# MATH 819 – HW3 (MODULES, FINITENESS CONDITIONS, LOCAL CONDITIONS)

Background on modules. Hand in **one** problem below:

(1) Let $R$ be a ring and $0 \to M' \to M \to M'' \to 0$ a short exact sequence of $R$-modules. Let $N$ be any $R$-module. Show that $M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$ is exact. (We say that $- \otimes N$ is a *right-exact functor*.)

**Comment.** The main mistake to avoid here is thinking that $a \otimes b = 0$ implies $a = 0$ or $b = 0$. This is false! For example $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = \mathbb{Z}/(2,3)\mathbb{Z} = 0$, so $1 \otimes 1 = 0$ in this module, even though $1 \neq 0$. □

(4) Let $X$ be a scheme and let $f \in \Gamma(X, \mathcal{O}_X)$. Let $X_f = \{p \in X : f(p) \neq 0\}$.
  (a) Show: for any affine open subset $U = \operatorname{Spec} R$, $X_f \cap U = D(f|_U)$. Conclude that $X_f$ is open.
  (b) If $X$ is quasicompact, show that $X_f$ is quasicompact. (Hint: Intersect $X_f$ with an affine cover of $X$ and use 3.6.H(a).)

**Solution.** (a): $X_f \cap U = \{p \in U : f(p) \neq 0\}$. This is equivalent to our definition of $D(f|_U) = \{p \in \operatorname{Spec} R : f \notin p\}$, because $f \in p \Leftrightarrow \frac{f}{1} \in pR_p \Leftrightarrow \overline{f} = 0 \in k(p)$. (Note that the direction $\frac{f}{1} \in pR_p \Rightarrow f \in p$ uses the fact that $p$ is prime.)

(b): Since $X$ is quasicompact, $X$ is a finite union of affine open sets $U_i$, so $X_f$ is covered by the finitely-many sets $X_f \cap U_i$. Each of these is a distinguished open in an affine scheme, hence is again affine, hence is quasicompact. Since a finite union of quasicompact sets is quasicompact, $X_f$ is quasicompact. □

(5) (Fitting ideals) Let $R$ be a ring and $M$ an $R$-module. A *free presentation* is an exact sequence

$$G \xrightarrow{\phi} F \to M \to 0,$$

where $G$ and $F$ are free $R$-modules. That is, $M$ is generated by (the images of) the generators of $F$, subject to the relations $\phi(g) = 0$ for each generator $g \in G$. We say $M$ is *finitely-presented* it has a presentation with $G \cong R^n$ and $F \cong R^m$ for some $n, m \in \mathbb{N}$. In this case $\phi$ is given by an $m \times n$ matrix of ring elements $(r_{ij})$, and $\phi(g_1, \ldots, g_n) = (r_{ij}) \cdot (g_j)$ (matrix-vector multiplication).
  (a) Assume $M$ is finitely-presented as above. Let $p \in \operatorname{Spec}(R)$. Tensor with $k(p)$ to get the right exact sequence

$$G(p) \xrightarrow{\phi(p)} F(p) \to M(p) \to 0.$$

Suppose that the matrix $\phi(p)$, of elements of $k(p)$, has rank $r$ as a matrix. What is the dimension of $M(p)$ as a $k(p)$-vector space?

(b) Prove that the set $\mathrm{Fitt}_{\geq r}(M) := \{p \in \operatorname{Spec} R : \dim_{k(p)} M(p) \geq r\}$ is closed in the Zariski topology. (Hint: consider the ideal of $R$ generated by determinants of minors of $\phi$.) Conclude there exists a nonempty open set $U \subseteq \operatorname{Spec} R$ such that $\tilde{M}|_U$ has constant rank (i.e. $\dim_{k(p)} M(p)$ is the same for all $p \in U$).

(c) Let $R = k[x, y, z]$ and let $M$ be the cokernel of the map $\phi : R^3 \to R^2$ given by the matrix $\begin{bmatrix} x & y & z \\ z & x & y \end{bmatrix}$. Describe the subsets of $\mathbb{A}^3$ on which $M$ has each possible rank.

*In (b), in fact more is true: the ideal of minors from part (c), the* Fitting ideal*, does not depend on the choice of presentation. Therefore, it gives not just a closed subset but a natural scheme structure on it. See e.g. Eisenbud,* Commutative Algebra with a View Toward Algebraic Geometry*, Corollary–Definition 20.4.*

**Solutions**. (a) Note that tensor product is right-exact. This means $F(p) \cong k(p)^m \to M(p)$ is surjective, with kernel equal to the image of $\phi(p)$. If $\phi(p)$ is given by a rank-$r$ matrix, then its image is $r$-dimensional, so the cokernel is has vector space dimension $m - r$.

(b) Let $I_k \subset R$ be the ideal generated by the $k \times k$ minors of $\phi$. Then:

$$p \in V(I_k) \Leftrightarrow I_k \subseteq p \Leftrightarrow I_k = 0 \mod p$$

and the last version is equivalent to "the $k \times k$ minors of $\phi$ are 0 mod $p$", i.e. $\phi(p)$ is given by a matrix of rank $\leq k - 1$. By part (a), this is equivalent to $M(p)$ being a vector space of dimension $\geq m - k + 1$. Solving, we find

$$\mathrm{Fitt}_{\geq r} = V(I_{m-r+1}).$$

This is a Zariski-closed subset since we have described it in terms of an ideal.

(c) The matrix has rank 0 (and hence $M$ has rank 2) at the origin $x = y = z = 0$. The matrix has rank 1 (and so $M$ does too) if $(x, y, z) \neq (0, 0, 0)$ but $x^2 = yz, y^2 = xz$ and $z^2 = xy$. This is the union of three lines through the origin (minus the origin). In fact it is the affine cone over the three points $[\zeta : \zeta^{-1} : 1] \in \mathbb{P}^2$ where $\zeta$ is a cube root of unity. Outside of these three lines, the matrix has rank 2 and $M$ has rank 0. In particular $M$ is only supported on those three lines. □

(6) Let $\mathscr{F}$ be a quasicoherent sheaf. Show that the condition "$\mathscr{F}(U)$ is a finitely-generated $\mathcal{O}_X(U)$-module" satisfies the requirements of the Affine Communication Lemma. If $\mathscr{F}$ has this property and $X$ is noetherian, $\mathscr{F}$ is called a **coherent sheaf**. (Adapt the proof for "locally noetherian".)

**Solution.** Suppose $X = \operatorname{Spec} R$ and $\mathscr{F} = \widetilde{M}$ for some $R$-module $M$.

*Step 1: Suppose $M$ is finitely-generated and $f \in R$. We show that $M_f$ is finitely-generated as an $R_f$-module.*

Proof: Let $m_1, \ldots, m_k$ generate $M$ over $R$. Then $\frac{m_1}{1}, \ldots, \frac{m_k}{1}$ generates $M_f$ over $R_f$. (Diagrammatically: we have a surjection $R^k \twoheadrightarrow M$. Localizing gives $R_f^k \twoheadrightarrow M_f$, which preserves surjectivity since localization is exact.)

*Step 2: Suppose $(f_1, \ldots, f_n) = (1)$ and $M_{f_i}$ is finitely-generated over $R_{f_i}$ for each $i$. We show that $M$ is finitely-generated over $R$.*

Proof: For each $i$, $M_{f_i}$ is generated by a finite set of elements $\{\frac{m_{ij}}{f_i^{k_{ij}}}\}$ over $R_{f_i}$. But then $\{\frac{m_{ij}}{1}\}$ generates $M_{f_i}$ over $R_{f_i}$ also.

Now combine all $m_{ij}$'s into one big finite set $\{m_1, \ldots, m_N\} \subset R$ and let $\phi : R^N \to M$ be the corresponding map. Then $\text{coker}(\phi)_{f_i} = \text{coker}(\phi_{f_i}) = 0$ for each $i$ since $\phi_{f_i}$ is manifestly surjective and localization preserves cokernels. Since the sets $D(f_i)$ cover $\operatorname{Spec} R$, it follows $\text{coker}(\phi) = 0$, that is, $\phi$ is surjective and $M$ is finitely-generated over $R$. □

(7) Let $\pi : X \to Y$ be a morphism of schemes. Show that the condition "$\pi^{-1}(U)$ is quasicompact" of open subsets $U \subseteq Y$ satisfies the hypotheses of the Affine Communication Lemma. We then call $\pi$ a **quasicompact morphism**.

Proceed as follows. Assume $Y = \operatorname{Spec} A$ is affine and $(f_1, \ldots, f_n) = (1)$ in $A$.

(i) If $\pi^{-1}(Y) = X$ is quasicompact, show that $\pi^{-1}(D(f_i))$ is quasicompact. (Pull $f_i$ to a global section on $X$ and use Problem 4.)

(ii) If $\pi^{-1}(D(f_i))$ is quasicompact for all $i$, show that $X$ is quasicompact.

**Solution.** For step 1, let $g_i = \pi^{\#}(f_i) \in \Gamma(X, \mathcal{O}_X)$. From the definition of morphism of schemes, values of regular functions pull back, i.e. if $\pi(p) = q$ we have a map of residue fields $\pi^{\#} : k(q) \to k(p)$ taking $f(q)$ to $g(p)$. So we have

$$\pi^{-1} D(f_i) = \{p \in X : f_i(\pi(p)) \neq 0\} = \{p \in X : g_i(p) \neq 0\} = X_{g_i}$$

in the notation of Problem 4. This is quasicompact by 4(b).

For step 2: the preimage of an open cover is an open cover, so the sets $X_{g_i}$ cover $X$. Since $X$ is covered by finitely-many quasicompact sets, $X$ is quasicompact. □

(8) Let $\pi : X \to Y$ be a morphism of schemes. Show that the condition "$\pi^{-1}(U)$ is affine" satisfies the conditions of the Affine Communication Lemma. We the call $\pi$ an **affine morphism**, a.k.a. a family of affine varieties.

(Adapt the proof that quasicoherence is a local property. For the second condition, you should be showing the following: let $Y = \operatorname{Spec} R$ and let $(f_1, \ldots, f_n) = (1) \in R$. Let $g_i = \pi^{\#}(f_i) \in \Gamma(X, \mathcal{O}_X)$. By assumption, each open subscheme $X_{g_i} = \{x \in X : g_i(x) \neq 0\}$ is affine. Show that for each $f \in R$, letting $g = \pi^{\#}(f)$, the natural map $\Gamma(X, \mathcal{O}_X)_g \to \Gamma(X_g, \mathcal{O}_X)$ is an isomorphism.)

**Solution sketch.** I'll just explain why this is the correct reduction. (The actual details of showing that the map $\Gamma(X, \mathcal{O}_X)_g \to \Gamma(X_g, \mathcal{O}_X)$ is an isomorphism are essentially identical to the proof from class.)

First note that Step 1 is easy: if $Y = \operatorname{Spec} R$ is affine and $\pi^{-1}(Y) = X = \operatorname{Spec} A$ is affine, and $f \in R$, then $\pi^{-1}(D(f)) = D(g)$ where $f \mapsto g$ via the ring map $R \to A$. This is affine since it is just $\operatorname{Spec} A_g$.

For Step 2, we are assuming each $X_{g_i}$ is affine. We want to show that $X$ is affine, so in fact we want to show $X \cong \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$. Recall that, for any affine scheme $\operatorname{Spec} A$, morphisms $X \to \operatorname{Spec} A$ are equivalent to ring maps $A \to \Gamma(X, \mathcal{O}_X)$. So the identity ring map of $A := \Gamma(X, \mathcal{O}_X)$ corresponds to a morphism $X \to \operatorname{Spec} A$.

Restricting $\operatorname{Spec} A$ to each $\operatorname{Spec} A_{g_i}$ in turn, we obtain the ring maps on global sections, which (by the work shown in the problem) are

$$A_{g_i} \xrightarrow{\sim} \Gamma(X_{g_i}, \mathcal{O}_X).$$

Since $X_{g_i}$ is assumed to be *affine*, the fact that these are ring isomorphisms implies that the corresponding scheme morphisms $X_{g_i} \to \operatorname{Spec} A_{g_i}$ are isomorphisms of schemes.

Finally, we use the general fact that "being an isomorphism is local on the target": if $\pi : X \to Y$ is any morphism of schemes, and $Y = \bigcup U_i$, then $\pi$ is an isomorphism if and only if $\pi^{-1}(U_i) \to U_i$ is an isomorphism for all $i$. (This is obvious for maps of sets; almost obvious for continuous maps of topological spaces, and then once we know $\pi$ is a homeomorphism of spaces, it's immediate that we have stalk isomorphisms at every point, hence an isomorphism of structure sheaves.) □