SOME SPECIAL VALUES OF COSINE

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1. Introduction

We all learn a few specific values of $\cos(x)$ (and $\sin(x)$) in high school – such as those in the following table:

x	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$	π
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0

Surely there are other 'nice' values of $\cos(m\pi)$, where $m \in \mathbb{Q}$? In fact, it turns out there are at least a couple that are more or less as nice as those in the above table, such as

$$\cos(\tfrac{\pi}{5}) = \tfrac{1}{4}(\sqrt{5}-1), \qquad \cos(\tfrac{\pi}{12}) = \tfrac{1}{2}(\sqrt{2}+\sqrt{6}), \qquad \cos(\tfrac{\pi}{8}) = \tfrac{1}{2}\sqrt{2+\sqrt{2}}. \tag{1.1}$$

We'll find these and a few others, using basic Galois theory and algebraic number theory. In particular, we'll classify the values of $m \in \mathbb{Z}$ such that $\cos(\frac{2\pi}{m})$ is, respectively, rational, quadratic, biquadratic and quartic (i.e., a nested quadratic). Our goal is to proceed algebraically as much as possible, so we avoid embedding into $\mathbb{C}$. In fact, the only truly analytic fact we need is

$$\cos(\tfrac{2\pi}{m}) \geq 0 \text{ if } m \geq 4. \tag{1.2}$$

We also observe that complex conjugation (always) restricts to the involution $\zeta \mapsto \zeta^{-1}$ of $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$.

2. Setup (and rational values of $\cos(\frac{2\pi}{m})$ for $m \in \mathbb{Z}$)

Our approach is based on the following fact (pointed out by Adam Kaye): let $\zeta = \zeta_m = e^{2\pi i/m} \in \mathbb{C}$ be a primitive m-th root of unity. Then

$$\alpha = \zeta + \zeta^{-1} = 2\,\mathrm{Re}(\zeta) = 2\cos(\tfrac{2\pi}{m}). \tag{2.1}$$

In particular, α is real, and ζ satisfies a quadratic polynomial over $\mathbb{Q}(\alpha)$,

$$\alpha\zeta = \zeta^2 + 1. \tag{2.2}$$

Assuming $m \neq 1, 2$, so that $[\mathbb{Q}(\zeta) : \mathbb{Q}] = \phi(m) \geq 2$, we therefore have the tower of fields

$$\begin{array}{c|l} & \mathbb{Q}(\zeta) \\ & \Big| \; 2 \\ \phi(m) & \mathbb{Q}(\alpha) = \mathbb{Q}(\zeta) \cap \mathbb{R} \\ & \Big| \; \frac{\phi(m)}{2} \\ & \mathbb{Q} \end{array}$$

In particular, the degree of $\cos(\frac{2\pi}{m})$ over $\mathbb{Q}$ is $\phi(m)/2$. We'll consider the cases where it is degree 4, since those are, for example, root extensions of $\mathbb{Q}$, hence reasonably tractable to work with.

2.1. **Rational values.** First of all, we see that $\alpha \in \mathbb{Q}$ if and only if $m = 1, 2$ or $\phi(m) = 2$, which is to say $m = 1, 2, 3, 4, 6$. We note that when m is odd, the identities

$$\zeta_{2m} = -\zeta_m^{(m+1)/2} \text{ and } \zeta_m = \zeta_{2m}^2 \tag{2.3}$$

show that the cyclotomic extension is the same and that α_{2m} is a Galois conjugate of α_m (the map $\zeta_m \mapsto \zeta_m^{(m+1)/2}$ is an automorphism of $\mathbb{Q}(\zeta_m)$). In particular, we can skip $m = 6$: we'll get it easily once we have done $m = 3$. For the others, we have the cyclotomic polynomials

$$\begin{aligned} f_1(x) &= x - 1, \\ f_2(x) &= \frac{x^2-1}{x-1} = x + 1, \\ f_3(x) &= \frac{x^3-1}{x-1} = x^2 + x + 1, \\ f_4(x) &= \frac{x^4-1}{x^2-1} = x^2 + 1. \end{aligned} \tag{2.4}$$

For the first two, we obtain $\zeta = 1, -1$, respectively, and so

$$\alpha = 2\cos(\tfrac{2\pi}{m}) = 2, -2 \text{ for } m = 1, 2, \tag{2.5}$$

as we learned in high school. For the other two, we observe that $-\alpha = -(\zeta + \zeta^{-1})$ is the linear term of the minimal polynomial of ζ! We conclude that

$$\alpha = -1, 0 \text{ for } m = 3, 4, \tag{2.6}$$

respectively, as we had hoped. Finally, for $m = 6$, we compute using our result from $m = 3$,

$$\alpha_6 = \zeta_6 + \zeta_6^{-1} = -\zeta_3^2 - \zeta_3^{-2} = -\alpha_3, \tag{2.7}$$

so $\alpha_6 = 1$. We have shown:

Corollary 1. *The only values of m for which $\cos(\frac{2\pi}{m}) \in \mathbb{Q}$ are $m = 1, 2, 3, 4, 6$. We have, respectively, $\cos(\frac{2\pi}{m}) = 1, -1, -\frac{1}{2}, 0, \frac{1}{2}$.*

3. Quadratic, Biquadratic and Quartic values

The quadratic values are straightforward, but will nicely illustrate the tools we'll use for the degree-4 case. We make use of the following standard facts:

Fact 1. *The cyclotomic field $\mathbb{Q}(e^{2\pi i/p})$, where p is an odd prime, contains $\sqrt{p}$ if $p \equiv 1 \bmod 4$ and $\sqrt{-p}$ if $p \equiv -1 \bmod 4$. The field $\mathbb{Q}(e^{2\pi i/8})$ contains $\sqrt{2}$.*

Fact 2. *The Galois group of the m-th cyclotomic field is isomorphic to the unit group $(\mathbb{Z}/m\mathbb{Z})^\times$. The unit u corresponds to the automorphism $\zeta \mapsto \zeta^u$. In particular, complex conjugation is always represented by the element $-1 \in \mathbb{Z}/m\mathbb{Z}$.*

Also, given a field extension L/K, we denote by N_K^L and Tr_K^L the norm and trace maps $L \to K$, respectively. Our approach is first to identify the extension $\mathbb{Q}(\alpha) = \mathbb{R} \cap \mathbb{Q}(\zeta)$ as a root extension (i.e. a quadratic, biquadratic or nested quadratic), so that α can be expressed in a familiar basis. We then use the trace and norm of certain intermediate fields to finish the computations.

3.1. **Quadratic values.** For this case, we need $\phi(m) = 4$, which yields $m = 5, 8, 10, 12$. As before, we save the case $m = 2 \cdot \text{odd}$ for the end.

For $m = 5$, the cyclotomic polynomial is

$$f_5(x) = 1 + x + x^2 + x^3 + x^4. \tag{3.1}$$

We have $(\mathbb{Z}/5\mathbb{Z})^\times \cong \mathbb{Z}/4\mathbb{Z}$, so there is exactly one quadratic subfield, which must be $\mathbb{Q}(\sqrt{5})$ by Fact 1. Hence $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{5})$, so $\alpha = a + b\sqrt{5}$ for some rational a, b. We will compute the trace and norm of α (which are, respectively, $2a$ and $a^2 - 5b^2$).

The nontrivial automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})$must come from $\zeta \mapsto \zeta^2$ (or, equivalently, ζ^3), since the other map restricts to the identity. Hence the Galois conjugate of α is $\zeta^2 + \zeta^{-2} = \zeta^2 + \zeta^3$, so the trace and norm are

$$\begin{aligned} \mathrm{Tr}_{\mathbb{Q}}^{\mathbb{Q}(\alpha)}(\alpha) &= \zeta + \zeta^2 + \zeta^3 + \zeta^4 = -1, \\ N_{\mathbb{Q}}^{\mathbb{Q}(\alpha)}(\alpha) &= (\zeta + \zeta^{-1})(\zeta^2 + \zeta^3) = \zeta^3 + \zeta^4 + \zeta + \zeta_2 = -1. \end{aligned} \tag{3.2}$$

Hence $2a = -1$ and $a^2 - 5b^2 = -1$, which gives

$$\alpha_5 = 2\cos(\tfrac{2\pi}{5}) = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{5}. \tag{3.3}$$

Finally, we know (analytically) that $\cos(\frac{\pi}{4})$ should be positive, so we conclude

$$\cos(\tfrac{2\pi}{5}) = \tfrac{1}{4}(\sqrt{5} - 1). \tag{3.4}$$

We also get the $m = 10$ case as a Galois conjugate, namely

$$\alpha_{10} = \zeta_{10} + \zeta_{10}^{-1} = -\zeta_5^3 - \zeta_5^{-3} - \sigma(\alpha), \tag{3.5}$$

the nontrivial Galois conjugate. We conclude

$$\cos(\tfrac{2\pi}{10}) = \tfrac{1}{4}(\sqrt{5} + 1). \tag{3.6}$$

For $m = 8, 12$, we can proceed similarly. The cyclotomic polynomials are

$$f_8(x) = x^4 + 1, \qquad f_{12}(x) = x^4 - x^2 + 1. \tag{3.7}$$

We observe that $\mathbb{Q}(\zeta_8)$ contains $\mathbb{Q}(\zeta_4) = \mathbb{Q}(i)$, and $\sqrt{2}$ by Fact 1, hence must be the biquadratic field $\mathbb{Q}(i, \sqrt{2})$. The real subfield is then $\mathbb{Q}(\sqrt{2})$. Likewise, $\mathbb{Q}(\zeta_{12})$ contains i and $\sqrt{-3}$, so must be $\mathbb{Q}(i, i\sqrt{3})$, with real subfield $\mathbb{Q}(\sqrt{3})$. As before, we compute the trace and norm, writing

$$\alpha_8 = a + b\sqrt{2}, \qquad \alpha_{12} = c + d\sqrt{3}, \qquad a, b, c, d \in \mathbb{Q}. \tag{3.8}$$

Note that the traces are $2a, 2c$ and the norms are $a^2 - 2b^2$ and $c^2 - 3d^2$.

By abuse of notation, we can write the nontrivial automorphism in both cases as the map $\zeta \mapsto \zeta^5$, so the remaining Galois conjugate is

$$\sigma(\alpha) = \zeta^5 + \zeta^{-5}. \tag{3.9}$$

Hence the trace and norm are

$$\begin{aligned} \mathrm{Tr}_{\mathbb{Q}}^{\mathbb{Q}(\alpha)}(\alpha) &= \zeta + \zeta^{-1} + \zeta^5 + \zeta^{-5}. \\ N_{\mathbb{Q}}^{\mathbb{Q}(\alpha)}(\alpha) &= \zeta^6 + \zeta^{-4} + \zeta^4 + \zeta^{-6}. \end{aligned} \tag{3.10}$$

For $m = 8$, we simplify using $\zeta_8^4 = -1$, which gives $\mathrm{Tr}(\alpha_8) = 0$ and $N(\alpha_8) = -2$. For $m = 12$, we instead simplify using $\zeta_{12}^6 = -1$, to get $\mathrm{Tr}(\alpha_{12}) = 0$, and $\zeta_{12}^4 = \zeta_{12}^2 - 1$, to get $N(\alpha_{12}) = -3$.

After solving for a, b, c, d, we obtain:

$$\alpha_8 = 2\cos(\tfrac{2\pi}{8}) = \pm\sqrt{2}, \qquad \alpha_{12} = 2\cos(\tfrac{2\pi}{12}) = \pm\sqrt{3}. \tag{3.11}$$

Finally, we know (analytically) that $\cos(\frac{\pi}{4})$ and $\cos(\frac{\pi}{6})$ should be positive. Putting these results together, we have the following:

Corollary 2. *The only $m \in \mathbb{Z}$ for which $\cos(\frac{2\pi}{m})$ is quadratic over $\mathbb{Q}$ are $m = 5, 8, 10, 12$. We have:*

m	5	8	10	12
$\cos(\frac{2\pi}{m})$	$\frac{1}{4}(\sqrt{5}-1)$	$\frac{\sqrt{2}}{2}$	$\frac{1}{4}(\sqrt{5}+1)$	$\frac{\sqrt{3}}{2}$.

3.2. **The Only Biquadratic Value.** It may come as a surprise that there is only one value of m making $\cos(\frac{2\pi}{m})$ biquadratic over $\mathbb{Q}$, namely $m = 24$. We expected (correctly) that these would be the easiest non-quadratic values, but all the others are all "nested quartics". The following facts explain why:

Fact 3 (Units mod m)**.** *The following facts determine the group structure of $(\mathbb{Z}/m\mathbb{Z})^\times$:*

(a) *If p^k is an odd prime power, then $(\mathbb{Z}/p^k)^\times \cong \mathbb{Z}/(p-1) \times \mathbb{Z}/p^{k-1}$. Moreover, any such isomorphism sends $-1 \in (\mathbb{Z}/p^k)^\times$ to the ordered pair $(\frac{p-1}{2}, 0)$ having order 2.*

(b) *For any $k \geq 2$, $(\mathbb{Z}/2^k)^\times \cong \mathbb{Z}/2 \times \mathbb{Z}/2^{k-2}$. Any such isomorphism sends $-1 \in (\mathbb{Z}/2^k\mathbb{Z})^\times$ to an element of order 2.*

(c) *If a, b are coprime, then $(\mathbb{Z}/ab)^\times \cong (\mathbb{Z}/a)^\times \times (\mathbb{Z}/b)^\times$, and this isomorphism maps -1 to $(-1, -1)$.*

Fact 4. *Consider the group $G = \mathbb{Z}/2^{k_1} \times \cdots \times \mathbb{Z}/2^{k_r}$ and the element $g = (2^{k_1-1}, \ldots, 2^{k_r-1})$, which has order two. Then the quotient $G/\langle g\rangle$ has the same decomposition into cyclic groups, except with the* **smallest** *k_i decremented by 1.*

In particular, if $G/\langle g\rangle$ is isomorphic to $(\mathbb{Z}/2)^n$, then every $k_i = 1$ and $n = r - 1$.

Thus Fact 4 shows that the only way for $\mathbb{Q}(\alpha_m)$ to be biquadratic (with Galois group $\mathbb{Z}/2\times\mathbb{Z}/2$) is if $\mathbb{Q}(\zeta_m)$ is 'triquadratic', with Galois group $(\mathbb{Z}/m\mathbb{Z})^\times \cong (\mathbb{Z}/2)^3$. By Fact 3, this can only occur for $m = 2^3 \cdot 3 = 24$. Note that the cyclotomic polynomial is then

$$f_{24}(x) = x^8 - x^4 + 1, \tag{3.12}$$

and that ζ_{24} also satisfies $\zeta_{24}^{12} = -1$.

For this case, we see that $\mathbb{Q}(\zeta_{24})$ contains $\mathbb{Q}(\zeta_{12})$ and $\mathbb{Q}(\zeta_8)$, hence is precisely $\mathbb{Q}(i, \sqrt{2}, \sqrt{3})$. The real subfield is thus $\mathbb{Q}(\alpha_{24}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, so we can write

$$\alpha_{24} = \zeta_{24} + \zeta_{24}^{-1} = a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}. \tag{3.13}$$

A direct calculation shows that the Galois conjugates of α_{24} are

$$\begin{aligned} \beta &= \zeta_{24}^5 + \zeta_{24}^{19}, \\ \gamma &= \zeta_{24}^7 + \zeta_{24}^{17}, \\ \delta &= \zeta_{24}^{11} + \zeta_{24}^{13}. \end{aligned} \tag{3.14}$$

We observe that $\alpha_{24} = -\delta$. The corresponding automorphism $\zeta_{24} \mapsto \zeta_{24}^{11}$ sends $\zeta_{12} \mapsto \zeta_{12}^{-1}$, hence fixes $\alpha_{12} = \sqrt{3}$. Thus it must be the map that negates $\sqrt{2}$ and $\sqrt{6}$. We equate coefficients, concluding that

$$a = c = 0. \tag{3.15}$$

By similar reasoning, the map $\zeta_{24} \mapsto \zeta_{24}^7$ fixes $\alpha_8 = \sqrt{2}$. Thus it negates $\sqrt{6}$, so

$$(\alpha + \gamma)^2 = (2b\sqrt{2})^2 = 8b^2 \tag{3.16}$$

and, using $\zeta_{24}^{12} = -1$,

$$(\zeta_{24} + \zeta_{24}^{-1} + \zeta_{24}^{7} + \zeta_{24}^{17})^2 = 2\zeta_{24}^{8} - 2\zeta_{24}^{4} + 4 = 2, \tag{3.17}$$

where the last equality is from the cyclotomic polynomial. A similar computation with $\alpha+\beta$ shows that $d = \pm\frac{1}{2}$, and so

$$\alpha_{24} = \pm\tfrac{1}{2}(\sqrt{2} \pm \sqrt{6}). \tag{3.18}$$

Both signs must be the same (and both give positive real numbers), but we have chosen our ζ's so that $\alpha_{24}^2 - 2 = \alpha_{12} = \sqrt{3}$, which forces the inner sign to be $+$.

Corollary 3. *The only $m \in \mathbb{Z}$ for which $\cos(\frac{2\pi}{m})$ is biquadratic is $m = 24$, and the resulting value is $\cos(\frac{2\pi}{24}) = \frac{1}{4}(\sqrt{2} + \sqrt{6})$.*

3.3. **Quartic Values.** Lastly, we consider the other values of m for which $\phi(m) = 8$, namely $m = 15, 16, 20, 30$. As usual, we will get $m = 30$ 'for free' as a Galois conjugate. In all three of these cases, the Galois groups are

$$\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) = \mathbb{Z}/2 \times \mathbb{Z}/4, \qquad \mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q}) = \mathbb{Z}/4. \tag{3.19}$$

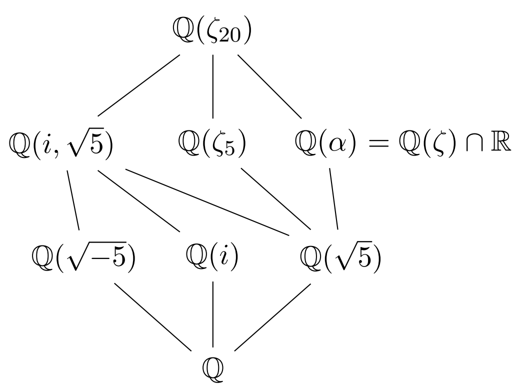
In particular, $\mathbb{Q}(\alpha)$ contains a unique quadratic subfield. In each case, we will identify the subfield as $\mathbb{Q}(\sqrt{r})$ for some r, then express $\mathbb{Q}(\alpha)$ as $\mathbb{Q}(\sqrt{r}, \sqrt{a + b\sqrt{r}})$. In other words, we will ultimately write

$$\cos(\tfrac{2\pi}{m}) = c_1 + c_2\sqrt{r} + c_3\sqrt{a + b\sqrt{r}} + c_4\sqrt{ar + br\sqrt{r}}. \tag{3.20}$$

We will work through the case $m = 20$; the cases $m = 15, 16$ are similar. We make the following observations:

- $\mathbb{Q}(\zeta_5), \mathbb{Q}(\sqrt{5}), \mathbb{Q}(i)$ are all subfields of $\mathbb{Q}(\zeta_{20})$;
- Since $\mathrm{Gal}(\mathbb{Q}(\zeta_{20})/\mathbb{Q}) \cong \mathbb{Z}/4 \times \mathbb{Z}/2$, there is a unique biquadratic subfield, which must therefore be $\mathbb{Q}(i, \sqrt{5})$.
- The other two degree-4 subfields have Galois groups isomorphic to $\mathbb{Z}/4$; these must be $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\zeta_5)$ (to distinguish them, note that $\mathbb{Q}(\alpha)$ is real and $\mathbb{Q}(\zeta_5)$ is not.)

Putting these thoughts together, we see that the lattice of subfields of $\mathbb{Q}(\zeta_{20})$ is the following:



In particular, we conclude that **the quadratic subfield of $\mathbb{Q}(\alpha_{20})$ is $\mathbb{Q}(\sqrt{5})$.**

By a similar analysis (identifying the biquadratic subfield and its real quadratic degree-2 subfield) we determine that the quadratic subfield of $\mathbb{Q}(\alpha_{15})$ is $\mathbb{Q}(\sqrt{5})$ and the quadratic subfield of $\mathbb{Q}(\alpha_{16})$ is $\mathbb{Q}(\sqrt{2})$.

Now we proceed as follows: we know that α_{20} is quadratic over $\mathbb{Q}(\sqrt{5})$. We will compute its (relative) norm and trace,

$$\mathrm{Tr}^{\mathbb{Q}(\alpha)}_{\mathbb{Q}(\sqrt{5})}(\alpha) = T = a + b\sqrt{5}, \tag{3.21}$$
$$N^{\mathbb{Q}(\alpha)}_{\mathbb{Q}(\sqrt{5})}(\alpha) = N = c + d\sqrt{5}.$$

These determine the minimal polynomial of α over $\mathbb{Q}(\sqrt{5})$, namely

$$\alpha^2 - T\alpha + N = 0, \tag{3.22}$$

so we can solve for α using the quadratic formula.

The Galois conjugates of α over $\mathbb{Q}$ are

$$\begin{aligned} \beta &= \zeta_{20}^{3} + \zeta_{20}^{17}, \\ \gamma &= \zeta_{20}^{7} + \zeta_{20}^{13}, \\ \delta &= \zeta_{20}^{9} + \zeta_{20}^{11}. \end{aligned} \tag{3.23}$$

We note that $\zeta \mapsto \zeta^9$ fixes $\zeta_{20}^4 + \zeta_{20}^{-4} = \alpha_5$. We proved earlier that $\mathbb{Q}(\alpha_5) = \mathbb{Q}(\sqrt{5})$, so in fact this is the identity automorphism of $\mathbb{Q}(\sqrt{5})$. Thus the nontrivial conjugate of α over $\mathbb{Q}(\sqrt{5})$ is δ. Consequently,

$$T = \alpha + \delta = \zeta_{20} + \zeta_{20}^{9} + \zeta_{20}^{11} + \zeta_{20}^{19} = 0, \tag{3.24}$$

$$N = \alpha \cdot \delta = \zeta_{20}^{10} + \zeta_{20}^{12} + \zeta_{20}^{8} + \zeta_{20}^{10} = -2 - (\zeta_{20}^{2} + \zeta_{20}^{-2}), \tag{3.25}$$

where we have used $\zeta_{20}^{10} = -1$. In particular,

$$\alpha_{20} = \pm\sqrt{\tfrac{5}{2} + \tfrac{1}{2}\sqrt{5}}. \tag{3.26}$$

As always, we take the positive root.

We apply this same method to the other values $m = 15, 16, 30$. Our results are summarized as follows:

Corollary 4. *The only $m \in \mathbb{Z}$ for which $\cos(\frac{2\pi}{m})$ is a "nested quartic" are $m = 15, 16, 20, 30$. We have:*

m	15	16	20	30
$\cos(\frac{2\pi}{m})$	$\frac{1}{8}(1 + \sqrt{5} + \sqrt{30 - 6\sqrt{5}})$	$\frac{1}{2}\sqrt{2 + \sqrt{2}}$	$\frac{1}{4}\sqrt{10 + 2\sqrt{5}}$	$\frac{1}{8}(-1 + \sqrt{5} + \sqrt{30 + 6\sqrt{5}})$.

Note: the field α_{15} and α_{30} generate the same field extension over $\mathbb{Q}$.