“CARRYING THE TENS” AND $\operatorname{Ext}^1(\mathbb{Z}/10, \mathbb{Z}/10)$

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Adding two-digit numbers in $\mathbb{Z}/100$ seems superficially similar to adding pairs of digits in $\mathbb{Z}/10 \oplus \mathbb{Z}/10$. The difference is whether or not we ‘carry the ten’ when the ones digit wraps around:

$$\begin{array}{rcr} \text{in } \mathbb{Z}/100: & \qquad & \text{in } \mathbb{Z}/10 \oplus \mathbb{Z}/10: \\ 27 & & (2,7) \\ +18 & & +(1,8) \\ =45 & & =(3,5) \end{array}$$

In the first, we increment the tens digit by 1. In the second, we don’t!

This writeup (phrased as a problem set) walks through an explanation of how the elements of $\operatorname{Ext}^1(\mathbb{Z}/10, \mathbb{Z}/10)$ correspond to **different ‘carrying’ rules**. In particular, the two situations above correspond to nonisomorphic short exact sequences,

$$0 \to \mathbb{Z}/10 \to \mathbb{Z}/10 \oplus \mathbb{Z}/10 \to \mathbb{Z}/10 \to 0, \tag{1}$$

$$0 \to \mathbb{Z}/10 \to \mathbb{Z}/100 \to \mathbb{Z}/10 \to 0. \tag{2}$$

The element $0 \in \operatorname{Ext}^1(\mathbb{Z}/10, \mathbb{Z}/10)$ will correspond to the split extension, and $1 \in \operatorname{Ext}^1(\mathbb{Z}/10, \mathbb{Z}/10)$ will correspond to the ordinary carrying rule for $\mathbb{Z}/100$. The remaining elements will correspond to alternative rules, resulting in weird ways to “add two-digit numbers”.

(a) Consider an arbitrary extension

$$0 \to \mathbb{Z}/10 \xrightarrow{\sigma} A \xrightarrow{\pi} \mathbb{Z}/10 \to 0,$$

and fix $x \in A$ such that $\pi(x) = 1$. **Show** that there is a unique $i_A \in \{0, \ldots, 9\}$ such that $10x = \sigma(i_A)$, and that any choice of $x \in \pi^{-1}(1)$ would have given the same i_A.

We call i_A the *carrying number for A*. Show that if $A = \mathbb{Z}/100$, then $i_A = 1$, but if $A = \mathbb{Z}/10 \oplus \mathbb{Z}/10$, then $i_A = 0$.

(b) Fix $x = x_A$ as in part (a). Define a function as follows:

$$[\cdot, \cdot] : \{0, \ldots, 9\} \times \{0, \ldots, 9\} \to A,$$

$$[ij] := \sigma(i) + j \cdot x_A.$$

Show that this is a bijection of sets (though it is **not** a homomorphism).

Using the $[ij]$ notation, we can think of elements of A as two-digit numbers, except that their addition does not behave normally. For example, show that

$$[09] + [01] = [i_A 0].$$

(Similarly, $[0a] + [0b] = [i_A 0]$ whenever $a + b = 10$.) If $i_A = 3$, show that $[27] + [18] = [65]$. So, we "carry an i_A whenever the ones digit wraps around."

(c) Let us add $[ij] + [i'j']$ in A. By part (b), this must equal $[i''j'']$ for some unique i'', j''. There are two cases:
- If $j + j' \leq 9$, then we have $i'' = i + i' \pmod{10}$ and $j'' = j + j'$.
- If $j + j' \geq 10$, then we have $i'' = i + i' + i_A \pmod{10}$ and $j'' = j + j' \pmod{10}$.

(d) Given another extension

$$0 \to \mathbb{Z}/10 \xrightarrow{\sigma'} B \xrightarrow{\pi'} \mathbb{Z}/10 \to 0,$$

with $i_A = i_B$, show that the choices of elements x_A, x_B yield an isomorphism,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/10 & \xrightarrow{\sigma} & A & \xrightarrow{\pi} & \mathbb{Z}/10 & \longrightarrow & 0 \\
 & & \| & & \downarrow f & & \| & & \\
0 & \longrightarrow & \mathbb{Z}/10 & \xrightarrow{\sigma'} & B & \xrightarrow{\pi'} & \mathbb{Z}/10 & \longrightarrow & 0
\end{array}$$

where f "sends $[ij]_A$ to $[ij]_B$", that is, $f(\sigma(i) + j \cdot x_A) = \sigma'(i) + j \cdot x_B$. (Hint: use (b) to show that f is a well-defined bijection, and use (*c*) to show that it is a homomorphism.) Conclude that the carrying number determines the extension up to isomorphism.

(e) Parts (a)-(d) show that any extension A has a carrying number $i_A \in \{0, \ldots, 9\}$, and that this determines the extension up to isomorphism. It remains to show that every carrying number is, in fact, possible.

We already know that $i_A = 0, 1$ correspond to the short exact sequences (1), (2), respectively. Now suppose the extension with carrying number n is given by

$$0 \to \mathbb{Z}/10 \xrightarrow{\sigma_n} A_n \xrightarrow{\pi_n} \mathbb{Z}/10 \to 0.$$

Let n, m be integers. Define

$$X = \big\{(a, a') : \pi_n(a) = \pi_m(a')\big\} \subseteq A_n \oplus A_m,$$
$$Y = \big\{(\sigma_n(i), -\sigma_m(i)) : i \in \mathbb{Z}/10\big\} \subseteq A_n \oplus A_m.$$

Show that $Y \subseteq X$, and that there is a short exact sequence

$$0 \to \mathbb{Z}/10 \xrightarrow{\sigma} X/Y \xrightarrow{\pi} \mathbb{Z}/10 \to 0,$$

where σ and π are defined by $\sigma(i) = (\sigma_n(i), 0)$ and $\pi(a, a') = \pi_n(a)$.

Thus, X/Y is a new extension, the **Baer sum of A_n and A_m**. Show, finally, that its carrying number is $n + m \pmod{10}$. Conclude that every carrying number $0, \ldots, 9$ can occur in some extension, and that the assignment

$$\mathrm{Ext}^1(\mathbb{Z}/10, \mathbb{Z}/10) \to \mathbb{Z}/10,$$
$$A \mapsto i_A,$$

is an isomorphism of abelian groups.

Remark. We already knew how these other extensions were supposed to work: they're just "adding two-digit numbers using a modified carrying number". The key thing that is *not* clear (and that is implied by Exercise (e)) is that this rule does not introduce new relations – that is, the resulting abelian group still has 100 distinct elements.

(f) Analyze the remaining carrying numbers, checking the following:

- $i_A = \{3, 7, 9\}$ gives an extension where the central group is isomorphic to $\mathbb{Z}/100$, but the extension as a whole is not isomorphic to the extension (2). (Hint: show that the element [01] has order 100.)
- $i_A = \{2, 4, 6, 8\}$ gives an extension where the central group is isomorphic to $\mathbb{Z}/50 \times \mathbb{Z}/2$. (Hint: show that the element [01] has order 50 and that every element has order dividing 50.)
- $i_A = 5$ gives an extension where the central group is isomorphic to $\mathbb{Z}/20 \times \mathbb{Z}/5$. (Hint: show that the element [01] has order 20 and that every element has order dividing 20.)

(g) Which step goes wrong if we try to reproduce these constructions on $\mathbb{Z}/3 \oplus \mathbb{Z}/5$? (Hint: Something must, since $\mathrm{Ext}^1(\mathbb{Z}/5, \mathbb{Z}/3) = 0$.)