MATH 819 – HW5 (SEPARATED AND PROPER; LOCALLY FREE SHEAVES)

Due date: In class Tuesday, April 4th

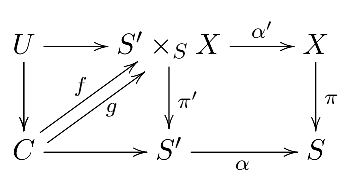
Reading: Vakil: Separated and proper (Chapter 11) – see also Hartshorne II.3. Vector bundles and locally free sheaves: Chapter 14.1-14.3 and 15. Global Spec and Global Proj: 18.1-18.2

(1) (Separatedness)
   (a) If $\pi : X \to S$ is separated, $S = \operatorname{Spec}(R)$ is affine, and $U, V \in X$ are affine open sets, then $U \cap V$ is affine. (Examine $\Delta^{-1}(U \times_S V)$.)
   Taking $X = S$ and $\pi = \mathrm{id}$, conclude: in an affine scheme, intersections of arbitrary affine open subschemes are affine. Then show $\mathbb{A}^2$ with two origins (over any base scheme) is not affine.
   (b) Show that separatedness is preserved by pullback, and properness is preserved by composition. (Use the definition, *or* assume X, S are locally noetherian and π is finite type and then use the valuative criterion).

Solutions. (a) We have $\Delta^{-1}(U \times_S V) = U \cap V$ as a set. Since S is affine, so is $U \times_S V$. Since π is separated, Δ is a closed embedding, in particular an affine morphism, so $\Delta^{-1}(U \times_S V) = U \cap V$ is again affine.

In the indicated special case, this shows that intersections of affines of $\operatorname{Spec} R$ are affine. (In fact, if $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$, then unwinding the formula, and using that $\Delta : S \to S \times_S S$ is an now an isomorphism, gives $U \cap V \cong \operatorname{Spec}(A \otimes_R B)$. I don't think it would have been obvious otherwise that this would represent an open subscheme of $\operatorname{Spec} R$.) Finally, if X is $\mathbb{A}^2$ with two origins, then $X = U \cup V$ where $U, V \cong \mathbb{A}^2$ are affine open sets, but $U \cap V = \mathbb{A}^2 \setminus \{(0, 0)\}$, which is not affine. Therefore X is not affine.

(b) Proof (sketch) via valuative criterion: For "separated is preserved by pullback", suppose we're given



with π separated. Composing f and g with α' gives two arrows $C \to X$. Since π is separated, $\alpha' \circ f = \alpha' \circ g$. But then by the universal property of fiber products, there is a *unique* map $C \to S' \times_S X$ making the diagram commute. Since both f and g work there, $f = g$.

For "properness is preserved by composition": first note that "finite type" is preserved by composition because it is a local property on source and target, so (reducing to the affine case) the point is that if $A \to B \to C$ are rings, B is a finitely-generated A-algebra, and C is a finitely-generated B-algebra, then C is a finitely-generated A-algebra (by the elements $b_i c_j$ ranging over the generators of C over B (including 1) and those of B over A (including 1)). Next, suppose we're given

$$\begin{array}{ccc} U & \longrightarrow & X \\ \Big\downarrow & \nearrow & \Big\downarrow \alpha \\ & & Y \\ & \nearrow & \Big\downarrow \beta \\ C & \longrightarrow & Z \end{array}$$

with α, β proper. Since β is proper, there's a unique map $C \to Y$ making the diagram commute. Then since α is proper, there's a unique map $C \to X$ making the diagram commute.

For the proofs directly from the definition: for separatedness, suppose we're given

$$\begin{array}{ccc} S' \times_S X & \xrightarrow{\alpha'} & X \\ \Big\downarrow \pi' & & \Big\downarrow \pi \\ S' & \xrightarrow[\alpha]{} & S \end{array}$$

with π separated, i.e. $\Delta_{X/S} : X \to X \times_S X$ is a closed embedding. Let $X' := S' \times_S X$. Ravi suggests proving that

$$\begin{array}{ccc} X' & \xrightarrow{\Delta_{X'/S'}} & X' \times_{S'} X' \\ \Big\downarrow & & \Big\downarrow \\ X & \xrightarrow{\Delta_{X/S}} & X \times_S X \end{array}$$

is Cartesian. Then $\Delta_{X'/S'}$ is the pullback of a closed embedding, hence again a closed embedding. For "composition of universally closed is universally closed", the proof is that pullback commutes with composition.

(2) Let R be a noetherian ring and M a finitely-generated R-module.
   (a) Let S be a multiplicative set and suppose $S^{-1}M = 0$. Show that there exists $f \in S$ such that $M_f = 0$.

(b) Let $p \in \operatorname{Spec}(R)$ and suppose M_p is a free R_p-module of rank n. Show that there exists $f \in R - p$ such that M_f is a free R_f-module of rank n.
(Hint: First find a map $R^n \to M$ such that $R_p^n \to M_p$ is an isomorphism. Then use (a).)

(c) Let X be a noetherian scheme and $\mathscr{F}$ a coherent sheaf on X. Show: $\mathscr{F}$ is locally free of rank $n \Leftrightarrow$ for all $p \in X$, $\mathscr{F}_p$ is a free $\mathcal{O}_{X,p}$-module of rank n.

Solution. (a) Let M have generators $m_1, \dots, m_k$. Since $S^{-1}M = 0$, there exist $f_1, \dots, f_k$ such that $f_i m_i = 0$ for each i. Let $f = \prod_i f_i$. Then $f m_i = 0$ for all i, so $M_f = 0$.

(b) Since M_p is free over R_p of rank n, M_p has a free basis $\left\{\frac{m_1}{s_1}, \dots, \frac{m_n}{s_n}\right\}$ for some $m_i \in M$ and $s_i \in R \setminus P$. Since the s_i are units in R_p, $\left\{\frac{m_1}{1}, \dots, \frac{m_n}{1}\right\}$ is again a basis. Let $\psi : R^n \to M$ be defined by $\psi(r_1, \dots, r_n) = \sum r_i m_i$. We have an exact sequence

$$0 \to \ker\psi \to R^n \to M \to \operatorname{coker}\psi \to 0.$$

Localizing to R_p, the middle map becomes an isomorphism. Since localization is exact, this shows $(\ker\psi)_p = 0 = (\operatorname{coker}\psi)_p$. By part (a), there exist $f, g \in R \setminus p$ such that $(\ker\psi)_f = 0 = (\operatorname{coker}\psi)_g$. Then over R_{fg}, both the cokernel and kernel vanish, so ψ_{fg} is an isomorphism.

(c) ($\Rightarrow$): Let $p \in X$. Since $\mathscr{F}_p$ is locally free of rank n, there is some affine neighborhood U of p such that $\mathscr{F}|_U \cong \mathcal{O}_U^{\oplus n}$. Then $\mathscr{F}_p \cong \mathcal{O}_{X,p}^{\oplus n}$ by localizing.

($\Leftarrow$): By part (b), for each p we can find U_p an open neighborhood of p on which $\mathscr{F}|_{U_p}$ is free of rank n. Therefore $\mathscr{F}$ is locally free of rank n.

(3) Let $\mathscr{E}, \mathscr{F}$ be locally free sheaves of ranks e and f on a noetherian scheme X.

(a) Show that $\mathscr{E} \otimes \mathscr{F}$, $\mathscr{E} \oplus \mathscr{F}$ and $\mathscr{H}om(\mathscr{E}, \mathscr{F})$ are all locally free, of ranks ef, $e+f$ and ef again. (Take for granted that $\mathscr{H}om(\mathscr{E}, \mathscr{F})$ is quasicoherent and given by $\operatorname{Hom}_R(E, F)$ on affine charts, when $\mathscr{F}$ is quasicoherent and $\mathscr{E}$ is coherent; see Vakil 1.6.G and 14.3.A.)

(b) Show that $\operatorname{Sym}^d(\mathscr{E})$ is locally free of rank $\binom{e+d-1}{d}$. (If $m_1, \dots, m_e$ is a basis for $\mathscr{E}$ locally, the monomials in $m_1, \dots, m_e$ give a basis for $\operatorname{Sym}^d(\mathscr{E})$.)

(c) Show that there is a surjective map of sheaves $\mathscr{E} \otimes \mathscr{H}om(\mathscr{E}, \mathscr{F}) \to \mathscr{F}$ given by "evaluation" $v \otimes \varphi \mapsto \varphi(v)$. (Show it exists via universal properties of $\otimes$ and gluing, then check it is surjective on sufficiently small affine charts.)

(d) Specialize part (b) to $\mathscr{F} = \mathcal{O}_X$; the sheaf $\mathscr{H}om(\mathscr{E}, \mathcal{O}_X)$ is called the *dual* of $\mathscr{E}$ and denoted $\mathscr{E}^*$ or $\mathscr{E}^\vee$.
Show: If $\mathscr{E}$ has rank 1, the map $\mathscr{E} \otimes \mathscr{E}^* \to \mathcal{O}_X$ is actually an isomorphism.

Solutions. For (a)(b)(d) and the surjectivity in (c), the thing we're asked to prove is a local property, so it suffices to check it on sufficiently small affine open sets. That is, you can immediately (if you wish) reduce to $X = \operatorname{Spec} R$ affine and

$\mathscr{E}, \mathscr{F}$ **free** (rather than just locally free), that is, $\mathscr{E} = \widetilde{E}$ where $E = R^e$ and $\mathscr{F} = \widetilde{F}$ where $F = R^f$.

For (b), the proof is called “stars and bars”.

For (c), Note that the given formula makes sense on every affine open set. Moreover, for a distinguished inclusion, $\operatorname{Spec} R_f \hookrightarrow \operatorname{Spec} R$, the diagram

$$\begin{array}{ccc} E \otimes \operatorname{Hom}_R(E,F) & \longrightarrow & F \\ \downarrow & & \downarrow \\ E_f \otimes \operatorname{Hom}_{R_f}(E_f, F_f) & \longrightarrow & F_f \end{array}$$

commutes. This means the formula defines a map of sheaves on the base, which therefore extends to a map of sheaves. We can check surjectivity locally: on a trivializing open set U as above, let $y \in F$. Then let $\phi : E \to F$ be defined by sending some basis element $x \in E$ to y. Then $\psi(y \otimes \phi) = x$, so ψ is surjective.

For (d), once we know the map exists, we can check it's an isomorphism locally. So locally $\mathscr{E} \cong R$ and $\mathscr{E}^* \cong R$, generated by the identity function $\mathrm{id} : R \to R$. (For any other $\phi \in \operatorname{Hom}(R,R)$, if $\phi(1) = s$ then ϕ is just multiplication by s, and so $\phi = s \cdot \mathrm{id}$.) Every element $r \otimes \phi$ is then just a multiple of $1 \otimes \mathrm{id}$, namely $r \otimes \phi = r \otimes (s \cdot \mathrm{id}) = rs(1 \otimes \mathrm{id})$.

The map is now $R \otimes \operatorname{Hom}(R,R) \to R$ given by $r \otimes \phi \mapsto \phi(r)$. This sends $1 \otimes \mathrm{id} \mapsto 1$ and $r(1 \otimes \mathrm{id}) \mapsto r$, so it is an isomorphism. (For an explicit inverse map, send $r \mapsto r(1 \otimes \mathrm{id})$.) □

(4) Let $R = k[S^4, S^3T, ST^3, T^4] \subset k[S,T]$ (i.e. the 4-th Veronese subring omitting S^2T^2). We reset the grading on S and count its four generators now as degree 1. You may take for granted that $S \cong \frac{k[X,Y,Z,W]}{(XW - YZ, Y^3 - ZX^2, Z^3 - YW^2)}$.
   (a) Verify that $\sqrt{(X,W)} = R_+$, so $D_+(X) \cup D_+(W) = \operatorname{Proj} R$.
   (b) Show that $D_+(X)$ and $D_+(W)$ are each isomorphic to $\mathbb{A}^1$ and that the gluing map simplifies to the usual $\mathbb{P}^1$ gluing map, so $\operatorname{Proj} R \cong \mathbb{P}^1$.
   (c) Examine $\widetilde{R(1)}$ on each of the charts $D_+(X)$ and $D_+(W)$: write down the generator and transition map. Recognize $\widetilde{R(1)}$ as what we would have called $\mathcal{O}(4)$ on $\operatorname{Proj} k[S,T]$. In particular, its global sections are

$$\Gamma(\operatorname{Proj} R, \widetilde{R(1)}) = k \cdot \{S^4, S^3T, S^2T^2, ST^3, T^4\},$$

even though $S^2T^2 \notin R$. This gives another example where the map $M_0 \to \Gamma(\widetilde{M}, \operatorname{Proj} R)$ isn't surjective. (It may be helpful to write S^2T^2 in terms of X, Y, Z, W on each chart.)

Solutions. (a) We have $Y^3 = ZX^2$ and $Z^3 = W^2Y \in (X,W)$, so $Y, Z \in \sqrt{(X,W)}$. Therefore $R_+ \subseteq \sqrt{(X,W)}$, which implies $D_+(X) \cup D_+(W) = \operatorname{Proj} R$.

(b) For $D_+(S^4)$, the ring is

$$\begin{aligned}(R_{S^4})_0 &= k[S^4, S^3T, ST^3, T^4, \tfrac{1}{S^4}]_0 \\ &= k[\tfrac{S^4}{S^4}, \tfrac{S^3T}{S^4}, \tfrac{ST^3}{S^4}, \tfrac{T^4}{S^4}] \\ &= k[1, \tfrac{T}{S}, \left(\tfrac{T}{S}\right)^3, \left(\tfrac{T}{S}\right)^4] = k[\tfrac{T}{S}].\end{aligned}$$

Similarly, for $D_+(T^4)$, the ring is $k[\frac{T}{S}]$. The transition maps are

$$k[\tfrac{T}{S}] \hookrightarrow k[\tfrac{T}{S}, \tfrac{S}{T}] \hookleftarrow k[\tfrac{S}{T}],$$

which we recognize as those of $\mathbb{P}^1$.

(c) On $D_+(S^4)$, $(R(1)_f)_0$ is $S^4 k[\frac{T}{S}]$, generated by S^4. On $D_+(T_4)$, the module is $T^4 k[\frac{S}{T}]$. We have

$$S^4 = T^4 \cdot \left(\tfrac{S}{T}\right)^4,$$

which is the transition function for $\mathcal{O}_{\mathbb{P}^1}(4)$. We therefore get 5 linearly independent global sections, including

$$S^4 \cdot \left(\tfrac{T}{S}\right)^2 = S^2T^2 = T^2\left(\tfrac{S}{T}\right)^2.$$

(5) (A valuative criterion) For any ring R and for $d \leq n$, let:

- $\mathrm{Mat}_{d\times n}(R)$ be the set of $d \times n$ matrices M with entries in R,
- $U_{d\times n}(R) \subset \mathrm{Mat}_{d\times n}(R)$ be the set of M such that the $d \times d$ minors of M generate the unit ideal in R, called *full rank* matrices.
- $GL_d(R) := U_{d,d}(R)$, the square matrices M such that $\det(M)$ is a unit.

One indirect "definition" of the *Grassmannian* $Gr(d, n)$ is to define, for all *affine* schemes $X = \mathrm{Spec}(R)$,

$$(*) \qquad \mathrm{Hom}(\mathrm{Spec}\, R, Gr(d, n)) := U_{d\times n}(R)/\sim,$$

where $M \sim AM$ for all $A \in GL_d(R)$. That is, by definition, a map $\mathrm{Spec}\, R \to Gr(d, n)$ "is" a full-rank $d \times n$ matrix over R, up to the equivalence relation of row operations.

For this problem, ignore the question of how $Gr(d, n)$ is a scheme and just work directly with the definition $(*)$ above. (This is essentially a definition via universal property of $Gr(d, n)$ as a quotient space.)

(a) Let k be a field. Show that $\mathrm{Hom}(\mathrm{Spec}\, k, Gr(d, n))$ — the k-points of $Gr(d, n)$ — is in bijection with the set of all d-dimensional subspaces $V \subset k^n$.

(b) Let $K = k(t)$, the field of rational functions, with valuation $\mathrm{val}(f)$ given by the order of vanishing of f at $t = 0$. Consider the matrix:

$$M = \begin{bmatrix} 1-t & 1-t^2 & t & t^2 \\ \frac{1}{t^2} & \frac{1+t}{t^2} & 1 & t \end{bmatrix} \in \mathrm{Mat}_{2\times 4}(k(t))$$

Check:

- Some minor of M is nonzero, so M represents a "morphism $\mathrm{Spec}\, k(t) \to Gr(2, 4)$". For generic t, we have a 2-dimensional subspace of k^4.

– If we set $t = 0$, the matrix is undefined. If we try rescaling the second row by t^2 and *then* set $t = 0$, the resulting matrix not full-rank over k. So it may seem that we can't "take the limit as $t \to 0$".

(c) Calculate the valuation of each minor of M. You should find two minors are identically zero (order $+\infty$), two are order -1 and two are order 0.

(d) Calculate $A^{-1}M$, where A is columns 1 and 3 of M. You should find that all nonzero entries now have nonnegative valuation, i.e., $A^{-1}M \in \mathrm{Mat}_{2\times 4}(k[t]_{(t)})$. *Now* set $t = 0$ and describe the resulting two-dimensional subspace of k^4. (Really what has happened is all *minors* now have nonnegative valuation. Since $A^{-1}M$ contains an identity matrix, this includes all individual *entries*.)

(e) Explain why every morphism $\operatorname{Spec} k(t) \to Gr(2,4)$ extends to a morphism $\operatorname{Spec} k[t]_{(t)} \to Gr(2,4)$. This is the "existence" part of the valuative criterion and is one way to prove that $Gr(2,4)$ is proper.

Solutions.

(a) By definition, a map $\operatorname{Spec} k \to Gr(d,n)$ is a row-equivalence class of $d \times n$ matrices. From linear algebra, two matrices over k have the same row-span if and only if they are row equivalent.

(b) The 13-minor is $\Delta_{13} = 1 - t - \frac{1}{t}$, so we see that for $t \neq 0$, this is nonzero. Thus for most t, the matrix has rank 2.

Rescaling the second row by t^2 and setting $t = 0$ gives $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$. This is rank 1, so it does not represent a map $\operatorname{Spec} k \to Gr(d,n)$.

(c) They are:

	Δ_{ij}	$\mathrm{val}_t(\Delta_{ij})$
Δ_{12}	0	∞
Δ_{13}	$-1/t + 1 - t$	-1
Δ_{14}	$-1 + t - t^2$	0
Δ_{23}	$-1/t - t^2$	-1
Δ_{24}	$-1 - t^3$	0
Δ_{34}	0	∞

(d) We have $A^{-1}M = \begin{bmatrix} 1 & 1+t & 0 & 0 \\ 0 & 0 & 1 & t \end{bmatrix}$. Now this represents a map $\operatorname{Spec} k[t] \to Gr(d,n)$ and the fiber at $t = 0$ is $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

(e) Here's what we did:

- We have a map $\operatorname{Spec} k(t) \to Gr(d,n)$, represented by a matrix M. Since M has full rank over the field $k(t)$, some minor is nonzero (as an element of $k(t)$).
- Let A be the submatrix whose minor has the lowest valuation. This valuation is not $+\infty$ (since some minor is nonzero).
- Then $A^{-1}M$ has all minors with nonnegative valuations, and an identity matrix in the columns coming from A. For any entry m_{ij} of $A^{-1}M$, if we swap out the i-th column of the identity submatrix for column j (containing m_{ij}), the resulting minor is exactly $\pm m_{ij}$. Therefore m_{ij} has valuation ≥ 0.

- Therefore $A^{-1}M$ has all entries in $k[t]_{(t)}$, hence represents a morphism $\operatorname{Spec} k[t]_{(t)} \to Gr(d,n)$. And of course we haven't changed the original morphism $\operatorname{Spec} k(t) \to Gr(d,n)$, since $A^{-1}M$ is row equivalent (over $k(t)$) to M.