# MATH 819 – HW3 (MODULES, FINITENESS CONDITIONS, LOCAL CONDITIONS)

**Due date:** In class Thursday, March 2nd

**Reading:** Vakil Sections 5.3, 6.1-6.4, 7.1-7.3 (Please note that I am following the December 2022 version)

Background on modules. Hand in **one** problem below:

(1) Let $R$ be a ring and $0 \to M' \to M \to M'' \to 0$ a short exact sequence of $R$-modules. Let $N$ be any $R$-module. Show that $M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$ is exact. (We say that $- \otimes N$ is a *right-exact functor*.)

(2) Give isomorphisms

$$R/I \otimes M \cong M/IM, \quad S^{-1}R \otimes M \cong S^{-1}M \text{ and } R/I \otimes R/J \cong R/(I + J)$$

and, for an $R$-algebra $A$, $A \otimes R[x] \cong A[x]$.

Combine the first two to obtain $k(p) \otimes M \cong M(p)$ for any $p \in \operatorname{Spec} R$.

If $S = \frac{R[x_1,\ldots,x_n]}{I}$ and $S' = \frac{R[y_1,\ldots,y_m]}{J}$ are finitely-generated $R$-algebras, show that $S \otimes_R S' = \frac{R[x_1,\ldots,x_n,y_1,\ldots,y_m]}{I+J}$ (think carefully about what "$I + J$" actually means.)

(3) Let $R$ be a ring and let $\phi : M \to N$ be a map of $R$-modules. Let $S \subseteq R$ be a multiplicative set. Show that $S^{-1}(\ker \phi) \cong \ker(S^{-1}\phi)$ and $S^{-1}(\operatorname{coker} \phi) \cong \operatorname{coker}(S^{-1}\phi)$. (Suggestion: Show that one satisfies the universal property of the other.)

Note: This is equivalent to showing: if $0 \to M' \to M \to M'' \to 0$ is a short exact sequence, then $0 \to S^{-1}M' \to S^{-1}M \to S^{-1}M'' \to 0$ is exact. That is, $S^{-1}(-)$ is an *exact functor*. If you prefer, you can show that instead.

Do **both** problems below:

(4) Let $X$ be a scheme and let $f \in \Gamma(X, \mathcal{O}_X)$. Let $X_f = \{p \in X : f(p) \neq 0\}$.
  (a) Show: for any affine open subset $U = \operatorname{Spec} R$, $X_f \cap U = D(f|_U)$. Conclude that $X_f$ is open.
  (b) If $X$ is quasicompact, show that $X_f$ is quasicompact. (Hint: Intersect $X_f$ with an affine cover of $X$ and use 3.6.H(a).)

(5) (Fitting ideals) Let $R$ be a ring and $M$ an $R$-module. A *free presentation* is an exact sequence

$$G \xrightarrow{\phi} F \to M \to 0,$$

where $G$ and $F$ are free $R$-modules. That is, $M$ is generated by (the images of) the generators of $F$, subject to the relations $\phi(g) = 0$ for each generator $g \in G$. We say $M$ is *finitely-presented* it has a presentation with $G \cong R^n$ and $F \cong R^m$ for some $n, m \in \mathbb{N}$. In this case $\phi$ is given by an $m \times n$ matrix of ring elements $(r_{ij})$, and $\phi(g_1, \ldots, g_n) = (r_{ij}) \cdot (g_j)$ (matrix-vector multiplication).

(a) Assume $M$ is finitely-presented as above. Let $p \in \operatorname{Spec}(R)$. Tensor with $k(p)$ to get the right exact sequence

$$G(p) \xrightarrow{\phi(p)} F(p) \to M(p) \to 0.$$

Suppose that the matrix $\phi(p)$, of elements of $k(p)$, has rank $r$ as a matrix. What is the dimension of $M(p)$ as a $k(p)$-vector space?

(b) Prove that the set $\mathrm{Fitt}_{\geq r}(M) := \{p \in \operatorname{Spec} R : \dim_{k(p)} M(p) \geq r\}$ is closed in the Zariski topology. (Hint: consider the ideal of $R$ generated by determinants of minors of $\phi$.) Conclude there exists a nonempty open set $U \subseteq \operatorname{Spec} R$ such that $\widetilde{M}|_U$ has constant rank (i.e. $\dim_{k(p)} M(p)$ is the same for all $p \in U$).

(c) Let $R = k[x, y, z]$ and let $M$ be the cokernel of the map $\phi : R^3 \to R^2$ given by the matrix $\begin{bmatrix} x & y & z \\ z & x & y \end{bmatrix}$. Describe the subsets of $\mathbb{A}^3$ on which $M$ has each possible rank.

*In (b), in fact more is true: the ideal of minors from part (c), the* Fitting ideal, *does not depend on the choice of presentation. Therefore, it gives not just a closed subset but a natural scheme structure on it. See e.g. Eisenbud,* Commutative Algebra with a View Toward Algebraic Geometry, *Corollary–Definition 20.4.*

"Affine-local property" problems: hand in **three** of these. Each one mirrors a proof we did in class. Make sure to read the statements of all four.

(6) Let $R$ be a ring and $M$ an $R$-module. Verify the claim from class, that $\widetilde{M}(D(f)) := M_f$ defines a sheaf on the distinguished base, hence produces a sheaf of modules. (Adapt the proof that the structure sheaf on $\operatorname{Spec} R$ is a sheaf on the base.)

(7) Let $\mathscr{F}$ be a quasicoherent sheaf. Show that the condition "$\mathscr{F}(U)$ is a finitely-generated $\mathcal{O}_X(U)$-module" satisfies the requirements of the Affine Communication Lemma. If $\mathscr{F}$ has this property and $X$ is noetherian, $\mathscr{F}$ is called a **coherent sheaf**. (Adapt the proof for "locally noetherian".)

(8) Let $\pi : X \to Y$ be a morphism of schemes. Show that the condition "$\pi^{-1}(U)$ is quasicompact" of open subsets $U \subseteq Y$ satisfies the hypotheses of the Affine Communication Lemma. We then call $\pi$ a **quasicompact morphism**.

Proceed as follows. Assume $Y = \operatorname{Spec} A$ is affine and $(f_1, \ldots, f_n) = (1)$ in $A$.

(i) If $\pi^{-1}(Y) = X$ is quasicompact, show that $\pi^{-1}(D(f_i))$ is quasicompact. (Pull $f_i$ to a global section on $X$ and use Problem 4.)
(ii) If $\pi^{-1}(D(f_i))$ is quasicompact for all $i$, show that $X$ is quasicompact.

(9) Let $\pi : X \to Y$ be a morphism of schemes. Show that the condition "$\pi^{-1}(U)$ is affine" satisfies the conditions of the Affine Communication Lemma. We the call $\pi$ an **affine morphism**, a.k.a. a family of affine varieties.

(Adapt the proof that quasicoherence is a local property. For the second condition, you should be showing the following: let $Y = \operatorname{Spec} R$ and let $(f_1, \ldots, f_n) = (1) \in R$. Let $g_i = \pi^\#(f_i) \in \Gamma(X, \mathcal{O}_X)$. By assumption, each open subscheme $X_{g_i} = \{x \in X : g_i(x) \neq 0\}$ is affine. Show that for each $f \in R$, letting $g = \pi^\#(f)$, the natural map $\Gamma(X, \mathcal{O}_X)_g \to \Gamma(X_g, \mathcal{O}_X)$ is an isomorphism.)

Vakil problems (optional):

- 3.6.VWXY (Noetherian modules – pure algebra)