MATH 819 – HW1 (GLUING, PRESHEAVES AND SHEAVES)

How much to do: You are not required to do every problem. You should aim to do *at least some* of the concrete problems and *at least some* of the formal manipulations. Modern algebraic geometry is built on many layers of *both* formalism *and* a large body of intuition-building examples.

Due date: In class Tuesday, Jan 23rd

Reading: Vakil Sections 2.1-2.5, 2.7 (equivalent to Hartshorne Section 2.1 – much more terse!). If interested in category theory, look at Vakil 1.1-1.5 only.

(1) (Gluing $\mathbb{P}^2$). Let $V = \Bbbk^3$ and let $\mathbb{P}^2 = \mathbb{P}(V) = \{\text{lines } \ell \subseteq V \text{ through } 0\}$. The homogeneous coordinates $[A : B : C]$ represent the line $\ell = \text{span}([A, B, C])$.

(a) Consider the subsets (coordinate charts)

$$U_0 = \{[1 : a_1 : a_2]\}, \quad U_1 = \{[b_0 : 1 : b_2]\}, \quad U_2 = \{[c_0 : c_1 : 1]\}.$$

Write down the transition functions $f_{01}(a_1, a_2)$, $f_{12}(b_0, b_2)$ and $f_{02}(a_1, a_2)$ and their inverses. Verify $f_{02} = f_{12} \circ f_{01}$.

(b) Let $\mathscr{L}^* = \{(\ell, f) : \ell \in \mathbb{P}^2 \text{ and } f : \ell \to \Bbbk \text{ is linear}\}$ (the *dual tautological line bundle on* $\mathbb{P}^2$). Glue $\mathscr{L}^*$ together using sets $X_i := U_i \times \Bbbk$.
(Modify the construction over $\mathbb{P}^1$ *done in class. For example, given* $(a_1, a_2, t) \in U_i \times \Bbbk$*, the vector* $[1, a_1, a_2]$ *is a basis for the line* $\ell = [1 : a_1 : a_2]$*, so* t *can correspond to the unique linear function* $\ell \to \Bbbk$ *sending* $[1, a_1, a_2] \mapsto t$*.)*
Define carefully the transition functions f_{01}, f_{12}, f_{02} and show $f_{02} = f_{12} \circ f_{01}$. (If can you do it nicely, show $f_{jk} \circ f_{ij} = f_{ik}$ for all i, j, k.)

Note: Part (a) is carried out for $\mathbb{P}^n$ as a scheme in Vakil 4.4.9. One lesson here is to be thoughtful in how you name your coordinates; also, coordinate-free definitions (as in $\mathbb{P}(V)$ and $\mathscr{L}^*$) are very valuable.

(2) Let X be a topological space and let $\mathscr{F}$ be given by $U \mapsto \{f : U \times U \to \mathbb{R}\}$, with the restriction maps $f \mapsto f|_{V \times V}$ when $V \subseteq U$.
Show that in general $\mathscr{F}$ fails the 'identity' axiom of sheaves. (You can take $X =$ two points with the discrete topology.)

(3) Let $X = \mathbb{R}$ and let $\mathcal{O}_X$ be the sheaf of *all* real-valued functions on X. Let $p \in X$. Show that $\mathcal{O}_{X,p}$ is not a local ring by showing $\mathfrak{m}_p = \{(f, U) : f(p) = 0\}$ is a maximal ideal, but that there are elements of $\mathcal{O}_{X,p} - \mathfrak{m}_p$ that are not units.

(4) Let X be a topological space and let $\phi : \mathscr{F} \to \mathscr{G}$ be a map of presheaves of abelian groups on X. Show that the presheaf image and presheaf cokernel are, indeed, presheaves, by defining appropriate restriction maps

$$\mathrm{im}(\phi(U)) \xrightarrow{?} \mathrm{im}(\phi(V)), \quad \mathrm{coker}(\phi(U)) \xrightarrow{?} \mathrm{coker}(\phi(V)),$$

whenever $V \subseteq U$ is an inclusion of open sets.

(5) Let X be a topological space and let $\phi : \mathscr{F} \to \mathscr{G}$ be a map of presheaves of abelian groups.
  (a) Give a natural isomorphism $(\ker \phi)_p \xrightarrow{\sim} \ker(\phi_p : \mathscr{F}_p \to \mathscr{G}_p)$.
  Conclude: if $\mathscr{F}, \mathscr{G}$ are sheaves, then $\ker \phi = 0$ if and only if the maps of stalks are injective.
  (b) Give a natural isomorphism $(\mathrm{coker}_{pre}\, \phi)_p \xrightarrow{\sim} \mathrm{coker}(\phi_p : \mathscr{F}_p \to \mathscr{G}_p)$.
  After January 17th, conclude: if $\mathscr{F}$ and $\mathscr{G}$ are sheaves, $\mathrm{coker}\, \phi = 0$ (which has the same stalks as $\mathrm{coker}_{pre}\, \phi$) if and only if the maps of stalks are surjective.

Book problems (Vakil):

- 2.1.A (Stalks of the sheaf of smooth functions on $\mathbb{R}$ are local rings.)
- 2.2.E (The constant sheaf is a sheaf)
- 2.2.G(a) (This example is historically why $f \in \mathscr{F}(U)$ is called a 'section')
- 2.2.H (The pushforward/direct image is a (pre)sheaf)
- 2.3.A (Maps $\phi : \mathscr{F} \to \mathscr{G}$ induce maps of stalks $\mathscr{F}_p \to \mathscr{G}_p$)
- 2.4.D (Injectivity is easier. For surjectivity, show that the germs of $g \in \mathscr{G}(U)$ lead to a set of compatible germs of $\mathscr{F}(U)$. Use 2.4.A in both parts (shown in class).)