Shifted tableau crystals

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joint with Maria Gillespie (UC Davis),
Kevin Purbhoo (Waterloo)

FPSAC Dartmouth
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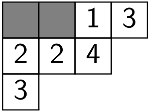
Ordinary and shifted tableaux

- Semistandard (skew) tableaux 

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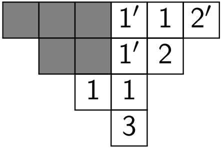
- **Shifted** (skew) tableaux 

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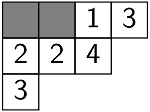
- Semistandard (skew) tableaux  :
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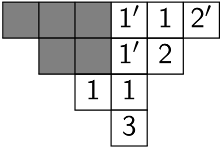
- **Shifted** (skew) tableaux  :
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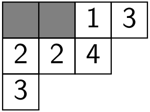
- Semistandard (skew) tableaux  :
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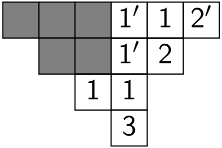
- **Shifted** (skew) tableaux  :
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    (For us: southwesternmost i/i' is always unprimed.)

From geometry to tableaux...

- Remarkable story connecting **geometry of curves**, **Schubert calculus**, **tableaux**

  [Shapiro-Shapiro, Eisenbud-Harris, Mukhin-Tarasov-Varchenko, Purbhoo, Speyer, Sottile, Halacheva-Kamnitzer-Rybnikov-Weekes, Osserman, Chan-López Martín-Pflueger-Teixidor i Bigas, Gillespie-L...]

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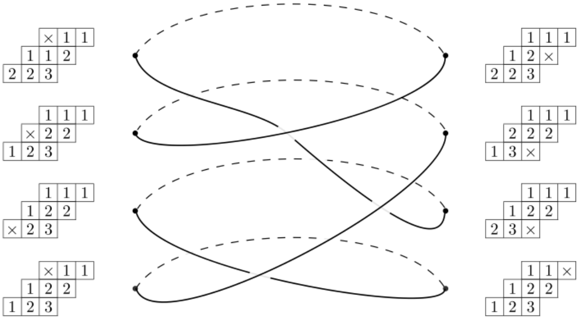

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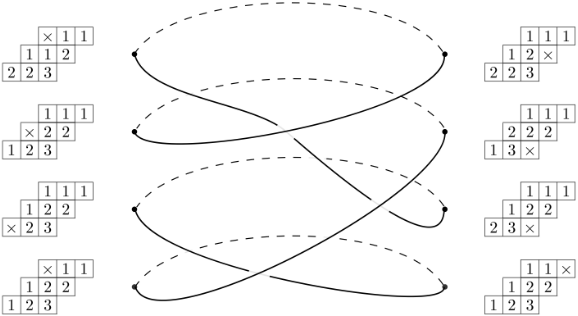

- Monodromy via (shifted) **tableaux** and **tableau algorithms**.

… to more tableaux

- Monodromy known in type A [Gillespie-L '16] – by jeu de taquin and crystal operators!

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- Key property: the monodromy operator is **coplactic**: it commutes with all (shifted) jeu de taquin slides.

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Question

What are the natural coplactic operators on shifted tableaux?

Natural operations on tableaux – type A

Any coplactic operation is determined by its action on **rectified** tableaux.

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Tableaux of shape $\lambda = (5, 3)$, organized by weight:

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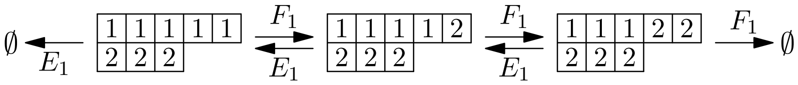

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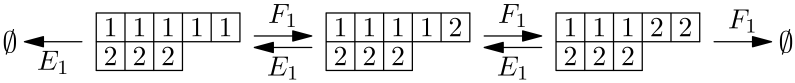
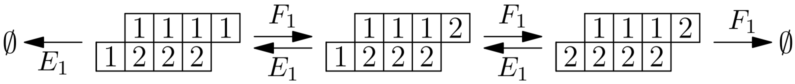


Action on general tableaux:



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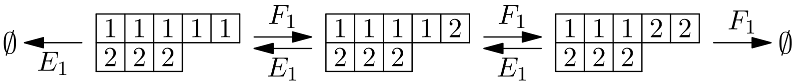

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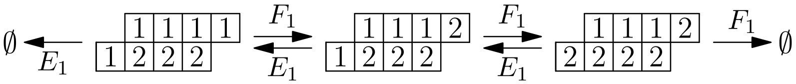

E_i, F_i: treat $i, i{+}1$ as $1, 2$.

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							1
					1	2	2
1	1	1	1	2			
2	2						

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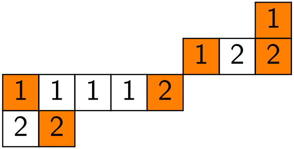

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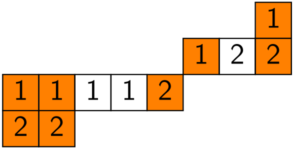

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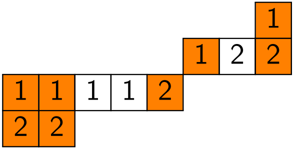

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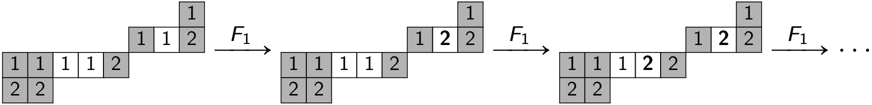
Result:



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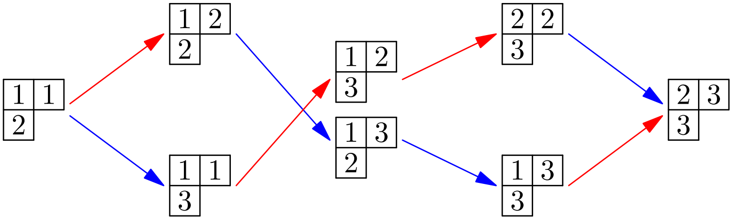

$F_1 = \color{red}{\longrightarrow}$, $F_2 = \color{blue}{\longrightarrow}$

(F_i : changes an $i \rightsquigarrow i{+}1$)

Uniquely determined JDT-invariant **graph structure** on tableaux.

Type B: *Two* operations on shifted Q-tableaux

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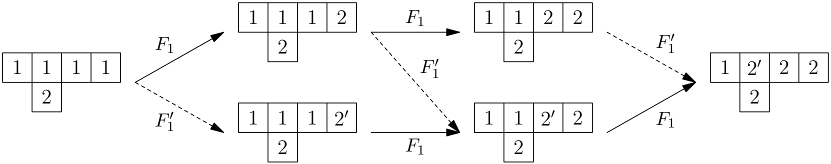

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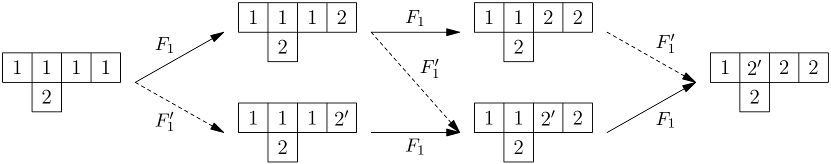

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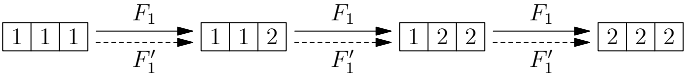

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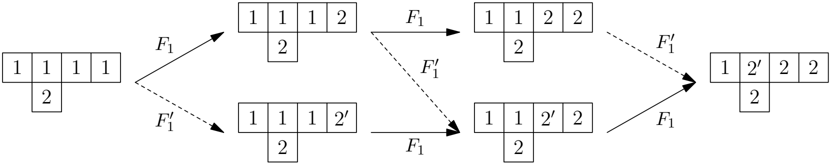

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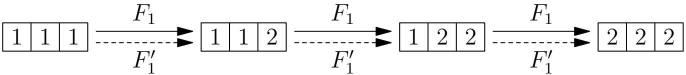

By coplacticity: unique operators $\xrightarrow{F_1}$, $\overset{F_1{}'}{\dashrightarrow}$ on all skew shifted tableaux.

Analog of a bracketing rule: F_i, F_i'

Theorem (Gillespie-L-Purbhoo '17)

There are direct definitions of F_i, F_i', depending only on $w = \mathrm{word}(T)$, via **first-quadrant lattice walks**.

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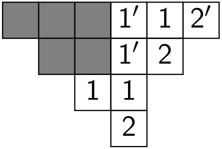
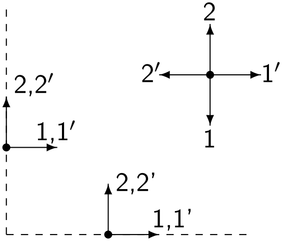
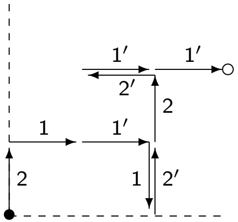


: $\mathrm{word}(T) = 2111'21'12'$.

First-quadrant lattice walks

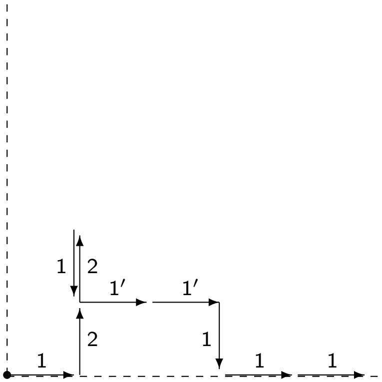




The **lattice walk** for $w = 211'12'22'1'1'$ ends at the point $(3, 2)$.

'Cancellation' away from the axes.

Example: repeated F_1, followed by one F_1':



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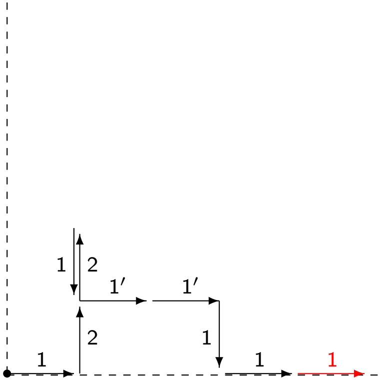

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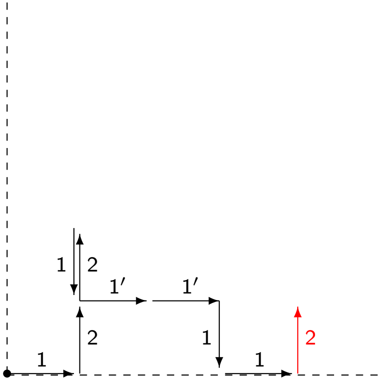

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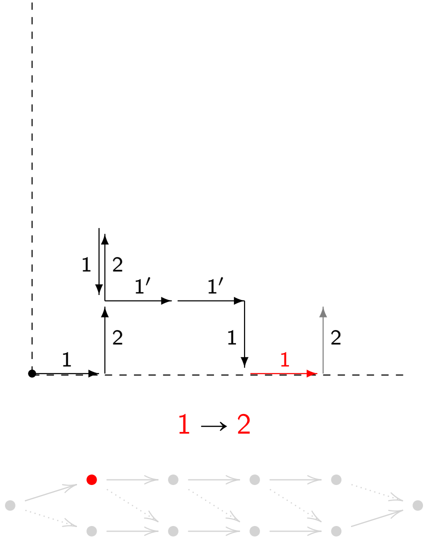

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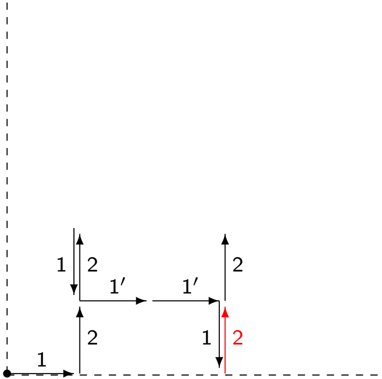

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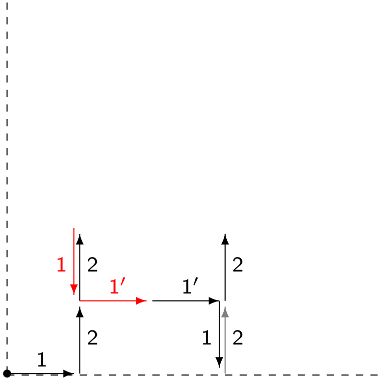

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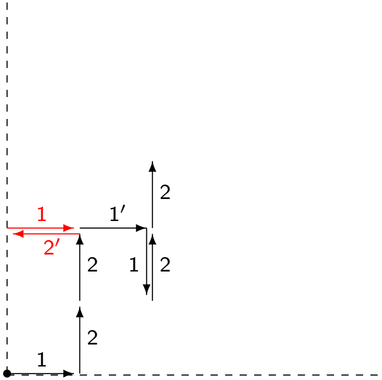

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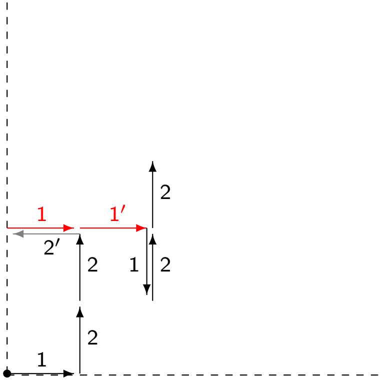

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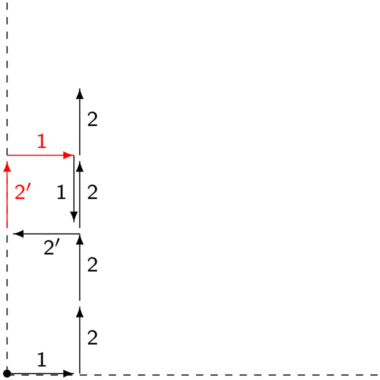

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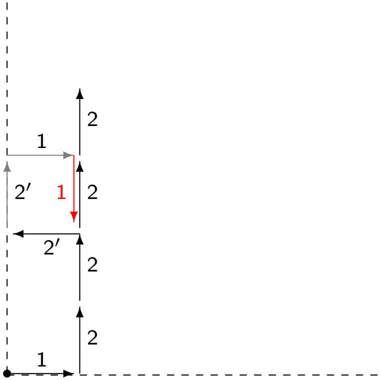

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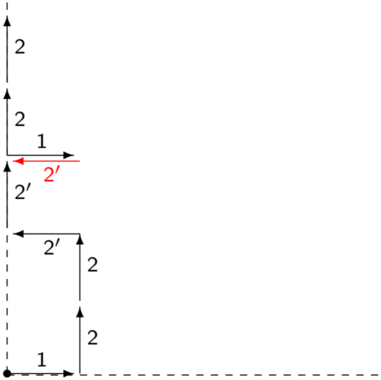

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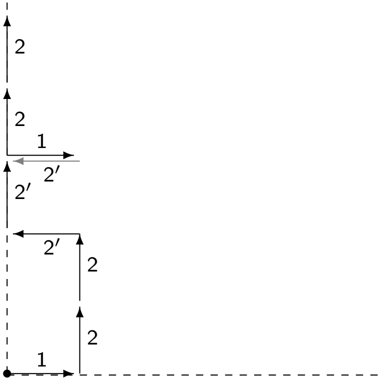

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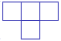
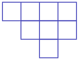
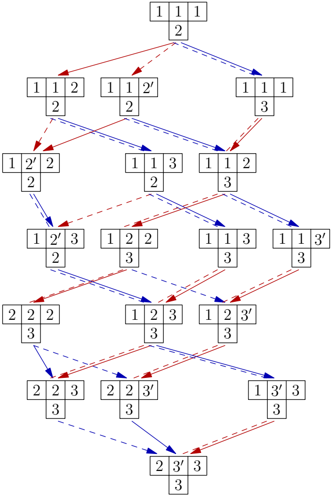
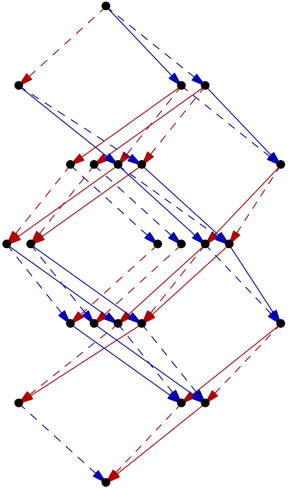
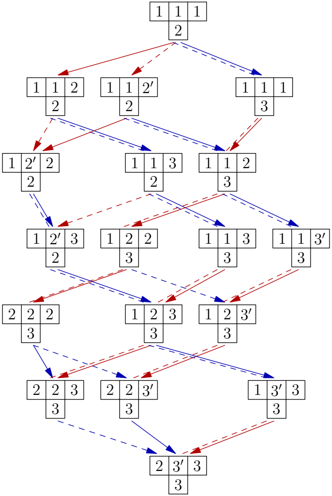
Example: the crystals $\mathcal{B}($$, 3)$ and $\mathcal{B}($$, 3)$





Legend: F_1 ⟶, F_1' ⇢, F_2 ⟶, F_2' ⇢

Features of shifted tableau crystals



Key features:

- Unique highest-weight element (type B LR tableau)
- Weighted characters are Schur Q-functions
- Connected components of $\mathrm{ShST}(\lambda/\mu, n)$ recover **skew LR rule** for Schur Q-functions,

$$\mathrm{ShST}(\lambda/\mu, n) \cong \bigsqcup_{\nu} \mathrm{ShST}(\nu, n)^{f^{\lambda}_{\nu,\mu}}$$

$$Q_{\lambda/\mu} = \sum_{\nu} f^{\lambda}_{\nu,\mu} Q_{\nu}.$$

Characterizing the graph structure

- F_i, F_i': the **unique** lowering operators on shifted Q-tableaux, coplactic for shifted JDT.
- Induced graph structure is intrinsic. What is this structure?

Characterizing the graph structure

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Theorem (Stembridge '03)

In type A, crystals are characterized by a short list of local axioms (relating F_i, F_j).

Similar statement for shifted tableau crystals:

Theorem (Gillespie-L '18)

Shifted tableau crystals are characterized by a short list of local axioms.

Four operators: $F_i, F_i', F_{i+1}, F_{i+1}' \rightsquigarrow$ 6 pairs.

Characterizing the graph structure – type A

Theorem (Stembridge '03)

Local axioms in type A:

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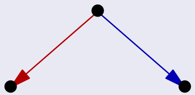

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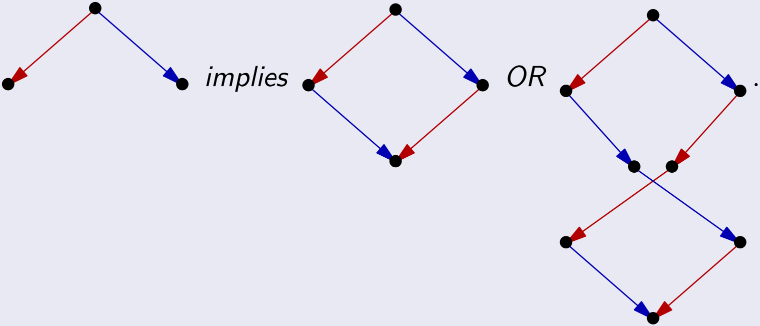

(depending on local data).

Axioms for shifted tableau crystals

Theorem (Gillespie-L '18)

Shifted Q-tableau crystals are characterized by a short list of local axioms (relating F_i, F_i', F_j, F_j').

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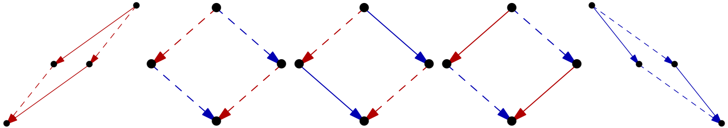
$$\{F_1', F_1\}, \{F_1', F_2'\}, \{F_1', F_2\}, \{F_1, F_2'\}, \{F_2', F_2\}$$



(Certain specific boundary-case exceptions.)

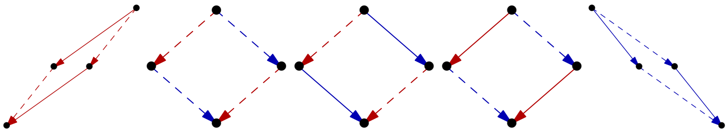
Axioms for shifted tableau crystals

Theorem (Gillespie-L '18)

Shifted Q-tableau crystals are characterized by a short list of local axioms (relating F_i, F'_i, F_j, F'_j).

Four operators. Other than $\{F_i, F_{i+1}\}$: they mostly commute.

$$\{F'_1, F_1\}, \{F'_1, F'_2\}, \{F'_1, F_2\}, \{F_1, F'_2\}, \{F'_2, F_2\}$$

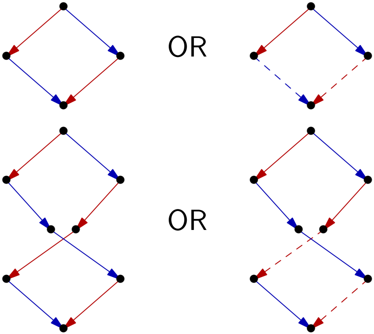


(Certain specific boundary-case exceptions.)

IDEA: Primed operator F'_i mostly "doubles" the crystal.

Axioms for shifted tableau crystals

The interesting pair: $\{F_1, F_2\}$. *Four* possibilities:



(depending on local data).

"Doubled" axioms from type A.

Using the axioms

Theorem (Gillespie-L '18)

Let G be a graph satisfying the local axioms for shifted tableau crystals. Then each connected component is $\cong \mathrm{ShST}(\sigma, n)$ for some σ.

Gives method to prove positivity of a generating function:

- Introduce operators F_i satisfying the axioms.

Using the axioms

Theorem (Gillespie-L '18)

Let G be a graph satisfying the local axioms for shifted tableau crystals. Then each connected component is $\cong \mathrm{ShST}(\sigma, n)$ for some σ.

Gives method to prove positivity of a generating function:

- Introduce operators F_i satisfying the axioms.
- Done in type A for affine Stanley symmetric functions [Morse-Schilling '15]
- **Q:** Can this be done for Schur Q positivity?

Thank you!

- *A crystal-like structure on shifted tableaux*. M. Gillespie, J. Levinson, and K. Purbhoo (2017). arXiv:1706.09969.
- *Schubert curves in the orthogonal Grassmannian* $\mathrm{OG}(n, 2n+1)$. M. Gillespie, J. Levinson, and K. Purbhoo (in preparation).
- *Axioms for shifted tableau crystals*. M. Gillespie and J. Levinson (2018). arXiv:1706.09969.