\mid U}\left(1+\frac{\frac{1}{q}-\frac{1}{q^{\frac{k}{h}}}}{1-\frac{1}{q^{\frac{1}{h}}}}\right)^{-1}\frac{\Omega(U)}{U^{\frac{1}{h}}}\\ &+O\left(x^{\frac{2h+1}{2h(h+1)}+\varepsilon}\sum_{\substack{U\in\mathcal{N}_k\\ U\leq B}}\Omega(U)U^{\frac{1}{k}-\frac{2h+1}{2h(h+1)}-\varepsilon}\right)\\ &+\sum_{\substack{n\in\mathcal{N}_h\\ n\leq x\\ B<U_k(n)\leq x}}\Omega(U_k(n))-f(k,h)x^{\frac{1}{h}}\sum_{\substack{U\in\mathcal{N}_k\\ B<U}}\prod_{q\mid U}\left(1+\frac{\frac{1}{q}-\frac{1}{q^{\frac{k}{h}}}}{1-\frac{1}{q^{\frac{1}{h}}}}\right)^{-1}\frac{\Omega(U)}{U^{\frac{1}{h}}}.\end{aligned}\tag{40}$$

We have the following estimate, analogous to (33):

$$\sum_{\substack{n\in\mathcal{N}_h\\ n\leq x\\ B<U_k(n)\leq x}}\Omega(U_k(n))\leq\sum_{\substack{U\in\mathcal{N}_k\\ B<U\leq x}}\left\lfloor\frac{x}{U}\right\rfloor\Omega(U)\leq\sum_{\substack{U\in\mathcal{N}_k\\ U\leq x}}\frac{x}{U}\Omega(U).\tag{41}$$

Applying Lemma 15 to (40) and (41), we have

$$\begin{aligned}\sum_{\substack{n\in\mathcal{N}_h\\ n\leq x}}\Omega(U_k(n)) =&f(k,h)x^{\frac{1}{h}}\sum_{U\in\mathcal{N}_k}\prod_{q\mid U}\left(1+\frac{\frac{1}{q}-\frac{1}{q^{\frac{k}{h}}}}{1-\frac{1}{q^{\frac{1}{h}}}}\right)^{-1}\frac{\Omega(U)}{U^{\frac{1}{h}}}\\ &+O\left(x^{\frac{2h+1}{2h(h+1)}+\varepsilon}B^{\frac{2}{k}-\frac{2h+1}{2h(h+1)}-\varepsilon}\log\log B\right)\\ &+O\left(x^{\frac{1}{k}}\log\log x\right)+O\left(x^{\frac{1}{h}}B^{\frac{1}{k}-\frac{1}{h}}\log\log B\right).\end{aligned}$$

We choose $B = x^{\frac{k}{k+2h(h+1)}}$ and get

$$\sum_{\substack{n\in\mathcal{N}_h\\ n\le x}} \Omega(U_k(n)) = f(k,h)x^{\frac{1}{h}} \sum_{U\in\mathcal{N}_k} \prod_{q|U} \left(1 + \frac{\frac{1}{q} - \frac{1}{q^{\frac{k}{h}}}}{1 - \frac{1}{q^{\frac{1}{h}}}}\right)^{-1} \frac{\Omega(U)}{U^{\frac{1}{h}}} + O\left(x^{\frac{1}{h} - \left(\frac{k}{h}-1\right)\frac{1}{k+2h(h+1)}+\varepsilon} \log\log x\right).$$

We now proceed to find a closed expression for

$$f(k,h) \sum_{U\in\mathcal{N}_k} \prod_{q|U} \left(1 + \frac{\frac{1}{q} - \frac{1}{q^{\frac{k}{h}}}}{1 - \frac{1}{q^{\frac{1}{h}}}}\right)^{-1} \frac{\Omega(U)}{U^{\frac{1}{h}}}.$$

We consider a generating function given by

$$\begin{aligned}
\mathcal{E}_{\Omega,k,h}(z) &= \sum_{n\in\mathcal{N}_k} \frac{z^{\Omega(n)}}{n^{\frac{1}{h}}} \prod_{q|n} \left(1 + \frac{\frac{1}{q} - \frac{1}{q^{\frac{k}{h}}}}{1 - \frac{1}{q^{\frac{1}{h}}}}\right)^{-1} \\
&= \prod_p \left(1 + \left(1 + \frac{\frac{1}{p} - \frac{1}{p^{\frac{k}{h}}}}{1 - \frac{1}{p^{\frac{1}{h}}}}\right)^{-1} \frac{z^k}{p^{\frac{k}{h}}} \left(1 + \frac{z}{p^{\frac{1}{h}}} + \frac{z^2}{p^{\frac{2}{h}}} + \cdots\right)\right) \\
&= \prod_p \left(1 + \left(1 + \frac{\frac{1}{p} - \frac{1}{p^{\frac{k}{h}}}}{1 - \frac{1}{p^{\frac{1}{h}}}}\right)^{-1} \frac{\frac{z^k}{p^{\frac{k}{h}}}}{1 - \frac{z}{p^{\frac{1}{h}}}}\right),
\end{aligned}$$

which is absolutely convergent over compact sets.

We will recover our term of interest by computing $\mathcal{E}'_{\Omega,k,h}(1)$, which we find by considering the logarithmic derivative:

$$\frac{\partial}{\partial z} \frac{\mathcal{E}'_{\Omega,k,h}(z)}{\mathcal{E}_{\Omega,k,h}(z)} = \sum_p \frac{\frac{\frac{kz^{k-1}}{p^{\frac{k}{h}}} - \frac{(k-1)z^k}{p^{\frac{k+1}{h}}}}{\left(1 - \frac{z}{p^{\frac{1}{h}}}\right)^2}}{\left(1 + \frac{\frac{1}{p} - \frac{1}{p^{\frac{k}{h}}}}{1 - \frac{1}{p^{\frac{1}{h}}}}\right) + \frac{\frac{z^k}{p^{\frac{k}{h}}}}{1 - \frac{z}{p^{\frac{1}{h}}}}}.$$

Therefore,

$$\frac{\partial}{\partial z} \frac{\mathcal{E}'_{\Omega,k,h}(z)}{\mathcal{E}_{\Omega,k,h}(z)}\bigg|_{z=1} = \sum_p \frac{\frac{k}{p^{\frac{k}{h}}} - \frac{k-1}{p^{\frac{k+1}{h}}}}{\left(1 - \frac{1}{p^{\frac{1}{h}}}\right)\left(1 + \frac{1}{p} - \frac{1}{p^{\frac{1}{h}}}\right)}.$$

By multiplying the above by $\mathcal{E}_{\Omega,k,h}(1)$ and by the coefficient $f(k,h)$, we get an expression for $E_{\Omega,k,h}$:

$$
\begin{aligned}
f(k,h)\mathcal{E}'_{\Omega,k,h}(1) =& f(k,h)\sum_p \frac{\frac{k}{p^{\frac{k}{h}}}-\frac{k-1}{p^{\frac{k+1}{h}}}}{\left(1-\frac{1}{p^{\frac{1}{h}}}\right)\left(1+\frac{1}{p}-\frac{1}{p^{\frac{1}{h}}}\right)}\prod_p\left(1+\left(1+\frac{\frac{1}{p}-\frac{1}{p^{\frac{k}{h}}}}{1-\frac{1}{p^{\frac{1}{h}}}}\right)^{-1}\frac{\frac{1}{p^{\frac{k}{h}}}}{1-\frac{1}{p^{\frac{1}{h}}}}\right)\\
=&\sum_p \frac{\frac{k}{p^{\frac{k}{h}}}-\frac{k-1}{p^{\frac{k+1}{h}}}}{\left(1-\frac{1}{p^{\frac{1}{h}}}\right)\left(1+\frac{1}{p}-\frac{1}{p^{\frac{1}{h}}}\right)}\\
&\times\prod_p\left(1+\frac{\frac{1}{p^{\frac{k}{h}}}}{1-\frac{1}{p^{\frac{1}{h}}}+\frac{1}{p}-\frac{1}{p^{\frac{k}{h}}}}\right)\left(1-\frac{1}{p}\right)\left(1+\frac{\frac{1}{p}-\frac{1}{p^{\frac{k}{h}}}}{1-\frac{1}{p^{\frac{1}{h}}}}\right)\\
=&\sum_p \frac{kp^{\frac{1}{h}}-k+1}{p^{\frac{k-h-1}{h}}\left(p^{\frac{1}{h}}-1\right)\left(p^{1+\frac{1}{h}}+p^{\frac{1}{h}}-p\right)}\prod_p\left(1+\frac{p-p^{\frac{1}{h}}}{p^2\left(p^{\frac{1}{h}}-1\right)}\right). \qquad (42)
\end{aligned}
$$

Equations (15) and (18) are proven analogously. Here instead we must consider

$$
f(k,h)\sum_{U\in\mathcal{N}_k}\prod_{q|U}\left(1+\frac{\frac{1}{q}-\frac{1}{q^{\frac{k}{h}}}}{1-\frac{1}{q^{\frac{1}{h}}}}\right)^{-1}\frac{\omega(U)}{U^{\frac{1}{h}}}.
$$

The corresponding generating function is given by

$$
\begin{aligned}
\mathcal{E}_{\omega,k,h}(z) =& \sum_{n\in\mathcal{N}_k}\frac{z^{\omega(n)}}{n^{\frac{1}{h}}}\prod_{q|n}\left(1+\frac{\frac{1}{q}-\frac{1}{q^{\frac{k}{h}}}}{1-\frac{1}{q^{\frac{1}{h}}}}\right)^{-1}\\
=&\prod_p\left(1+\left(1+\frac{\frac{1}{p}-\frac{1}{p^{\frac{k}{h}}}}{1-\frac{1}{p^{\frac{1}{h}}}}\right)^{-1}\frac{z}{p^{\frac{k}{h}}}\left(1+\frac{1}{p^{\frac{1}{h}}}+\frac{1}{p^{\frac{2}{h}}}+\cdots\right)\right)\\
=&\prod_p\left(1+\left(1+\frac{\frac{1}{p}-\frac{1}{p^{\frac{k}{h}}}}{1-\frac{1}{p^{\frac{1}{h}}}}\right)^{-1}\frac{\frac{z}{p^{\frac{k}{h}}}}{1-\frac{1}{p^{\frac{1}{h}}}}\right),
\end{aligned}
$$

which is absolutely convergent.

In order to find $\mathcal{E}'_{\omega,k,h}(1)$, we consider the logarithmic derivative:

$$
\frac{\partial}{\partial z}\frac{\mathcal{E}'_{\omega,k,h}(z)}{\mathcal{E}_{\omega,k,h}(z)} = \sum_p \frac{\frac{1}{p^{\frac{k}{h}}}}{1-\frac{1}{p^{\frac{1}{h}}}+\frac{1}{p}-\frac{1}{p^{\frac{k}{h}}}+\frac{z}{p^{\frac{k}{h}}}}.
$$

Therefore,

$$
\left.\frac{\partial}{\partial z}\frac{\mathcal{E}'_{\omega,k,h}(z)}{\mathcal{E}_{\omega,k,h}(z)}\right|_{z=1} = \sum_p \frac{\frac{1}{p^{\frac{k}{h}}}}{1-\frac{1}{p^{\frac{1}{h}}}+\frac{1}{p}}.
$$

By multiplying the above by $\mathcal{E}_{\omega,k,h}(1)$ and by the coefficient $f(k,h)$, we get an expression for $E_{\omega,k,h}$:

$$\begin{aligned}
f(k,h)\mathcal{E}'_{\omega,k,h}(1) =& f(k,h)\sum_p \frac{\frac{1}{p^{\frac{k}{h}}}}{1-\frac{1}{p^{\frac{1}{h}}}+\frac{1}{p}}\prod_p\left(1+\left(1+\frac{\frac{1}{p}-\frac{1}{p^{\frac{k}{h}}}}{1-\frac{1}{p^{\frac{1}{h}}}}\right)^{-1}\frac{\frac{1}{p^{\frac{k}{h}}}}{1-\frac{1}{p^{\frac{1}{h}}}}\right)\\
=&\sum_p \frac{\frac{1}{p^{\frac{k}{h}}}}{1-\frac{1}{p^{\frac{1}{h}}}+\frac{1}{p}}\prod_p\left(1+\frac{\frac{1}{p^{\frac{k}{h}}}}{1-\frac{1}{p^{\frac{1}{h}}}+\frac{1}{p}-\frac{1}{p^{\frac{k}{h}}}}\right)\left(1-\frac{1}{p}\right)\left(1+\frac{\frac{1}{p}-\frac{1}{p^{\frac{k}{h}}}}{1-\frac{1}{p^{\frac{1}{h}}}}\right)\\
=&\sum_p \frac{1}{p^{\frac{k-h-1}{h}}\left(p^{1+\frac{1}{h}}+p^{\frac{1}{h}}-p\right)}\prod_p\left(1+\frac{p-p^{\frac{1}{h}}}{p^2\left(p^{\frac{1}{h}}-1\right)}\right).
\end{aligned}$$

This concludes the proof of Theorem 5. □

Proof of Corollary 6. Recall that $n = L_k(n)U_k(n)$ and this implies

$$\Omega(n) = \Omega(L_k(n)) + \Omega(U_k(n)).$$

Combining equations (14) and (27), we immediately obtain equation (19).

Equation (20) follows from

$$\omega(n) = \omega(L_k(n)) + \omega(U_k(n)) = \omega(L_k(n)) + \omega(U_k(n))$$

by combining equations (15) and (28).

□

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