Syzygies, Hilbert polynomials and matrix varieties

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Equations and geometry

I am an algebraic geometer (and combinatorialist).

My way to understand interesting spaces X:

$$\text{equations defining } X \qquad \leftrightarrow \qquad \text{geometry of } X$$

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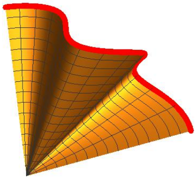
homogeneous polynomials $x^2y^3 + 2y^5 - x^4y = 0$		X is scale-invariant: $x \in X \implies t \cdot x \in X$



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This is not always true.

Motivating example (from linear algebra)

Rank-deficient matrices:

$$\begin{array}{ccc} \mathbb{C}^6 & = & \{2 \times 3 \text{ matrices } M\} \\ \cup\!\mid & & \cup\!\mid \\ Z & = & \{\mathrm{rank}(M) < 2\} \end{array}$$

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Relevant to **matrix factorization**, e.g. Netflix preference model:

$$\underbrace{(\text{users} \times \text{movies})}_{\text{HUGE(!), sparse}} = \underbrace{(\text{users} \times \text{features}) \cdot (\text{features} \times \text{movies})}_{\mathrm{rank} \, \leq \, |\text{features}|}$$

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Goal: understand Z.

- equations, topology, dimension...

Rank-deficient 2×3 matrices

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- All three are necessary: $\begin{bmatrix} * & * & 0 \\ * & * & 0 \end{bmatrix}, \begin{bmatrix} * & 0 & * \\ * & 0 & * \end{bmatrix}, \begin{bmatrix} 0 & * & * \\ 0 & * & * \end{bmatrix}$

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Problem: this is false! In fact $\dim(Z) \geq 4$:

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Even worse for $k \times n$ matrices:

- kn variables, $\binom{n}{k}$ equations $\rightsquigarrow kn - \binom{n}{k}$ is negative!
- (Actually $\dim(Z) = (k-1)(n+1)$.)

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Eventual idea: study **syzygies** (polynomial relations) between the defining equations.

Hilbert (and abstract algebra) to the rescue

Look at all all possible polynomial equations for Z:

- Polynomials in $R = \mathbb{C}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

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Hilbert's idea (1890):

- Study the **quotient ring** R/I_Z (set $f = 0$ for $f \in I_Z$)
- Intuition: the larger R/I_Z is, the larger the space Z.

Hilbert's idea (cont'd)

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- Specifically, look at the vector space

  $(R/I_Z)_d := \{$ homogeneous degree-d elements of $R/I_Z \}$.

  Then, as $d \to \infty$: $\quad \text{vdim}_{\mathbb{C}}(R/I_Z)_d \approx d^{\dim(Z)-1}$.

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Warm-up: let's try $R = \mathbb{C}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ (for $Z = \mathbb{C}^6,\ I_Z = (0)$).

Claim: $\operatorname{vdim}_{\mathbb{C}}(R_d) = O(d^{6-1}) = O(d^5)$.

How big is a polynomial ring?

Compute $\mathrm{vdim}_{\mathbb{C}}(R_d) = \{$ homog. deg-d poly's in 6 variables $\}$:

- Count monomials of total degree d

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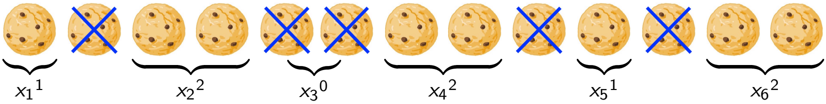

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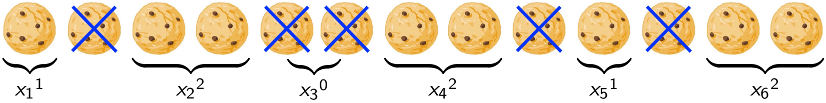

- Total # : $\binom{d+5}{5} = \frac{(d+5)(d+4)\cdots(d+1)}{5!} = \frac{1}{5!}d^5 + \cdots$.

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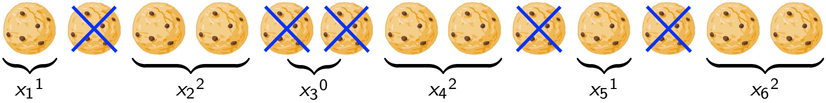

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Theorem (Hilbert)

Let R be a graded ring, X the corresponding geometric space. Then $\mathrm{vdim}_{\mathbb{C}}(R_d)$ is eventually a polynomial of degree $\dim(X) - 1$.

Rings, modules and gradings

Let R be a ring. An R-**module** M is a "vector space over R":

- Can **add**: $m_1 + m_2 \in M$
- Can **scalar-multiply** by R: $r \cdot m \in M$

Think: ideals, quotient rings, $R^n := \{(r_1, \ldots, r_n) : r_i \in R\}$
R^n is a "rank-n free module", generators $(0, \ldots, 1_i, \ldots, 0)$.

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A module (or ring) is **graded** if its elements "have degrees":
Think: I, R/I given by homogeneous polynomials.

- $M \supseteq M_d = \{\text{deg-}d \text{ homogeneous elements}\}$,
- Every $m \in M$ decomposes as $m = \sum_d m_d$
- Multiplying: $\deg(r \cdot m) = \deg(r) + \deg(m)$.

Hilbert's approach for R/I_Z

Back to R/I_Z, where $I_Z = (\det_{[23]}, \det_{[13]}, \det_{[12]})$.

Claim: $\mathrm{vdim}_{\mathbb{C}}(R/I_Z)_d$ is a polynomial.

$$R \xrightarrow{f_0} R/I_Z$$

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Is f_1 injective? If yes, **we'd be done**:

$$\operatorname{vdim}_{\mathbb{C}}(R/I_Z)_d = \operatorname{vdim}_{\mathbb{C}}(R_d) - 3 \cdot \operatorname{vdim}_{\mathbb{C}}(R_{d-2})$$

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Sadly, it is **not** injective, $\ker(f_1) \neq 0$.

Syzygies!

Let's describe $\ker(f_1)$ = the module of **syzygies**.
Elements of $\ker(f_1)$ are **relations** among the generators of I_Z:

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Complete, minimal generators for $\ker(f_1)$.

Hilbert's approach for R/I_Z

Back to R/I_Z.

$$R(-2)^3 \xrightarrow{f_1} R \xrightarrow{f_0} R/I_Z$$

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Back to R/I_Z.

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$$(1,0) \mapsto (a_{11}, -a_{12}, a_{13})$$
$$(0,1) \mapsto (a_{21}, -a_{22}, a_{23})$$

We had $\ker(f_1) \neq 0$ with two generators.

So, set up f_2 with $\text{image}(f_2) = \ker(f_1)$.

Is f_2 injective? If yes, **we'll be done**.

It is!

Hilbert's approach for R/I_Z

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\end{aligned}$$

In particular: $\dim(Z) = 3 + 1 = 4.$

Hilbert's Syzygy Theorem

The sequence

$$0 \to R(-3)^2 \xrightarrow{f_2} R(-2)^3 \xrightarrow{f_1} R \xrightarrow{f_0} R/I_Z$$

is called a (graded) **free resolution** of R/I_Z.

The property $\text{image}(f_{i+1}) = \ker(f_i)$ is called "exactness".

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Theorem (Hilbert's Syzygy Theorem, 1890)

Every *graded R-module M has a graded free resolution, terminating after* **finitely-many** *steps (at most* $\dim(R)$ *needed).*

Therefore, for $d \gg 0$, *vdim*$_{\mathbb{C}}(M_d)$ *is a polynomial, called the* **Hilbert polynomial** $\mathrm{hilb}_M(d)$.

What does the Hilbert polynomial tell us about X?

- Topology: $\deg(\mathrm{hilb}_R) = n \rightsquigarrow$ **dimension** of $X = n + 1$
- Algebra/geometry:

  $n! \cdot$ (leading coefficient) $\rightsquigarrow$ the **degree** of X

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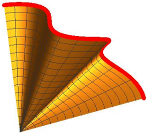
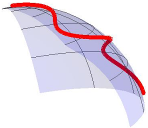
And more:

$\mathrm{hilb}_R(0) =$ **Euler characteristic** of the **projective variety** of X



(homogeneous eqns: X scale-invariant) $\rightsquigarrow$



projectivization of X

Improving on the Hilbert polynomial

Hilbert polynomials are great, but there was more data:

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*The **minimal** graded free resolution is **unique**.*

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Record #, deg of syzygies at each step: the **Betti table** $\beta(R/I_Z)$:

$\cdots$	2	1	0	$\beta_{i,d}$
–	–	–	1	$d = 0$
–	–	–	–	$d = 1$
–	–	3	–	$d = 2$
–	2	–	–	$d = 3$

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- Defining equations, deformation theory, smoothness

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Example (3D interpolation)

Fix 7 points in 3D, $Z = \{p_1, \ldots, p_7\} \subseteq \mathbb{C}^3$.

Is there a low-degree curve $(x(t), y(t), z(t)) \subseteq \mathbb{C}^3$ through all 7?

Hilbert polynomial of Z can't tell (same for any 7 points).
Betti table can: **Yes** if $\beta_{2,3} \neq 0$, **no** otherwise!

Which Betti tables are possible?

Modern Goal: Classify Betti tables. Which tables can occur?

$\rightsquigarrow$ **Combinatorics** to describe behaviors of spaces, equations.

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$$\text{Exactness} \rightsquigarrow \underbrace{\sum_d \beta_{i,d}}_{\text{column } i} \leq \underbrace{\sum_d \beta_{i+1,d}}_{\text{column } i+1} + \underbrace{\sum_d \beta_{i-1,d}}_{\text{column } i-1}.$$

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Boij-Söderberg theory (2006)

Study Betti tables *up to rational multiple*:

$$BS_n := \mathbb{Q}_{\geq 0} \cdot \{\beta(M) : \text{ graded modules } M\}.$$

BS_n is called the **cone of Betti tables**.

Boij-Söderberg theory is growing rapidly

Graded Betti tables are fully understood:

Theorem (Eisenbud-Schreyer 2008)

The Betti cone BS_n is **rational polyhedral**, *with explicitly-known* **extremal rays** *and* **facets**.

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Facets: via geometry of $\mathbb{P}^n$: cohomology of vector bundles
Duality: Betti tables $\longleftrightarrow$ "cohomology tables"

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- Multigraded polynomials, toric rings, (some) homogeneous coordinate rings
- Partial results, many **open questions**: describe rays, facets...
  - Bi-graded Betti tables over $R = \mathbb{C}[x_0, x_1, y_0, y_1]$ ($\mathbb{P}^1 \times \mathbb{P}^1$)
    - $x_0^3 y_0 + x_0 x_1^2 (y_0 + y_1) + x_1^3 y_0 = 0$ (bi-degree $(3, 1)$)

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- Computer-aided exploration in Macaulay2, Sage...
- Cohomology of vector bundles on $\mathbb{P}^n \times \mathbb{P}^m$, on toric varieties, on curves...

Syzygies of matrix varieties

My interest: matrices and subvarieties $Z \subset \mathbb{C}^{kn} = \mathrm{Mat}_{k \times n}$.

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    $$\rightsquigarrow Z = \{M : \dim(\mathrm{rowspan}(M) \cap P_i) \geq d_i \text{ for each } i\}.$$
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  - Tangent spaces of varieties $V \subset \mathbb{P}^{n-1}$
- Study GL_k-**equivariant** modules, resolutions, Betti tables
  - Record GL_k-action on syzygies, not just the degrees

Syzygies of matrix varieties

Representations of GL_k correspond to **Young diagrams**:

$$\ydiagram{3,1} \quad \longleftrightarrow \quad V_{\ydiagram{3,1}}(\mathbb{C}^k) \otimes R.$$

The GL_k-equivariant free resolution for R/I_Z:

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Questions:

- Is the Betti cone $BS_{k,n}$ rational polyhedral?
  - $\mathrm{Mat}_{k \times k}$, $\mathrm{pdim}(M) = 1$: yes! [Ford-L.-Sam '16]
- What are its extremal rays, facets?
- Cohomology of vector bundles on **Grassmannians** $Gr(k, n)$
  - Duality exists in this setting [Ford-L. '16]

Thank you!