Monodromy and K-theory of Schubert Curves

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joint with Maria Gillespie (UC-Davis)

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Variation on a theme of Schubert

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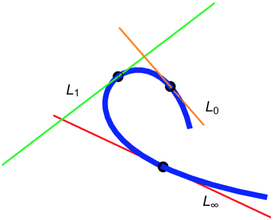

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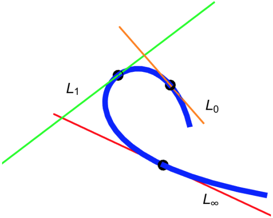

Define $\mathcal{S}$ by:

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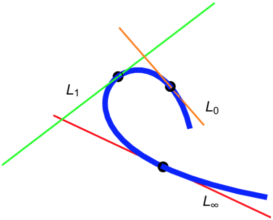

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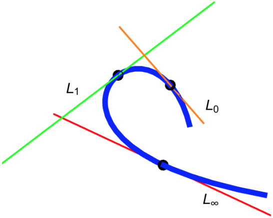

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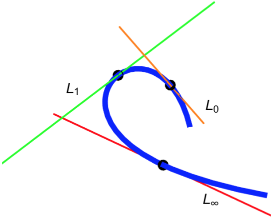

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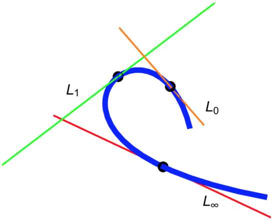

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  - **Monodromy**: Sweep t around $\mathbb{RP}^1 \Rightarrow$ points switch places.

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- The **rational normal curve** in $\mathbb{P}^{n-1}$:

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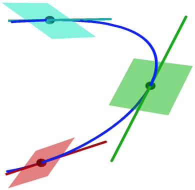

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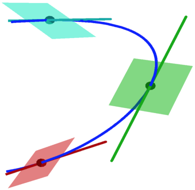

- $\mathcal{S}$ will be a locus in $Gr(k, \mathbb{C}^n)$, defined using $\mathscr{F}_{t=0,1,\infty}$.

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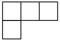
    $$\lambda = (3,1) \Longleftrightarrow$$ 

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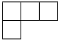
    $\lambda = (3,1) \Longleftrightarrow$ 

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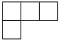
    $\lambda = (3,1) \Longleftrightarrow$ 

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- For **tangent flags** $\mathscr{F} = \mathscr{F}_t$, $\Omega_\lambda(\mathscr{F}_t)$ used to study degenerations of curves [Eisenbud-Harris '80s]

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The **Schubert curve** $\mathcal{S} \subseteq G(k, n)$ is

$$\mathcal{S} = \mathcal{S}(\alpha, \beta, \gamma) = \Omega_\alpha(\mathscr{F}_{t=0}) \cap \Omega_\beta(\mathscr{F}_{t=1}) \cap \Omega\gamma(\mathscr{F}_{t=\infty}).$$

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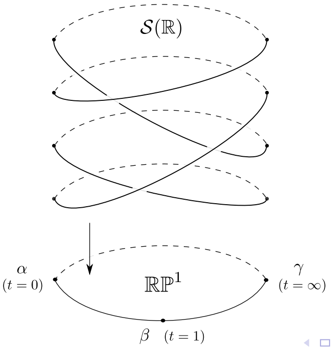
- **"Reality"** and **Monodromy**:
  Fourth Schubert condition $\lambda = \square$ at $t \in \mathbb{RP}^1$
  vary $t \rightsquigarrow$ sweep out $\mathcal{S}(\mathbb{R})$ (!!)

Real geometry of Schubert curves

Theorem (L., extending Speyer, Mukhin-Tarasov-Varchenko; see also Purbhoo (and Eisenbud-Harris . . .))

$\mathcal{S} = \mathcal{S}(\alpha, \beta, \gamma)$ a Schubert curve, $\mathcal{S}(\mathbb{R})$ its real points.

- *There is a map $f : \mathcal{S} \to \mathbb{P}^1$, inducing a smooth covering of real points, $\mathcal{S}(\mathbb{R}) \to \mathbb{RP}^1$. (Note: $f^{-1}(t) = \mathcal{S} \cap \Omega_{\square}(\mathscr{F}_t)$.)*



Conventions on Young tableaux

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- **Yamanouchi tableau** of **shape** ν/μ:

  Semistandard, whose reverse row word is **ballot**.

$T =$

	μ		1	1	1
			2	2	
	1	2	3	3	
2	3	4	4		
3	5				

$\mu = (3, 3, 1)$
$\nu = (6, 5, 5, 4, 1)$

- **Content** $= (\#1\text{'s}, \#2\text{'s}, \cdots)$
- **Word** $=$ 111223321443253

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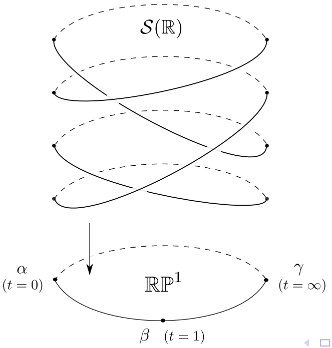

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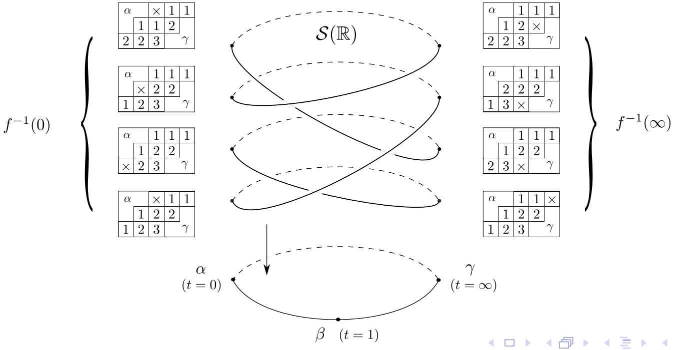

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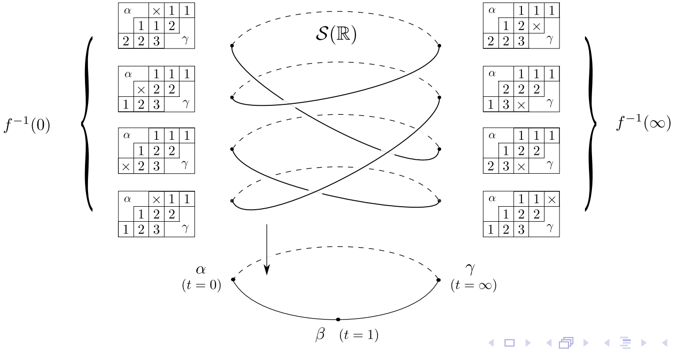

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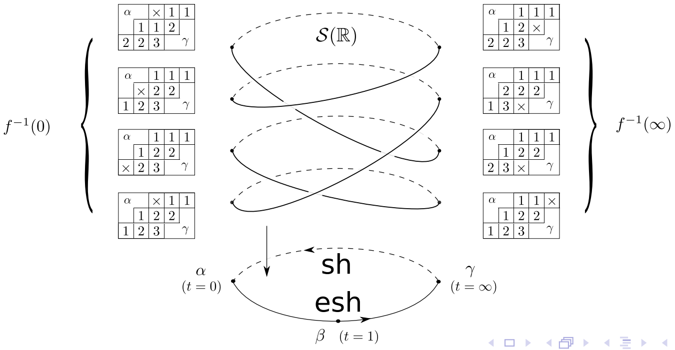
- *The arcs of* $\mathcal{S}(\mathbb{R})$ *lying over* $\mathbb{R}_-$ *and* $\mathbb{R}_+$ *induce* **shuffling (JDT)** *and* **evacuation-shuffling** *on tableaux* $(\mathrm{sh}, \mathrm{esh})$.



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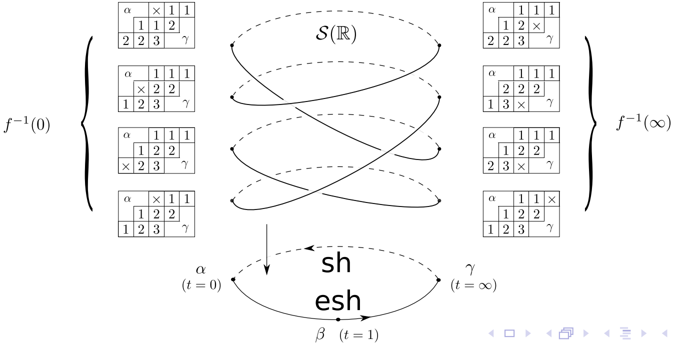
- *Monodromy operator:* $\omega = \mathrm{sh} \circ \mathrm{esh}$.
- *Orbit structure of ω fully characterizes $\mathcal{S}(\mathbb{R})$!*



Shuffling and Evacuation-Shuffling

Two bijections:

$$\mathrm{LR}(\alpha, \square, \beta, \gamma) \underset{\mathrm{sh}}{\overset{\mathrm{esh}}{\rightleftarrows}} \mathrm{LR}(\alpha, \beta, \square, \gamma)$$

- **Shuffling**, or **JDT**: Slide $\boxtimes$ through the tableau using jeu de taquin.

α	α	α	1	1	1
α	α	1	2	2	2
α	2	3	3	γ	γ
1	3	4	4	γ	γ
3	4	5	$\times$	γ	γ

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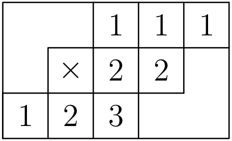

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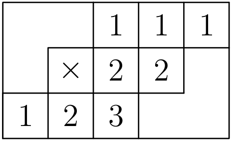

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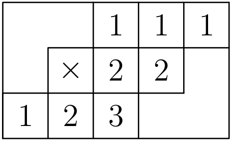

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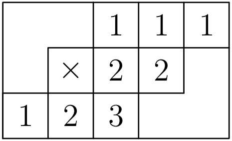

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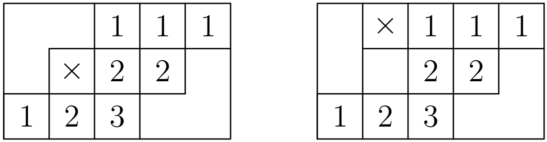

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  Conjugation (rsr^{-1}) of **shuffling** by **rectification**.
  1. **Rectify:** Treat $\times$ as 0.

		1	1	1
	$\times$	2	2	
1	2	3		

T

	$\times$	1	1	1
	2	2	2	
1	3	1		

Shuffling and Evacuation-Shuffling

Two bijections:

$$\mathrm{LR}(\alpha, \square, \beta, \gamma) \underset{\text{sh}}{\overset{\text{esh}}{\rightleftarrows}} \mathrm{LR}(\alpha, \beta, \square, \gamma)$$

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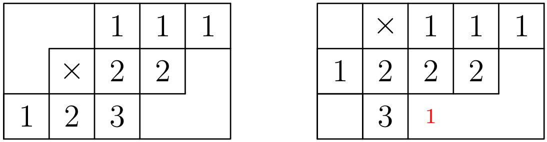

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	$\times$	1	1	1
1	2	2	2	
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1	2	3		

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$\times$		1	1	1
1	2	2	2	
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$\times$	1		1	1
1	2	2	2	
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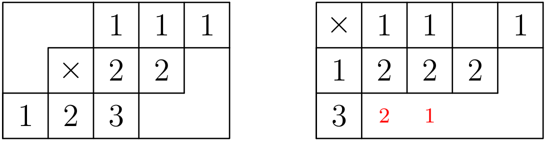

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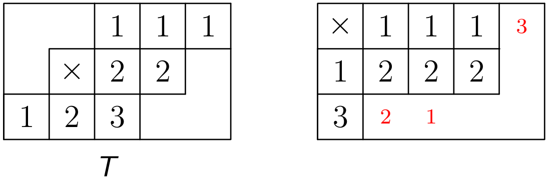

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	$\times$	2	2	
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T

1	1	1	1	3
$\times$	2	2	2	
3	2	1		

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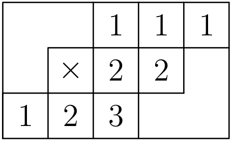

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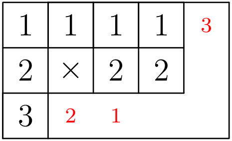

Shuffling and Evacuation-Shuffling

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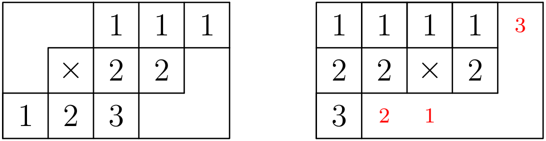

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		1	1	1
	$\times$	2	2	
1	2	3		

T

1	1	1	1	3
2	2	2	$\times$	
3	2	1		

Shuffling and Evacuation-Shuffling

Two bijections:

$$\mathrm{LR}(\alpha, \Box, \beta, \gamma) \underset{\mathrm{sh}}{\overset{\mathrm{esh}}{\rightleftarrows}} \mathrm{LR}(\alpha, \beta, \Box, \gamma)$$

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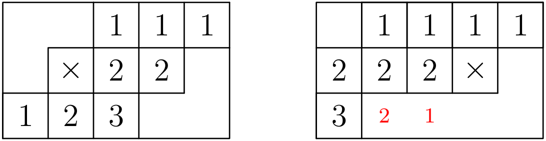

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Shuffling and Evacuation-Shuffling

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	1	1	1	1
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2		3		

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2	2	3		

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		1	1	1
	$\times$	2	2	
1	2	3		

T

		1	1	1
	1	2	$\times$	
2	2	3		

$\mathrm{esh}(T)$

Questions about $\omega = \mathrm{sh} \circ \mathrm{esh}$

Three motivating problems:

1. Find an easier algorithm.

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   - In general: likely hard!
   - Related: **promotion** on **standard tableaux**
     - orbits $\longleftrightarrow$ components of $S(\alpha, \underbrace{\square, \ldots, \square}_{|\gamma/\alpha|-1}, \gamma)(\mathbb{R})$.

Questions about $\omega = \text{sh} \circ \text{esh}$

Three motivating problems:

1. Find an easier algorithm.
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   - In general: likely hard!
   - Related: **promotion** on **standard tableaux**
     - orbits $\longleftrightarrow$ components of $S(\alpha, \underbrace{\square, \ldots, \square}_{|\gamma/\alpha|-1}, \gamma)(\mathbb{R})$.
3. Connection to K-theoretic Schubert calculus
   - Combinatorial identities involving $\chi(\mathcal{O}_S)$ and ω
   - $\chi(\mathcal{O}_S)$ computed by **genomic tableaux** [Pechenik-Yong '14]

Local, linear-time algorithm for evacuation-shuffling

Theorem (Gillespie,L.)

Start at $i = 1$.

					1	1	1	1	1
					2	2	2	2	
		×	1	2	3	3			
	1	1	2	3	4	4			
2	3	3	3	4	5	5			
3	4	4							

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- **Phase 1** *(move $\boxtimes$ down-and-left):*
  - *Switch $\boxtimes$ with* **nearest** *i in reading order.*

    If no i available, go to Phase 2.

Phase 1
($i = 1$)

					1	1	1	1	1
					2	2	2	2	
		×	1	2	3	3			
	1	1	2	3	4	4			
2	3	3	3	4	5	5			
3	4	4							

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Phase 1
($i = 2$)

					1	1	1	1	1
					2	2	2	2	
		1	1	2	3	3			
	1	$\times$	2	3	4	4			
2	3	3	3	4	5	5			
3	4	4							

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Phase 1
($i = 3$)

					1	1	1	1	1
					2	2	2	2	
		1	1	2	3	3			
	1	2	2	3	4	4			
×	3	3	3	4	5	5			
3	4	4							

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Phase 1
($i = 4$)

					1	1	1	1	1
					2	2	2	2	
		1	1	2	3	3			
	1	2	2	3	4	4			
3	3	3	3	4	5	5			
×	4	4							

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- **Phase 2** *(move $\boxtimes$ right-and-up):*
  - **If** *suffix from $\boxtimes$ is not tied for $(i, i+1)$, switch $\boxtimes$ with* **nearest** *i in reading order. Repeat until tied. Increment i, repeat.*

Phase 2
$(i = 4)$

					1	1	1	1	1
					2	2	2	2	
		1	1	2	3	3			
	1	2	2	3	4	4			
3	3	3	3	4	5	5			
×	4	4							

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				1	1	1	1	1
				2	2	2	2	
		1	1	2	3	3		
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3	3	3	3	4	5	5		
4	×	4						

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Phase 2
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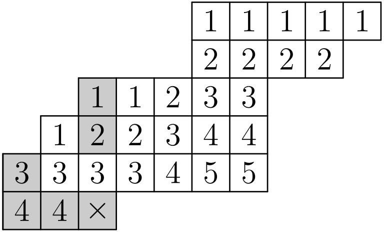

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					1	1	1	1	1
					2	2	2	2	
		1	1	2	3	3			
	1	2	2	3	4	4			
3	3	3	3	$\times$	5	5			
4	4	4							

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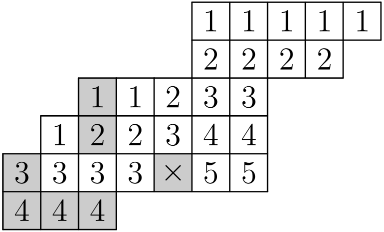
Phase 2
($i = 5$)



Local, linear-time algorithm for evacuation-shuffling

Theorem (Gillespie,L.)

Start at $i = 1$.

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Phase 2
($i = 5$)

					1	1	1	1	1
					2	2	2	2	
		1	1	2	3	3			
	1	2	2	3	4	4			
3	3	3	3	5	$\times$	5			
4	4	4							

Local, linear-time algorithm for evacuation-shuffling

Theorem (Gillespie,L.)

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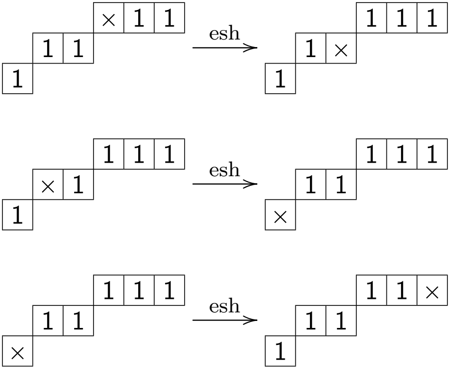
Phase 2
($i = 5$)

					1	1	1	1	1
					2	2	2	2	
		1	1	2	3	3			
	1	2	2	3	4	4			
3	3	3	3	5	5	$\times$			
4	4	4							

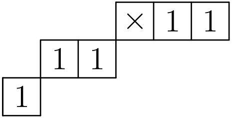
Proof of local rule

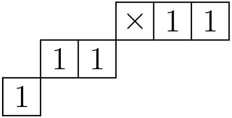
- First: "Pieri case", β = horizontal strip = .
- Claim:



Proof of local rule

- Proof of Pieri Case:





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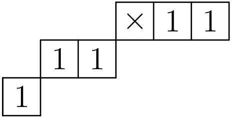

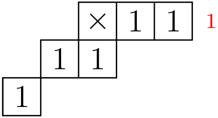

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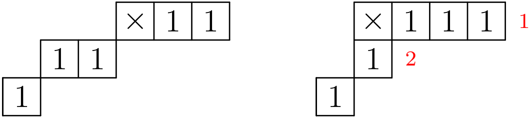

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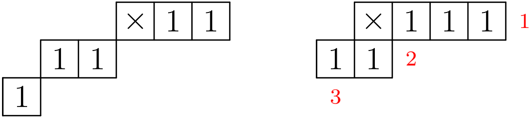

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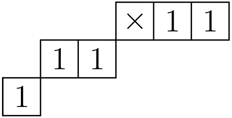

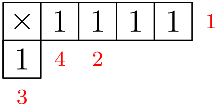

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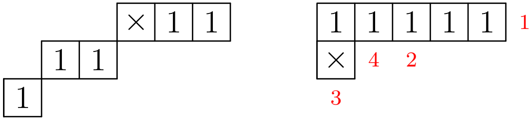

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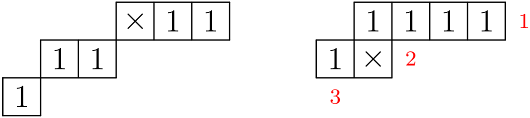

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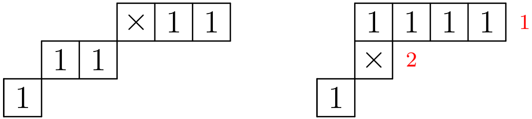

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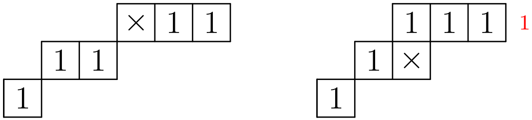

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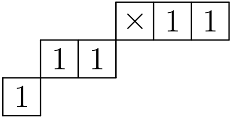

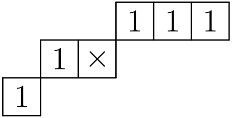

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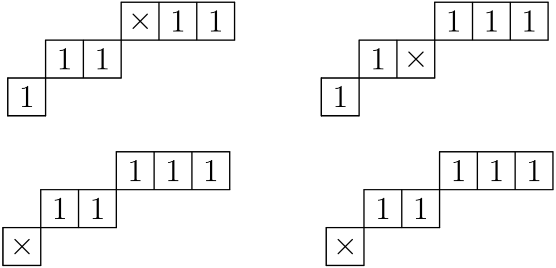

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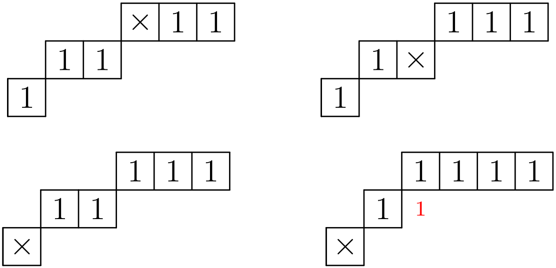

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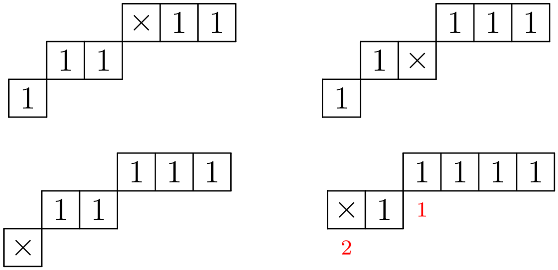

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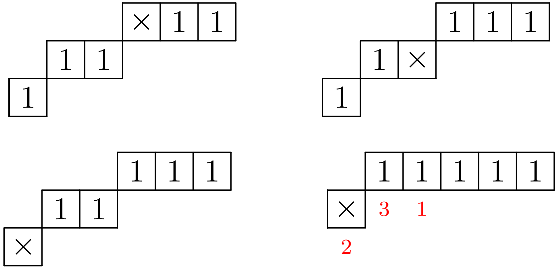

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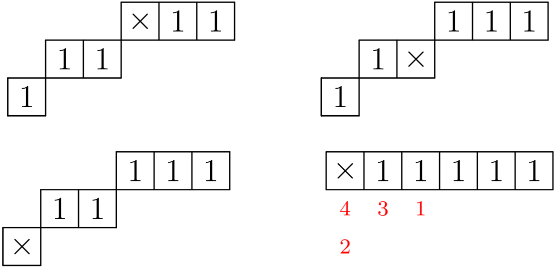

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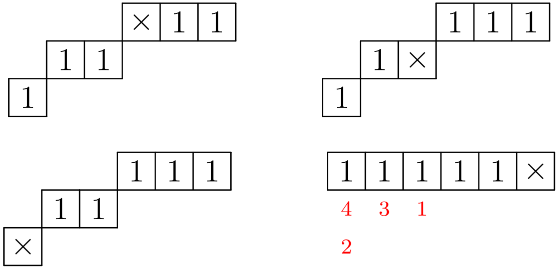

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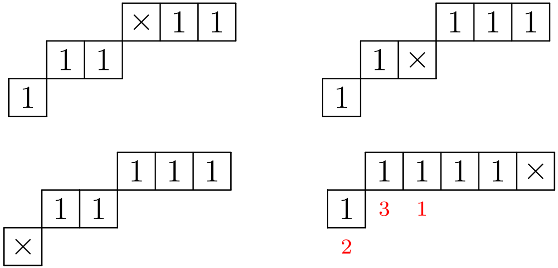

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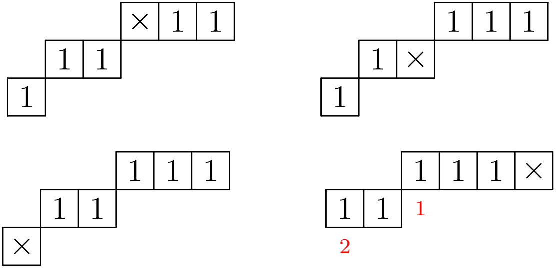

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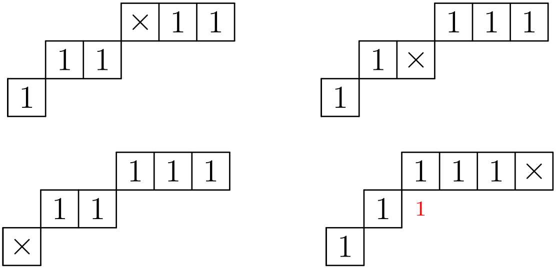

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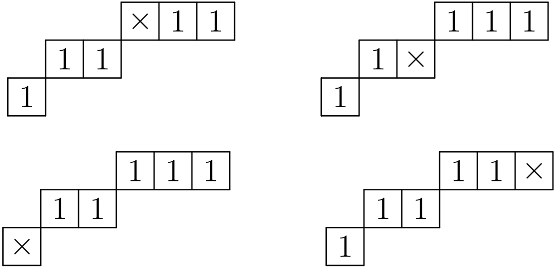

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$\times$	1	1	1	1	1
1	2	2	2	2	
2	3	3	3	3	
4	4	4			
5					

						1	1	1
			$\times$	1	1	2	2	
			1	2	2	3		
			3	3	4			
		4	4					
2	3	5						

- Switch $\boxtimes$ with **nearest** square...
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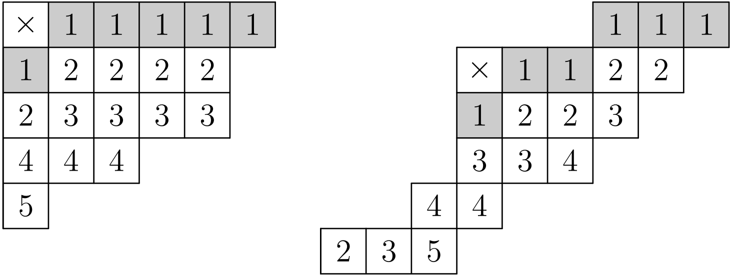

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2	3	3	3	3	
4	4	4			
5					

						1	1	1
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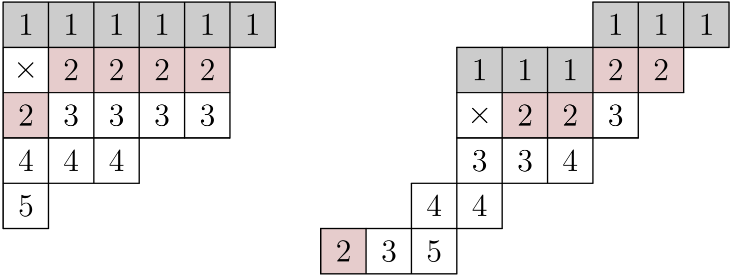

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2	2	2	2	2	
$\times$	3	3	3	3	
4	4	4			
5					

						1	1	1
			1	1	1	2	2	
			2	2	2	3		
			3	3	4			
		4	4					
$\times$	3	5						

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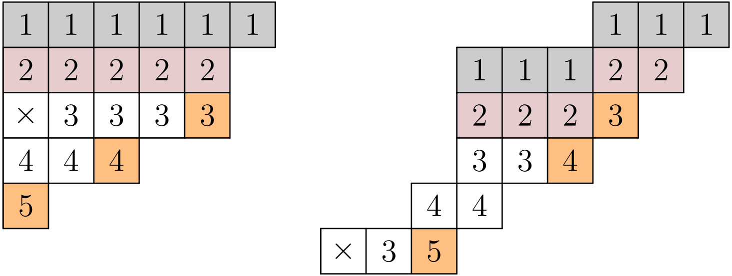

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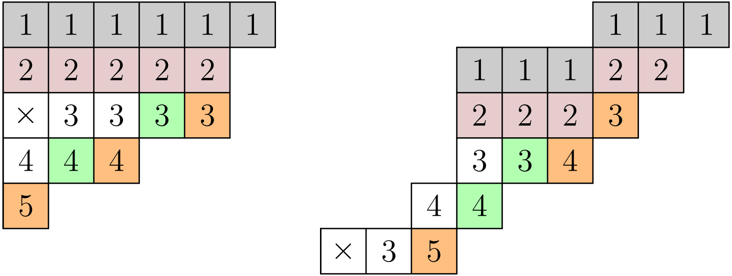

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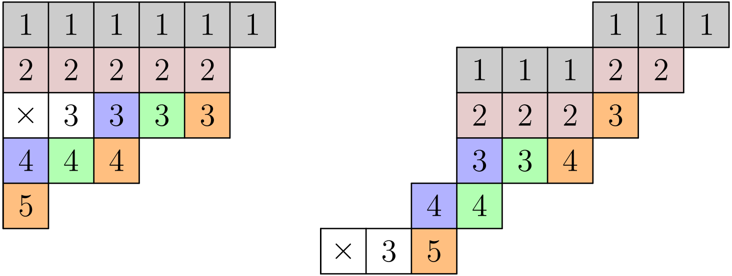

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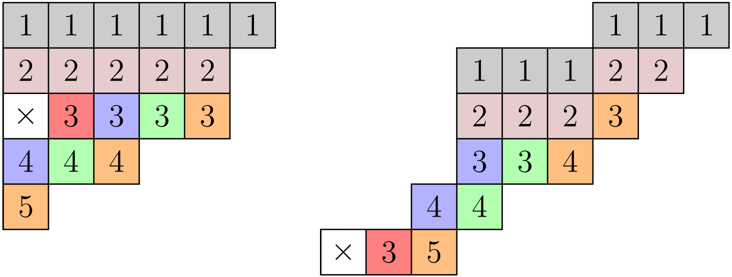

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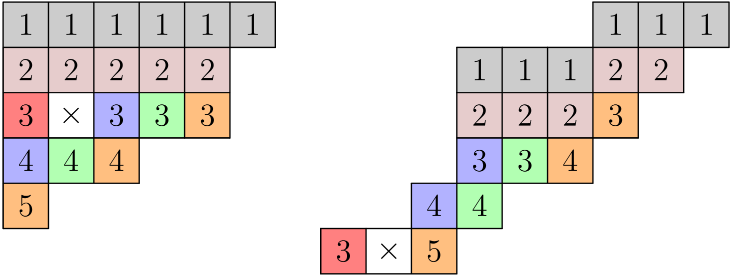

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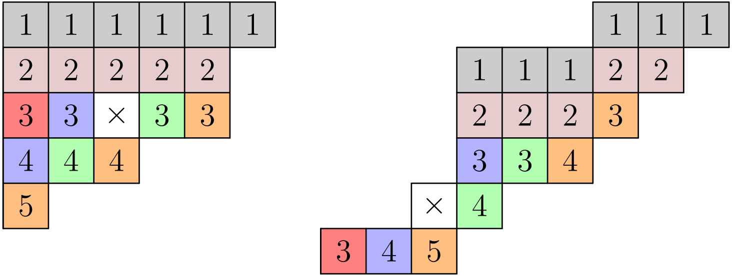

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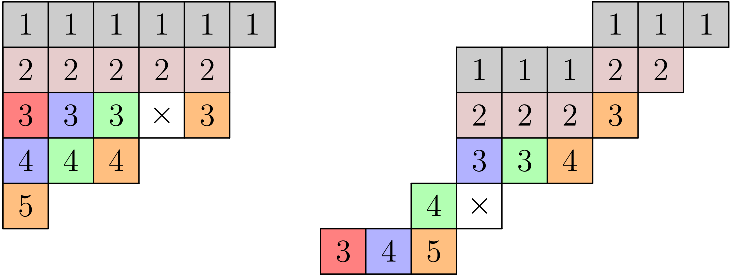

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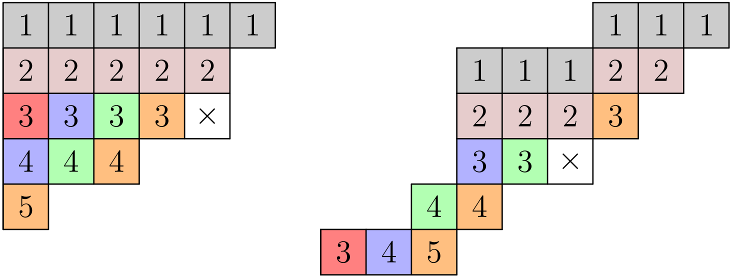

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Application: K-theory and $\chi(\mathcal{O}_{\mathcal{S}})$

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K-theoretic Schubert calculus:

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- $k^{\square\square\square}_{\alpha,\beta,\gamma} = |K(\gamma^c/\alpha; \beta)| =$ genomic tableaux [Pechenik-Yong '15]

Genomic tableaux

- **Genomic tableau**: T with shaded entries $\{\boxtimes, \boxtimes'\}$, where:
  (i) $\boxtimes, \boxtimes'$ non-adjacent, contain same entry i,
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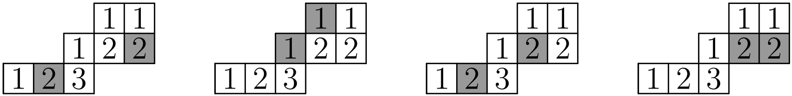

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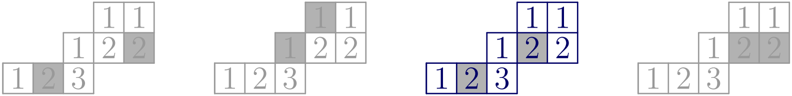

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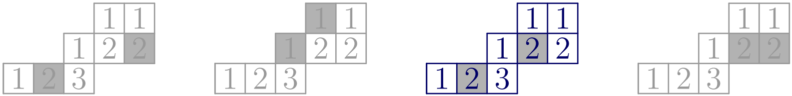

- ***K*-theoretic content:** $\beta = (4, 2, 1)$

Generating genomic tableaux

Theorem (Gillespie, L.)

Two bijections

$$K(\gamma^c/\alpha;\beta) \leftrightarrow \{\textit{non-adjacent moves in esh (Phase 1)}\}$$
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						1	1	1
			×	1	1	2	2	
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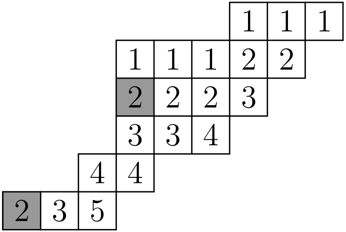

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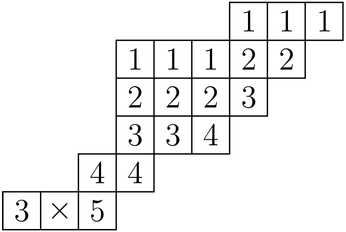

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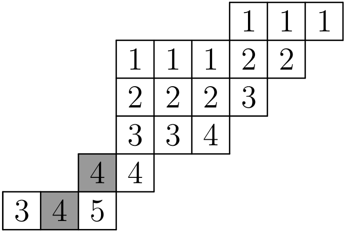

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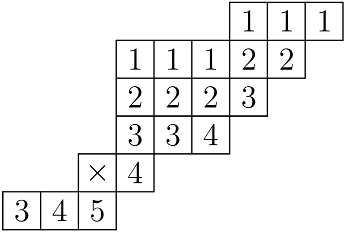

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						1	1	1
			1	1	1	2	2	
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$$K(\gamma^c/\alpha;\beta) \leftrightarrow \{\textit{non-adjacent moves in esh (Phase 2)}\}$$

						1	1	1
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			2	2	2	3		
			3	3	4			
		4	4					
3	4	5						

Generating genomic tableaux

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(Not true of general maps of real algebraic curves!)

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Schubert curves over $\mathbb{C}$ [Gillespie, L.]:

- $\mathcal{S}$ with arbitrarily many $\mathbb{C}$-connected components
- $\mathcal{S}$ integral, with arbitrarily high genus $g_a(\mathcal{S})$

Application: sign, rlength of ω

- **Reflection length** of $\sigma \in S_N$
  $= \min\{r : \sigma = \tau_1 \cdots \tau_r\}$ with τ_i arbitrary transpositions
  $= N - \#\mathrm{orbits}(\sigma)$

- **Sign** $\mathrm{sign}(\sigma) = \mathrm{rlength}(\sigma) \pmod 2$.

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Is there a combinatorial explanation?

The sign and reflection length of ω

Corollary (Gillespie,L.)

Independent proofs of

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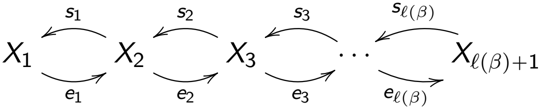

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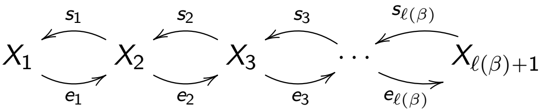

Each $s_i \circ e_i$ has very simple orbit structure and

$$\mathrm{sign}(\omega) \equiv \sum \mathrm{sign}(s_i \circ e_i) \pmod 2,$$
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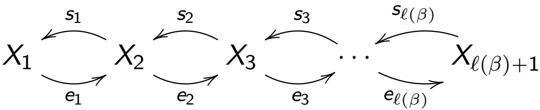

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- **Conjecture**. In every orbit $\mathcal{O}$ of ω, at least $|\mathcal{O}| - 1$ genomic tableaux are generated (in each Phase).

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- Local rules for esh, ω in general:
  - Shifted tableaux for $OG(n, 2n+1)$ [with Kevin Purbhoo]

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Geometry:

- Schubert curves in $OG(n, 2n+1)$, $LG(2n)$ [Purbhoo]
- Higher dimensions: "Schubert surfaces", 3-folds, ...

PREVIEW: Schubert curves in OG(n,2n+1)

(with Maria Gillespie and Kevin Purbhoo)

- Odd orthogonal Grassmannian (type C):
  - Symmetric form $\langle -, - \rangle$ on $\mathbb{C}^{2n+1}$
  - $V \subseteq \mathbb{C}^{2n+1}$ is **isotropic** if $\langle v_1, v_2 \rangle = 0$ for all $v_1, v_2 \in V$.
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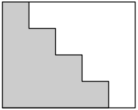
- 'Halved' combinatorial picture: 
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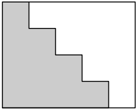
- 'Halved' combinatorial picture: 
- Similar story, giving $\mathcal{S}(\alpha, \beta, \gamma) \subset OG(n, 2n+1)$
- **Thm** (G-L-P). Topology of $\mathcal{S}$ determined by **shifted** JDT, esh.
  - Local esh: Phase 1 resembles Type A, Phase 2 does not!
  - Instead, Phase 2 uses crystal-like (coplactic) operators on words.

Schubert curves in OG(n,2n+1)

- Strict partitions α, β, γ, shifted (ballot) SSYTs T
- Alphabet $1' < 1 < 2' < 2 < \cdots$, allowed to have $\begin{array}{|c|}\hline 1' \\ \hline 1' \\ \hline \end{array}$ and $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}$.
  - Convention: first $\{i, i'\}$ in reading order is an i.
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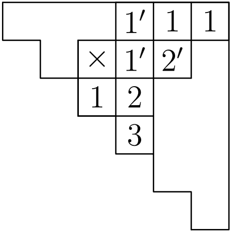

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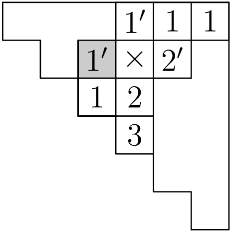

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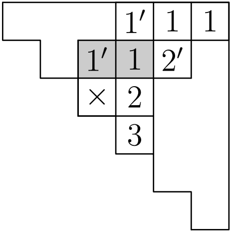

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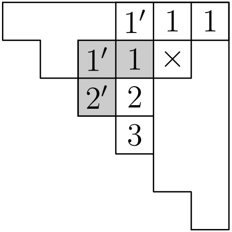

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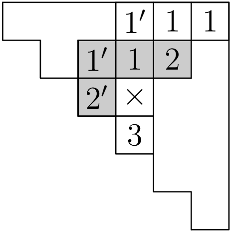

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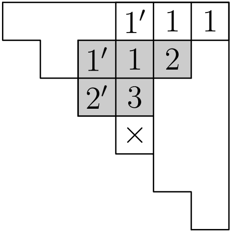

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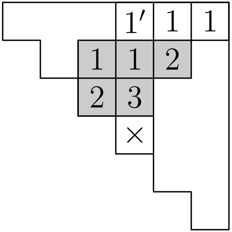
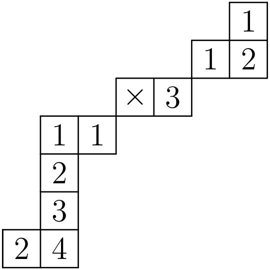


Example of Phase 2 (coplactic operators)

- Operators E_i, F_i, E'_i, F'_i for raising and lowering weights
  - F: converts an $i \to i+1$, possibly also moves a prime
  - F': converts an $i \to (i+1)'$ (can omit in computation)
- Apply $F_1, F_2, F_3, \ldots$ (essentially "$\lim_{x \to \infty} F_x$")



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						1
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			$1'$	3		
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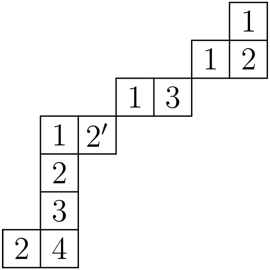

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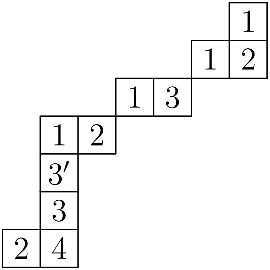

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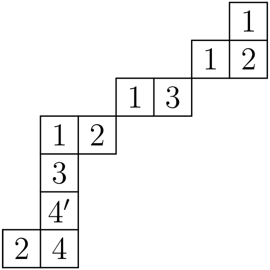

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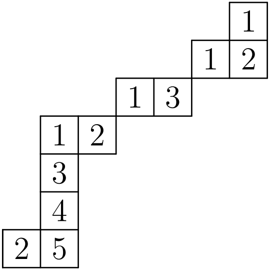

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						1
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	3					
	4					
2	6					

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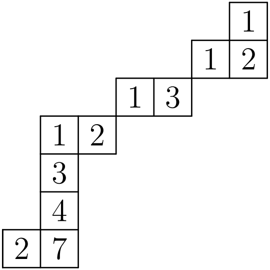

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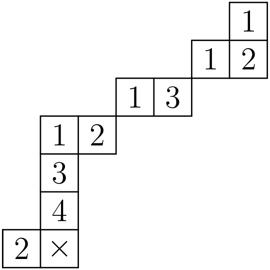

Thank you!