MATH 819 – HW6 (DIVISORS AND MAPS TO PROJECTIVE SPACE)

Due date: Friday, April 21st

Reading: Vakil: Ch. 15, 16.2, 17.4: Divisors, line bundles and maps to projective space

(1) Let $C = \operatorname{Proj} \frac{k[X,Y,Z]}{(Y^4 - X^4 - Z^4)}$, a quartic curve in $\mathbb{P}^2$. Let $i : C \hookrightarrow \mathbb{P}^2$ denote the closed embedding of C in $\mathbb{P}^2$.
   (a) Find the divisor of the rational function $f = Z/(X - Y)$.
   (b) For which $d \in \mathbb{Z}$ is f a global section of $O_C(d \cdot [1:1:0])$?
   (c) Explain the following assertions:
   - $i^*\mathcal{O}_{\mathbb{P}^2}(1) \cong \mathcal{O}_C(4 \cdot [1:1:0])$,
   - $\mathcal{O}_C(4 \cdot [1:1:0])$ is very ample, and
   - $\mathcal{O}_C([1:1:0])$ is ample.

   (d)* (Optional!) Show that C is smooth and of genus 3.

(2) Let $V \subset \Gamma(\mathcal{O}_{\mathbb{P}^1}(3))$ be the linear system spanned by S^3, ST^2, T^3. Is V basepoint-free? Does V give a closed embedding?

(3) Let $\mathbb{P}(k[X,Y,Z]_2) \cong \mathbb{P}^5$ be the projective space of conics in $\mathbb{P}^2$.
   Let $V \subset k[X,Y,Z]_2$ be the linear series of conics passing through $[0:1:1]$, $[1:0:1]$ and $[1:1:0]$. Find a basis for V, list any base points, and write down the corresponding rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^n$.

(4) Vakil 16.2.B,C(a) (look up Vakil 8.2.H), and E on globally generated sheaves.
   Note: an $\mathcal{O}_X$-module map $\mathcal{O}_X \to \mathscr{F}$ is the same as a choice of element of $\Gamma(\mathscr{F}, X)$. (Just like an R-module map $R \to M$ is the same as an element of M.)

(5) (The relationship between complete and incomplete linear series)
   (a) Let $p = [0 : \cdots : 0 : 1] \in \mathbb{P}^n$. Show that $k[X_0, \ldots, X_{n-1}] \hookrightarrow k[X_0, \ldots, X_n]$ gives a morphism $\pi_p : \mathbb{P}^n \setminus \{p\} \to \mathbb{P}^{n-1}$. This is called "projection from p".
   If $H \subseteq \mathbb{P}^{n-1}$ is a hyperplane, show that $\overline{\pi_p^{-1}(H)}$ is a hyperplane containing p. Conclude that π_p comes from an incomplete linear series in $\Gamma(\mathcal{O}_{\mathbb{P}^n}(1))$.
   (b) Let $0 \to K \to V \to W \to 0$ be a short exact sequence of k-vector spaces. Forget about schemes for a moment and think of $\mathbb{P}(V)$ as the set of one-dimensional subspaces of V. Give a map of sets $\pi_K : \mathbb{P}(V) \setminus \mathbb{P}(K) \to \mathbb{P}(W)$.
   (c) Continuing (b), describe a k-algebra map $\operatorname{Sym}(W^*) \hookrightarrow \operatorname{Sym}(V^*)$ giving π_K as a morphism of schemes. This is "linear projection away from $\mathbb{P}(K)$".

(d) Let $\mathscr{L}$ be a line bundle on a projective k-scheme X. Let $V \subset \Gamma(\mathscr{L}, X)$ be a basepoint-free linear series. Show that the morphism $|V|$ is the morphism $|\mathscr{L}|$ corresponding to the *complete* linear series, followed by a linear projection: if $\dim_k \Gamma(\mathscr{L}, X) = n$ and $\dim_k V = m$,

$$\begin{array}{ccc} X & \xrightarrow{|\mathscr{L}|} & \mathbb{P}^{n-1} \\ & {\scriptstyle |V|}\searrow & \downarrow{\scriptstyle \pi} \\ & & \mathbb{P}^{m-1} \end{array}$$