MATH 819 – HW2 (SHEAFIFICATION, STRUCTURE SHEAVES, BASICS OF SCHEMES)

Instructor problems:

(1) (Gluing and generic points) Recall the gluing of $\mathbb{P}^2$ from HW1. In this problem, view the open sets U_i and U_{ij} as affine schemes (i.e. including non-closed points).

(a) The following are equivalent to one of the gluing maps from HW1:

$$U_2 \supset U_{21} \xleftarrow{\sim} U_{12} \subset U_1$$

$$k[x,y] \hookrightarrow k[x,y,1/y] \xrightarrow{\phi} k[x,z,1/z] \hookleftarrow k[x,z]$$

$$x \mapsto x/z,$$

$$y \mapsto 1/z.$$

Let $(f) = (y^2 - x^3 - 1) \in \operatorname{Spec}(k[x,y])$ and $(g) = (z - x^3 - z^3) \in \operatorname{Spec}(k[x,z])$. (You may assume these ideals are prime.) Show that ϕ identifies these ideals. *Something similar happens in the third chart, so our gluing maps for* $\mathbb{P}^2$ *also glue together* $\operatorname{Spec} k[x,y]/(f)$ *and* $\operatorname{Spec} k[x,z]/(g)$ *(and one more) to give an elliptic curve* $C \subseteq \mathbb{P}^2$. *Overall, the underlying set of the scheme* $\mathbb{P}^2$ *has just* one *non-closed "generic point" corresponding to* C.

(b) Which points of $\mathbb{P}^2$ are *not* in U_2? (Hint: various closed points but only *one* non-closed point.)

Solution to (b). The gluing isomorphism

$$U_2 \supset U_{21} \xleftarrow{\sim} U_{12} \subset U_1$$

$$k[x,y] \hookrightarrow k[x,y,1/y] \xrightarrow{\phi} k[x,z,1/z] \hookleftarrow k[x,z]$$

identifies the open subset $U_{12} \subset U_1$ with points coming from U_2. Since U_{12} is the distinguished open set $D(z) \subset U_1$, its complement is $V(z)$, which has only one non-closed point (namely $p = (z)$ itself). Similarly, from

$$U_2 \supset U_{20} \xleftarrow{\sim} U_{02} \subset U_0$$

$$k[x,y] \hookrightarrow k[x,y,1/x] \xrightarrow{\phi} k[y,z,1/z] \hookleftarrow k[y,z]$$

we see that $U_{02} \subset U_0$ is $D(z)$, so the only non-closed point of U_0 not in U_2 is $V(z)$. This represents the same non-closed point of $\mathbb{P}^2$ as $V(z)$ on the U_1 chart.

In summary, the points of P^2 not in U_2 are:

- The points corresponding to homogeneous coordinates $[x : y : 0]$,
- The non-closed point $V(z)$, which is the generic point of the line $[x : y : 0]$. $\square$

(2) (Pictures of nonreducedness) In this problem, for an ideal I of a ring R, by abuse of notation write $\mathbb{V}(I)$ for the scheme $\operatorname{Spec} R/I$. The *scheme-theoretic intersection* $\mathbb{V}(I)\cap\mathbb{V}(J)$ is by definition $\mathbb{V}(I+J) = \operatorname{Spec} R/(I+J)$ (**not** $\operatorname{Spec} R/\sqrt{I+J}$.) We'll examine these concepts further in class.

Let $X = \operatorname{Spec} \dfrac{k[x,y,z]}{(x,z)^2}$. Draw a picture of X (it looks like a tube in $\mathbb{A}^3$).

(a) Let $X' = X \cap \mathbb{V}(x)$. Draw a picture that makes it clear that $X' \subseteq \mathbb{V}(x)$. Does your picture suggest X' is contained in $\mathbb{V}(x+z)$? Check: is $x+z \in (x,z)^2+(x)$?

(b) Let $X'' = X \cap \mathbb{V}(xy - z)$. Draw a picture of X'' in $\mathbb{A}^3$ (it may be helpful to first plot $xy - z = 0$, which is a quadric surface containing the y-axis. Try `https://math3d.org`.)
Let $H = \mathbb{V}(ax + bz)$ be *any* plane containing the y-axis (assume a and b are not both 0). Show that X'' is *not* contained in H. That is, the nonreduced structure of X'' "twists around", following the shape of $\mathbb{V}(z - xy)$.

(c) Let $Y = \operatorname{Spec} \dfrac{k[x,y]}{xy}$, the union of the x and y axes. Let Z be a tangent vector at the origin pointing in *any* direction (figure out the equations for Z). Show that $Z \subseteq Y$. So Y contains the entire 2D "tangent space" at the origin.
This should be surprising – you might have expected Y to only contain the vertical and horizontal tangent vectors. In general for schemes we only have

$$(Y_1 \cup Y_2) \cap Z \supseteq (Y_1 \cap Z) \cup (Y_2 \cap Z).$$

Solutions.

(a) To see that $x + z \notin (x,z)^2 + (x)$, we can write $(x,z)^2 + (x) = (x^2, xz, z^2, x) = (x, z^2)$. Then it's clear that $x + z \notin (x, z^2)$.

(b) To see that $ax + bz \notin (x,z)^2 + (xy - z)$, it's convenient to mod out and ask whether $ax + bz = 0$ in the quotient ring

$$\frac{k[x,y,z]}{(x,z)^2 + (xy - z)} = \frac{k[x,y,z]}{(x^2, xz, z^2, xy - z)} \cong \frac{k[x,y]}{(x^2, x^3y, x^2y^2)} = \frac{k[x,y]}{(x^2)}.$$

Under this isomorphism (where we have set $z = xy$), our element is $ax + bz = ax + bxy = x(a + by)$. This is not a multiple of x^2 in $k[x,y]$, so it is not zero in the quotient by x^2.

(c) A tangent vector along the line $ax + by = 0$ would be given by the ideal $(x,y)^2 + (ax + by)$, i.e. it would be $\operatorname{Spec} \frac{k[x,y]}{(x,y)^2+(ax+by)}$. This is just $(x^2, xy, y^2, ax + by)$ and contains the ideal (xy), so it is a subscheme of $\frac{k[x,y]}{(xy)}$. Indeed we can see directly that $(xy) \subseteq (x,y)^2$, so the entire "fat neighborhood" $\operatorname{Spec} \frac{k[x,y]}{(x,y)^2}$ is contained in the union of the two axes... despite the fact that $\frac{k[x,y]}{(xy)}$ is reduced. I don't have a good picture of this.

Pictures for (a), (b):