Intersection Theory!

(Math 583B)

Intersection Theory.

(Day 1). What is it good for?

① Enumerative geometry.

② Invariants, obstructions, ...

History:

1800s: "Italian school" of AG.

O M G !

Castelnuovo, Severi, Enriques,
Cremona, Segre, Albanese,
Bertini, del Pezzo, Veronese,
Picard, Schubert, Noether (♂),
Kähler...

Amazing calculations:

e.g. # twisted cubics tangent to 12 quadric surfaces

$$= 5,819,539,783,680$$

But, subtle errors crept in (informal arguments about "genericity", edge cases, limits...)

Hilbert (1901 ICM): Put this on rigorous footing.

$\rightsquigarrow$ 15^{th} problem.

$\rightsquigarrow$ Fulton. (1984).

1900s: Another "Who's Who":

Zariski, Lefschetz, Todd, Chow,
Samuel, Chevalley, Verdier,
Serre, Grothendieck, Weil, Kleiman...
Mumford, MacPherson, Fulton.

Examples.

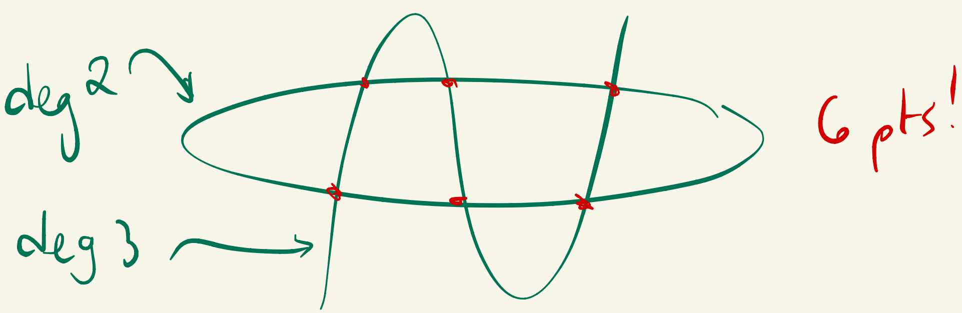
① Bézout's Theorem.

Let $C, D \subseteq \mathbb{P}^2_{\mathbb{C}}$ be integral curves.

$C \neq D$.

degrees c, d.

If $C \cap D$ is reduced, $\# C \cap D = c \cdot d$.



Comments:

1. $c \cdot d$ is a product. This suggests $\exists$ a ring structure where "multiplication = intersection".
   $\rightsquigarrow$ The Chow ring of $\mathbb{P}^2$.

2. We didn't care which C, D we had, only their degrees,
   $\rightsquigarrow$ Essentially a way to classify plane curves.

2 Lines on a cubic surface.

Thm. Let $C \subseteq \mathbb{P}^3$ be a general cubic surface.
Then C contains exactly 27 lines.

It turns out this is a special case of a theorem about vector bundles.

Thm. Let X = proper variety, $\dim = n$.
Let E vec bdle on X, rank n (same)
globally generated.
$s \in H^0(E)$ a general section.

Then s vanishes at a fixed # of pts on X, dependent only on E (not s).

$\rightsquigarrow$ This gives an <u>invariant</u> of E,

called the <u>top Chern class</u>

<u>$c_n(E)$</u>. ← (this will be an element of the Chow Ring of X.)

<u>e.g.</u> $E = TX, \Omega X$

$c_n(E)$ is an invariant <u>of X</u>.

e.g. (27 lines) (in $\mathbb{P}^3$)

$X = Gr(2,4)$ full rank 2×4 matrix

$= \{\text{lines in } \mathbb{P}^3\} = \{ [\begin{smallmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{smallmatrix}] \} / GL_2$

Smooth variety. dim 4.

E = "vector bundle of cubic polynomials on $\mathbb{P}^3$"

$s \in H^0(E)$ "cubic function on each line"

s vanishes at $\ell \in X \iff \text{function}|_\ell \equiv 0$.

$\iff$ cubic surface contains ℓ.

$$c_4(E) = 27\, \underbrace{[\text{point}]}_{\text{Chow ring of } Gr(2,4)}.$$

e.g. E, F vecbdles on X.

Does there exist a map of vector bundles of rank $\geq k$ (at each pt)?

Obstruction: A certain combination of Chern classes of E, F must vanish.

③ Let $S_1, S_2, S_3 \subseteq \mathbb{P}^3$ be distinct smooth surfaces of degrees s_1, s_2, s_3.

Suppose $S_1 \cap S_2 \cap S_3 = C$, a smooth curve.

(e.g. twisted cubic is cut out by 3 quadrics.)

Thm. $g(C) = \frac{1}{2}\left(1 + \deg(C)\left(s_1+s_2+s_3-4\right) - s_1 \cdot s_2 \cdot s_3\right)$

genus

(canonical bdle $\mathcal{O}(-4)$ on $\mathbb{P}^3$)

Topology, not just enumeration.

This is a special case of the excess intersection formula.

3 eqns to get codim $= 2$.

(4) Beautiful combinatorial theory.

<u>Thm</u>. $L_1, L_2, L_3, L_4 \subseteq \mathbb{P}^3$ general lines.

How many lines intersect all 4?

<u>Ans = 2</u>.

<u>Special case of</u>:

Let $H_1, \ldots, H_{k(n-k)}$ dim-$(k-1)$ planes in $\mathbb{P}^{n-1}$

How many complementary planes intersect all of them?

<u>Ans</u>: The number of <u>Standard</u> Young <u>tableaux</u>

$$\overset{\longleftarrow n-k \longrightarrow}{} \qquad k \updownarrow \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array}\, , \quad \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline \end{array}\, , \ \ldots \qquad \text{increasing.}$$

<u>Thm</u>: twisted cubics through 12 general lines?

<u>Ans</u>: 80160 "Quantum Chow ring".

Day 2: Cycles, multiplicity.

"scheme": separated, finite type / k = field.
(Hausdorff)

"variety": integral scheme,

Although most applications care mainly about nice spaces (e.g. smooth proj. var.), it is necessary to develop the theory for all schemes, even messy ones.

X = scheme.

Def: A k-cycle on X ($k \in \mathbb{N}$) is a ($\mathbb{Z}$-) formal sum of k-dimensional subvarieties of X.

$Z_k(X)$ = free ab. gp of k-cycles on X.

$$Z(X) = \bigoplus_k Z_k(X).$$

$$\left.\begin{aligned} Z_0(X) &= \text{formal sums of pts.} \\ Z_1(X) &\quad \text{--- curves} \\ &\ \vdots \end{aligned}\right\} \text{infinite rank}$$

If $\dim(X) = k$, $Z_k(X) = \mathbb{Z}^{\#\text{ irred. comp. of dim } k}$ } finite.

~~By definition~~, $Z(X) \equiv Z(X^{red})$.

literally the same formal sums.

To track ~~nonreduced structure~~ (multiplicity!), we need to put coefficients on our sums.

Counting with multiplicity.

⊛ Length:

R = ring.

M = R-module.

The *length* of M, $\ell_R(M)$, is the largest n

such that $\exists$ a chain of submodules

$$0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_n \ (\subseteq M).$$

If finite, we say M has <u>finite length</u>.

<u>Main facts</u>:

① Additive in short exact sequences:

$$0 \to A \to B \to C \to 0$$

$$\ell_R(B) = \ell_R(A) + \ell_R(C).$$

② R = k-algebra

R <u>local</u> ring with maximal ideal $\mathfrak{m}$

If $R/\mathfrak{m} = k$, then

$$\ell_R(M) = \dim_k(M).$$

X = scheme.

$X' \subseteq X$ irreducible component. (viewed w/ reduced structure).

$\mathcal{O}_{X',X}$ = local ring of X along X'.

↳ Compute from <u>any</u> affine chart intersecting X':

$X \supseteq \mathcal{U} = \operatorname{Spec} R$.

$\mathfrak{p}$ = minimal prime ideal corresp. to X' $(X' \cap \mathcal{U})$.

$\mathcal{O}_{X',X} = R_{\mathfrak{p}}$ <u>Artinian</u> ring (dim 0).

$$= \left\{ \frac{f}{g} : g \not\equiv 0 \text{ on } X' \cap \mathcal{U} \right\}$$

<u>If</u> $R_{\mathfrak{p}}$ is reduced, this is just the function field $k(X')$.

Otherwise, we're remembering nonreduced structure occurring "generically" along X':

$\mathcal{O}_{X',X}$ reduced.

"generically reduced"

$\mathcal{O}_{X',X}$ nonreduced.

"generically nonreduced"

Def: The multiplicity of X' in X is

$$\operatorname{mult}(X',X) := \ell_{\mathcal{O}_{X',X}}(\mathcal{O}_{X',X}) < \infty.$$

The fundamental cycle of X is

$$[X] := \sum_{\substack{X' \subseteq X \\ \text{irred.} \\ \text{comp.}}} \operatorname{mult}(X',X)\,[X'].$$

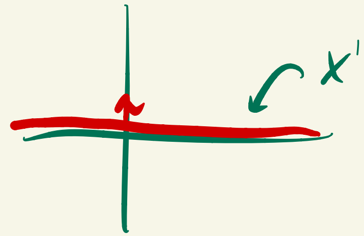
So $[X] \neq [X^{red}]$ even though $Z(X) = Z(X^{red})$.

Likewise: If $Y \subseteq X$ subscheme

$[Y] \in Z(Y)$ same as $[Y] \in Z(X)$.

Ex:

① $X = \operatorname{Spec} \dfrac{k[x,y]}{y^2, xy}$



$p = (y)$

R_p = invert other stuff ⇝ invert x.

⇝ implies $y = 0$.

$= k(x)$.

length 1. $[X] = [X']$.

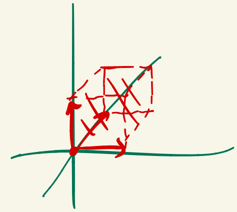
② $X = \operatorname{Spec} \dfrac{k[x,y]}{f^2}$ $f = y^2 - x^3 - 1$ elliptic curve.

$p = (f)$.

$R_p = \left(\dfrac{k[x,y]}{f^2}\right)_{(f)}$ length $= 2$.

$[X] = 2[E]$.

③ $X = \operatorname{Spec} \underbrace{\dfrac{k[x,y,z]}{(x,y,z)^2}}_{R \text{ (already local)}}$.



$R/\mathfrak{m} = k$.

$\ell(R) = \dim_k(R)$: basis: $1, x, y, z$

$= \underline{4}$. $[X] = 4[pt]$.

④ A bad example.

In $\mathbb{P}^4$: coords $[X:Y:Z:W:T]$.

$A =$  $= \{X=Y=0\} \cup \{Z=W=0\}$.

$B =$  $= \{X-Z = Y-W = 0\}$.

Later: In the Chow ring of $\mathbb{P}^4$:

· all 2D planes are equivalent

· all points are equivalent.

$$[A]\cdot[B] = \left(2\,[\text{2D-plane}]\right)\cdot[\text{2D-plane}]$$
$$= 2\,[pt].$$

If B were a general plane, $A \cap B$ would indeed be 2 pts.

⊛ See next page for "geometric" picture.

Now compute $A \cap B$:

Chart $T=1$.

$k[x,y,z,w]$: $I_A = (x,y) \cap (z,w)$
$= (xz, xw, yz, yw)$.

$I_B = (x-z, y-w)$.

$$\frac{k[x,y,z,w]}{I_A + I_B} = \frac{k[x,y]}{x^2, xy, y^2} \quad \begin{pmatrix} z=x \\ w=y \end{pmatrix}$$

: basis $1, x, y$.

length $= \underline{\underline{3}}$.

$$[A]\cdot[B] = 2\,[pt].$$

$$\neq$$

$$[A \cap B] = 3\,[pt].$$

$\rightsquigarrow$ Turns out this is because A isn't Cohen-Macaulay.

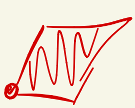
(Thm for later: If A, B are ~~smooth~~ Cohen-Macaulay, and the dimension of $A \cap B$ is correct, then $[A]\cdot[B] = [A\cap B]$.)

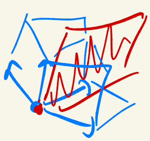
How did the geometry work?

A:

pt + full 4D Zariski tangent space.

B:





$\rightsquigarrow A \cap B$: pt + 2D tangent space.

$\Rightarrow$ length 3.

Day 3. Rational functions & Rational equivalence

Divisors of rational functions

X = variety, dim n.

$Z_{n-1}(X)$ is called the group of Weil divisors.

$f \in K(X)^*$ rational function.

$f : X \dashrightarrow \mathbb{P}^1$

The divisor of f, $\operatorname{div}(f) \in Z_{n-1}(X)$ is given

by: $$\operatorname{div}(f) = \sum_{\substack{D \subseteq X \\ \text{codim } 1, \\ \text{irreducible}}} \operatorname{ord}_D(f)\,[D].$$

order of vanishing.

>0 @ zeros of f

<0 @ poles of f

($= 0$ for all but finitely many D)

$\rightsquigarrow$ $\operatorname{div}(f) = [\text{zeros}] - [\text{poles}]$.

(If $f : X \to \mathbb{P}^1$, same as

$[f^{-1}(0)] - [f^{-1}(\infty)]$.)

$\underline{\mathrm{ord}_D(f)}$: Look in the local ring $\underbrace{\mathcal{O}_{D,X}}_{\text{1-dim'l ring}}$

$\rightsquigarrow$ we can write $f = \frac{a}{b}$ with $a, b \in \mathcal{O}_{D,X}$.

$$\mathrm{ord}_D(f) = \underbrace{\ell(\mathcal{O}_{D,X}/(a))}_{\mathrm{ord}_D(a)} - \underbrace{\ell(\mathcal{O}_{D,X}/(b))}_{\mathrm{ord}_D(b)}.$$

(doesn't depend on the choices.)

If X is <u>normal</u>, $\mathcal{O}_{D,X}$ is a DVR.

$\Rightarrow$ Principal maximal ideal (t),

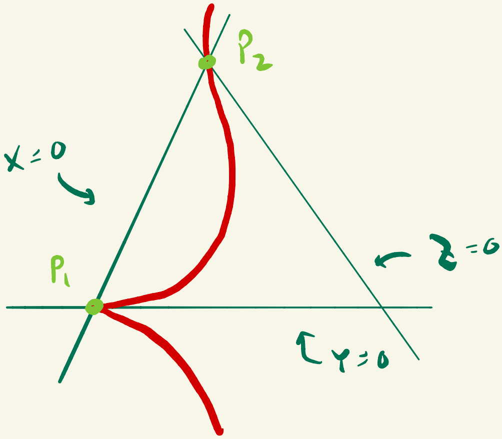
$$\left.\begin{aligned} a &= t^{k_a} \cdot u \\ b &= t^{k_b} \cdot u' \end{aligned}\right\} f = t^{k_a - k_b} u u'^{-1}$$

Hence "divisor". $D \leftrightarrow t$.

Ex: $C = \mathbb{V}(Y^2Z - X^3) \subseteq \mathbb{P}^2$

cuspidal cubic.

$f = \dfrac{Y}{X} \in k(C)^*$.



$Y=0 \rightsquigarrow [0:0:1]$ P_1

$X=0 \rightsquigarrow [0:1:0]$ P_2

chart $Z=1$: $\mathcal{O}_{P_1,C} = \left(\dfrac{k[x,y]}{y^2-x^3}\right)_{(x,y)}$ $f = \dfrac{y}{x}$

$\mathcal{O}_{P_1,C}/y = \dfrac{k[x]}{x^3}$ $\ell = 3$.

$\mathcal{O}_{P_1,C}/x = \dfrac{k[y]}{y^2}$ $\ell = 2$

$\mathrm{ord}_{P_1}(f) = 3 - 2 = 1$.

(**Comment**: Not possible to write $f = \dfrac{a}{b}$ $\begin{matrix} \leftarrow \mathrm{ord} = 1 \\ \leftarrow \mathrm{ord} = 0 \end{matrix}$ or else $\mathcal{O}_{P_1,C}$ would be a DVR!)

chart $Y=1$: check $\mathrm{ord}_{P_2}(f) = -1$.

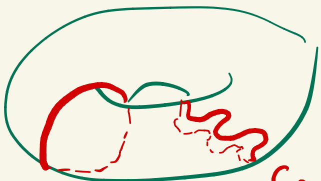
$\rightsquigarrow \mathrm{div}(f) = [P_1] - [P_2]$.

Obs: Total degree of div(f) = 0.

(Thm (Hartshorne II.6.10)
On any proper curve, div(f) has total degree 0.)

Rational Equivalence

$Z_k(X)$ is analogous to the group of topological cycles in singular homology.

T = 

C_1 C_2

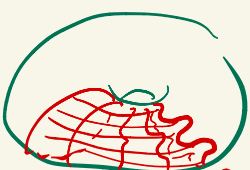
$C_1, C_2 \in Z_1^{top}(T)$

$\Downarrow$

C_1, C_2 have homology classes

$[C_1], [C_2] \in H_1^{top}(T)$

In this example, $[C_1] = [C_2]$ because we can deform C_1 into C_2:



C_1 C_2 Z

$\underbrace{\partial Z}_{\text{boundary}} = C_1 - C_2$.

$\Rightarrow [C_1] = [C_2]$.

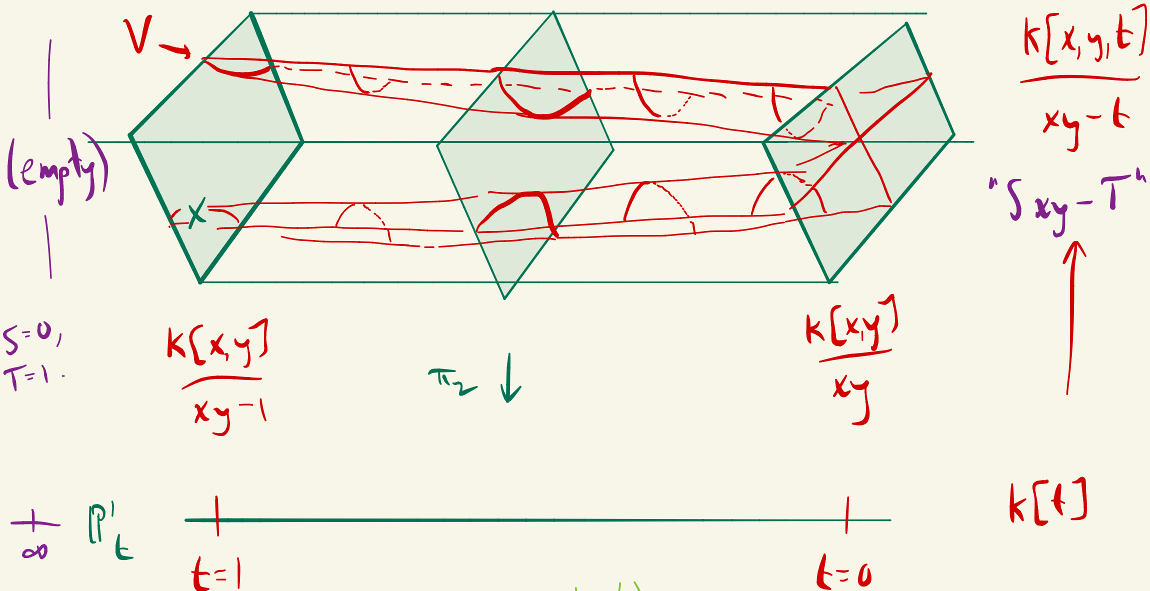
The algebraic analog of homotopy / homology equivalence is called <u>rational equivalence</u>.

Two definitions: $\rightarrow$ more topological $\Updownarrow$ $\rightarrow$ more algebraic

X = scheme.

<u>Def 1</u>: Let $V \subseteq X \times \mathbb{P}^1$ be a sub<u>variety</u>, not contained in a fiber $X \times \{t\}$.

($\circledast$ implies: $\pi_2 : V \to \mathbb{P}^1$ is <u>flat</u>.)



(Fundamental cycles!)

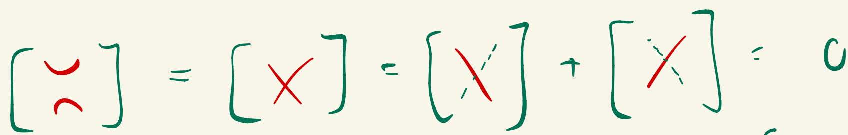
$\rightsquigarrow$ We say $[V(0)]$ and $[V(\infty)]$ (or $[V(1)]$, or any fiber) are rationally equivalent.

⇝ More generally we say $\alpha, \beta \in Z_k(X)$ are rationally equivalent if there exist subvarieties $V_1, \dots, V_\ell \subseteq X \times \mathbb{P}^1$ of dim $k+1$, such that

$$\alpha - \beta = \sum_i [V_i(0)] - [V_i(\infty)].$$

i.e. the transitive closure of the "basic" equivalences.

Ex: In $\mathbb{A}^2$ (above),



(V(1)) (V(0)) (V(∞)) (empty)

Def: The k^{th} Chow group of X is

$$A_k(X) := Z_k(X) / \equiv.$$

Day 4. Rational equivalence (cont'd).

X = scheme.

Last class: defined rational equivalence on X via families over $\mathbb{P}^1$: $V \subseteq X \times \mathbb{P}^1$.

Alternate def:

Let $V' \subseteq X$ be a subvariety of dim $k+1$.
($\underline{\text{not}}$ $X \times \mathbb{P}^1$)

Let $f \in k(V')^*$.

We then say $\operatorname{div}(f) \in Z_k(V') \subseteq Z_k(X)$

is *rationally equiv. to zero*.

(*Think:*

$$\begin{tikzcd} \overline{\Gamma(f)} \arrow[d] \arrow[dr] & \subseteq V' \times \mathbb{P}^1 \subseteq X \times \mathbb{P}^1 \\ f : V' \arrow[r, dashed] & \mathbb{P}^1 \end{tikzcd}$$

)

We'll use this as the definition today.

$\rightsquigarrow$ The k^{th} *Chow group of X* is

$$A_k(X) := Z_k(X) \Big/ \left\langle \operatorname{div}(f) : \begin{array}{l} f \in k(V')^* \\ V' \subseteq X \text{ subvar.} \end{array} \right\rangle .$$

$$A(X) = \bigoplus_k A_k(X).$$

Remarks.

① If $Y \overset{i}{\hookrightarrow} X$ is a closed embedding, we have

$$A(Y) \xrightarrow{i_*} A(X).$$

(A subvariety $V \subseteq Y$ is also $\subseteq X$.)

As with $Z(X)$, we have $A(X^{red}) \equiv A(X)$.

Soon we'll generalize this to any proper morphism.

② X = variety dim n

$$A_{n-1}(X) = \text{Weil divisors} \Big/ \begin{matrix}\text{principal divisors} \\ \text{div}(f) : f \in k(X)\end{matrix}$$

$$= Cl(X). \quad \text{Divisor class group.}$$

③ There's a natural forgetful map

($X/\mathbb{C}$ proper). $A_k(X) \longrightarrow H_{2k}^{top}(X)$ by "forgetting the algebra".

In general this is neither injective nor surjective.

Examples.

(1) C = proper (integral) curve.

$A_1(C) = \mathbb{Z} \cdot [C]$.

$A_0(C) = Cl(C)$.

Note: Since C is proper, every $\mathrm{div}(f)$ on C has total degree 0.

So there's a well-defined surjective homomorphism

$\deg : A_0(C) \to \mathbb{Z}$.

(2) $\mathbb{P}^2$: $A_2(\mathbb{P}^2) = \mathbb{Z} \cdot [\mathbb{P}^2]$.

$A_1(\mathbb{P}^2) = Cl(\mathbb{P}^2) \cong \mathbb{Z} \cdot [\text{line}]$ (Hartshorne II.6.4).

$A_0(\mathbb{P}^2) = \mathbb{Z} \cdot [pt]$.

any two pts are connected by a straight line $\cong \mathbb{P}^1$.

So $A_0(\mathbb{P}^2)$ is generated by the class of a point.

Since degrees of pts on curves are well-defined, no multiple of $[pt]$ is $\cong 0$.

③ $\mathbb{A}^n$: $A_n(\mathbb{A}^n) = \mathbb{Z} \cdot [\mathbb{A}^n]$.

$A_k(\mathbb{A}^n) = 0$ for $k < n$.

Proof: Enough to show $[Z] = 0$ for $Z \subseteq [\mathbb{A}^n]$ irred dimension $k < n$.

By translating, assume $\vec{0} \notin Z$.

Scale Z by $t \in k$.
(field)

In the limit, $\lim_{t \to \infty} (t \cdot Z) = $ empty.

$\rightsquigarrow$ Check details (Problem Session). $\boxtimes$

Functoriality:

The Chow groups are functorial two ways:

① They are covariant for proper morphisms:

(today) $p: X \to Y$ proper $\rightsquigarrow$ $p_*: A(X) \to A(Y)$.

($\approx$ singular homology).

② They are contravariant for flat morphisms.

(after) $f: X \to Y$ flat $\rightsquigarrow$ $f^*: A(Y) \to A(X)$.

($\approx$ singular cohomology).

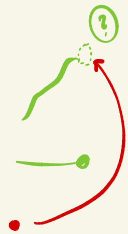
① Pushforward:

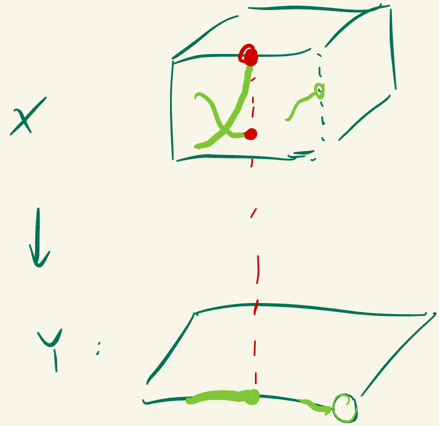
Say $p: X \to Y$ proper morphism.

proper:

start with:

get limit:



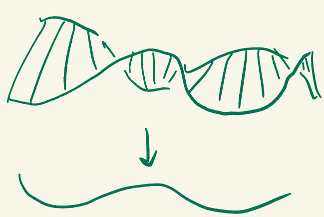


Examples:

- Any morphism $X \to Y$, if X itself is proper.
- Any projection $X \times F \to X$ where the fiber F is a proper variety.
- Closed embeddings $X \hookrightarrow Y$.
- Properness is local on the base

  So any $X \to Y$ locally like

  ex. ruled surface:



S fibers are $\mathbb{P}^1$'s.

$\downarrow$

C smooth curve

· Compositions & base changes of the above.

$$\begin{tikzcd} V \arrow[r, hook, "\text{proper}"] \arrow[dr, "\text{proper}"'] & X \times \mathbb{P}^1 \arrow[d, "\text{proper}"] \\ & X \end{tikzcd}$$

(compositions).

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "\text{proper}"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "\text{proper}"] \\ Y' \arrow[r, "\text{any map}"'] & Y \end{tikzcd}$$

base change:

$$\begin{tikzcd} X \times \mathbb{P}^1 \arrow[r] \arrow[d, "\text{proper}"'] \arrow[dr, phantom, "\square"] & \mathbb{P}^1 \arrow[d, "\text{proper}"] \\ X \arrow[r] & \{\cdot\} \end{tikzcd}$$

Day 5: Proper pushforward.

Let $p: X \to Y$ be proper.

Let $V \subseteq X$ be a subvariety, dim k.

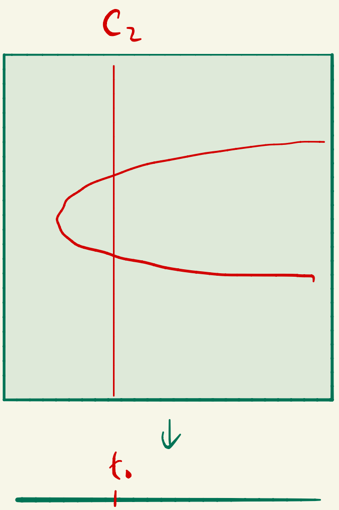
Proper maps are closed $\Rightarrow$ $p(V) \subseteq Y$ closed subvariety.
dim $\leq k$.

- If $\dim p(V) = k$, $k(V)$ is a finite degree field extension of $k(p(V))$.

Def: The pushforward $p_* : Z_k(X) \to Z_k(Y)$ is

$$p_*[V] = \begin{cases} \deg(V : p(V)) \cdot [p(V)] & \text{if } \dim p(V) = k. \\ 0 & \text{if } \dim p(V) < k. \end{cases}$$

Ex: $X = \mathbb{P}^1_x \times \mathbb{P}^1_t$



C_1 : $\dfrac{k[x,t]}{x^2-t} \cong k[x]$, field $k(x)$.

Fraction field:

$\cong k(\sqrt{t})$.

deg. 2 field extension.

$\mathbb{P}^1_t$

$k[t] \underbrace{\hphantom{xxxx}}_{}$ field $k(t)$

$\rightsquigarrow p_*[C_1] = 2[\mathbb{P}^1_t]$.

$\rightsquigarrow p_*[C_2] = 0$ (not $[t_0]$!).

⊛ properness will ensure that p_* descends to a map
$A_k(X) \to A_k(Y)$.

Foreshadowing: It's often useful to look at the generic fiber of our map $X \to Y$, i.e. base change (localize) to the fraction field of Y.

In charts: our example becomes

$$\text{tr deg. } 1 \left\{ \begin{array}{ll} k(t)[x] & \text{tr. deg. } 2. \\ \uparrow & \\ k(t) & \text{tr. deg } 1. \end{array} \right.$$

This becomes a variety over $k(t)$.

⊛ In fact a curve over $k(t)$.

just $\mathbb{P}^1$ over $k(t)$ in this example.

$\underset{(t-t_0)}{\Rightarrow} C_2$ disappears (equation $t-t_0$ is a unit in $k(t)$.)

$\underset{(x^2-t)}{\Rightarrow} C_1$ stays + becomes a "fuzzy point" (degree 2 field extension).

Similar to $\mathbb{R}[x]/_{x^2+1}$

$$\begin{array}{c} \uparrow \\ \mathbb{R} \end{array}$$

still true that $p_*[C_1] = 2[\operatorname{Spec} k(t)]$.

<u>Theorem</u>: If $\alpha \in Z_k(X)$ is rat'ly eq. to zero, then so is $p_*\alpha \in Z_k(Y)$.

Thus p_* descends to $A_k(X) \to A_k(Y)$.

<u>Proof</u>: By def, $\alpha = \sum \operatorname{div}(f_i)$ for some subvarieties $V_i \subseteq X$ dim $k+1$, $f_i \in k(V_i)^*$.

Enough to consider $\alpha = \operatorname{div}(f)$ on <u>one</u> subvariety $V \subseteq X$ dim $k+1$.

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d, "p"'] & X \arrow[d, "p"] \\ p(V) \arrow[r, hook] & Y \end{tikzcd}$$

Enough to show $p_*\alpha$ rat'ly eq. to 0 in $Z_k(p(V))$ rather than $Z_k(Y)$.

$\leadsto$ Reduce to $V = X$, $p(V) = Y$. & both X, Y are <u>varieties</u>, $\dim(X) = k+1$, $\dim Y \leq k+1$.

p surjective.

$\alpha = \mathrm{div}(f)$ for $f \in k(X)^*$.

Three cases depending on $\dim(Y)$.

① If $\dim(Y) \le k-1$ $(\dim X - 2)$.

Then $A_k(Y) = Z_k(Y) = 0$ so $p_*\alpha = 0$. ✓

② If $\dim(Y) = k+1 = \dim X$.

So $k(X)$ is a finite field ext'n of $k(Y)$.

$f \in$

Recall norms: $N : k(X)^* \to k(Y)^*$

$N(f) = \det\left(k(X) \xrightarrow{\cdot f} k(X) \right)$

map of $k(Y)$ vector spaces.

$\alpha = \mathrm{div}(f)$.

Fact (Fulton A.3) $p_* \mathrm{div}(f) = \mathrm{div}(N(f))$.

properties of local rings & determinants!

So $p_*\alpha = 0$ in $A_k(Y)$.

③ If $\dim(Y) = \dim(X) - 1$
$= k$.

$$A_k(Y) = Z_k(Y) = \mathbb{Z} \cdot [Y].$$

$$\mathrm{div}(f) = \sum n_i [D_i] \text{ on } X$$

$$p_* \mathrm{div}(f) = \underbrace{\sum n_i \deg(D_i : Y)}_{} [Y]$$

dropping any D_i that didn't contribute (didn't dominate Y).

$$= m \cdot [Y].$$

need this to be 0!

⊛ This is a local calculation (mults / orders / degrees) over $k(Y)$.

So, base change to $k(Y)$:

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, phantom, "\square"] & X' := X \times_Y \operatorname{Spec} k(Y) \arrow[l] \arrow[d] \\ Y & \operatorname{Spec} k(Y) \arrow[l] \end{tikzcd}$$

Since $\dim(X)$ was $k+1$, and $\dim Y = k$,

So X' is a curve over $k(Y)$.

Actually a proper curve since properness is preserved by base change.

So f is now a rat'l function on the proper curve $X'/k(Y)$

(*) The total degree of $\operatorname{div}(f)$ on any <u>proper curve</u> is 0!

So $m = 0$. $\blacksquare$

<u>Exer</u>: Check functoriality: $X \xrightarrow[\text{proper}]{p} Y \xrightarrow[\text{proper}]{q} Z$, $(q \circ p)_* = q_* \circ p_*$ on Chow groups.

Consequence:

If X is a proper scheme, the structure map $X \xrightarrow{\pi} \operatorname{Spec} k$ is proper. This gives the <u>degree map</u>

$$\deg : A_0(X) \to A_0(\operatorname{Spec} k) = \mathbb{Z}.$$

(π_*)

point count weighted by multiplicity & field ext'n degree.

(same as <u>length</u> if $k = \bar{k}$.)

$$\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$$

$$\pi_*[\mathbb{C}] = 2\,[\mathbb{R}].$$

Day 6. Excision and Flatness.

The simplest example of a flat morphism is an open embedding $\mathcal{U} \xrightarrow{j} X$, where $\mathcal{U}$ is an open subscheme.

We can't push forward, but we can restrict to $\mathcal{U}$:

$$j^* : Z_k(X) \to Z_k(\mathcal{U})$$
$$[Z] \longmapsto \begin{cases} [Z \cap \mathcal{U}] & \text{if nonempty} \\ 0 & \text{otherwise} \end{cases}$$

This respects rational equivalence:
if $V \subseteq X$ is a subvariety,

$$\begin{array}{c} j^* \operatorname{div}(f) \\ (f \in k(V)^*) \end{array} = \begin{cases} 0 & \text{if } V \cap \mathcal{U} \text{ empty} \\ \operatorname{div}(f|_{V\cap\mathcal{U}}) & \text{if } V \cap \mathcal{U} \neq \phi. \end{cases}$$

Same local calculations, just discard any terms D_i disjoint from $\mathcal{U}$.

Soon we'll generalize to all flat morphisms.

But first:

Theorem (Excision):

X = scheme, $\underset{\text{open}}{U} \xhookrightarrow{j} X \quad Z \xhookrightarrow{i} X \quad Z = X \setminus U$ closed.

There is a right-exact sequence:

$$A(Z) \xrightarrow{i_*} A(X) \xrightarrow{j^*} A(U) \to 0 .$$

⊛ Generalizes Hartshorne II.6.5 only for divisors on varieties.

Proof: j^* surjective: given $V \subseteq U$ subvariety, let $\bar{V}$ = closure of V in X,

$\rightsquigarrow$ then $\bar{V} \cap U = V$.

$j^* \circ i_* = 0$: true on cycles.

$\ker j^* \subseteq \operatorname{im} i_*$: Let $\alpha \in Z_k(X)$ such that $j^*\alpha = 0$ in $A_k(U)$

i.e. $j^*\alpha = \sum \operatorname{div}(f_i)$ for some subvarieties $V_i \subseteq U$, rat'l fns $f_i \in k(V_i)$.

(in $Z_k(U)$.)

$\rightsquigarrow$ Take closures of each $V_i \rightsquigarrow \bar{V_i} \subseteq X$.

$\rightsquigarrow$ Think of each f_i on V_i as $\bar{f}_i$ on $\bar{V}_i$.

i.e. same element $f_i \in k(V_i)^* \longleftrightarrow \bar{f}_i \in k(\bar{V}_i)$.

"terms from the extra local rings".

But, $\mathrm{div}(\bar{f}_i)$ may have extra terms $[D]$ where $D \subseteq X \setminus U = Z$.

So, $j^*\left(\alpha - \sum \mathrm{div}(\bar{f}_i)\right) = 0$ in $Z_k(U)$ as a cycle on U.

$\alpha - \sum \mathrm{div}(\bar{f}_i) = i_* \beta$ for some cycle $\beta \in Z_k(Z)$.

Descend from $Z_k(X)$ to $A_k(X)$:

$$\alpha - 0 = i_* \beta. \text{ in } A_k(X). \qquad \blacksquare$$

Corollary (Chow groups of $\mathbb{P}^n$).

(HW) $$A(\mathbb{P}^n) = \bigoplus_{d=0}^{n} \mathbb{Z} \cdot [\text{dim-}d \text{ plane}].$$

$\hookrightarrow$ Based on decomposition $\underset{\text{open}}{\mathbb{P}^n = \mathbb{A}^n} \sqcup \underset{\text{closed}}{\mathbb{P}^{n-1}}$.

Flat families.

A morphism $f: X \to Y$ is *flat* if all the maps of local rings

$$\mathcal{O}_{x,X} \leftarrow \mathcal{O}_{y,Y} \quad \text{are flat.}$$
$(y = f(x))$

Equivalently: in any open affine chart, $\overset{\mathcal{O}_Y}{R} \to \overset{\mathcal{O}_X}{S}$ is flat.

Flat morphisms are algebraic "continuous families".

Fiber dimension: Given $x \in X$, $y = f(x) \in Y$,

$$\underbrace{\dim \mathcal{O}_{x, f^{-1}(y)}}_{\text{fiber dimension}} = \underbrace{\dim \mathcal{O}_{x,X} - \dim \mathcal{O}_{y,Y}}_{\text{ambient dimensions.}}$$

We always assume a flat morphism has a constant relative dimension $r \geq 0$. This means:

(*) whenever $V \subseteq Y$ is a subvariety of dim k,
then $f^{-1}(V)$ is pure of dimension $k+r$.

$\rightsquigarrow [f^{-1}(V)] \in Z_{k+r}(Y)$.
(fundamental cycle)

Fiber degree: If f is proper and flat of rel dimension 0,
then f is finite and the fiber degree is constant.

($\uparrow$ Finite map = finite fibers + proper)

(over $k=\bar{k}$: length of the fiber is constant.)

Example:

$$\begin{array}{c} \mathbb{Q}[x,t]/_{x^2-t} \\ \uparrow \\ \mathbb{Q}[t] \end{array}$$

- if t has $\pm\sqrt{\ }$ in $\mathbb{Q}$: fiber is two pts.
- if $t=0$: fiber is double pt.
- if $t \neq$ square, fiber is $\operatorname{Spec} \mathbb{Q}(\sqrt{t})$.

length 1 but degree 2.

$$\left(\begin{array}{c} \operatorname{length}_{\mathbb{Q}(\sqrt{t})}(\mathbb{Q}(\sqrt{t})) = 1 \\ \deg(\mathbb{Q}(\sqrt{t}):\mathbb{Q}) = \operatorname{length}_{\mathbb{Q}}(\mathbb{Q}(\sqrt{t})) = 2 \end{array} \right)$$

Core examples:

1. Open embeddings $U \xhookrightarrow{j} X$.
2. Projections $X \times F \to X$ (F pure of dimension r).
3. Flatness is local on the base, so anything locally like 2:
   vector bundles $E \to X$ (locally $\mathbb{A}^n \times X \to X$)
   projective bundles $\mathbb{P}(E) \to X$ (locally $\mathbb{P}^{n-1} \times X \to X$).
   proper and flat.
   fiber bundles.

to be continued.

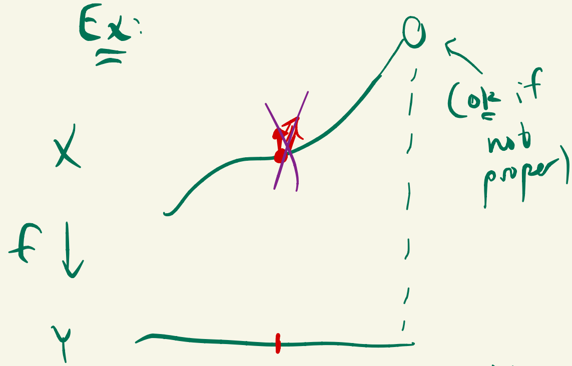
(Day 7: Flatness cont'd)

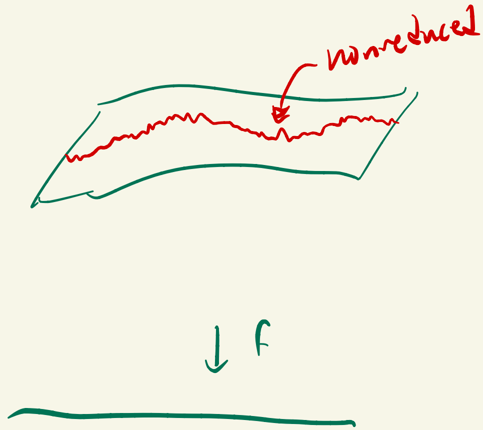
④ Flatness / sm curve.

$f: X \to Y$, Y sm curve.

Then f is flat $\iff$ every irred. and/or embedded component of X dominates Y.

Ex:





- X is not flat over Y.
- but, X^{red} is flat.

f is flat.

very easy to achieve flatness if it's not already present.

Also $U \subseteq X$ open is flat.
(delete the bad fibers)

⑤ Universal Families.

For projective morphisms $\begin{array}{c} X \subseteq Y \times \mathbb{P}^n \\ f \downarrow \\ Y \text{ reduced.} \end{array}$, i.e. families of projective varieties.

f is flat iff the Hilbert polynomial is the same for all fibers.

↳ $Z = f^{-1}(y)$ fiber.

$\mathrm{Hilb}_Z(d) = \dim H^0(\mathcal{O}_Z(d))$

(for $d \gg 0$)

ex: Conics in $\mathbb{P}^2_{[X:Y:Z]}$

↳ specified by 6 coefficients:

$$\begin{array}{ccc} Y^2 & YZ & Z^2 \\ XY & & XZ \\ & X^2 & \end{array}$$

⇝ $\mathbb{P}^5$ = space of conics.

very simple example of a moduli space,

$\langle F \rangle \in \mathbb{P}^5 \iff$ conic in $\mathbb{P}^2$,

"Universal Family": $\mathbb{P}^5 \times \mathbb{P}^2 \supseteq \Phi := \{(F,p) : F(p) = 0\}$,

$$\begin{tikzcd} & \Phi \arrow[dl, "\pi_1"'] \arrow[dr] & \\ \mathbb{P}^5 & & \mathbb{P}^2 \end{tikzcd}$$

Fibers of π_1: $F \in \mathbb{P}^5$.

$$\pi_1^{-1}(F) = [F] \times \underbrace{\{p : F(p) = 0\}}_{\text{actual conic in } \mathbb{P}^2!}$$

This family is flat b/c $\mathbb{P}^5$ is reduced and the Hilbert polynomial is always $\mathrm{Hilb}(d) = 2d+1$.

$$\left(0 \to \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_C \to 0. \right)$$

it's flat <u>even</u> though some fibers are

2 lines — or — double line.

⑥ Compositions & base changes of flat morphisms are flat.

e.g. if $\begin{array}{c} X \\ f \downarrow \\ Y \end{array}$ is flat, the restriction of f to any subscheme $Z \subseteq Y$ is flat.

$$\begin{tikzcd} X \arrow[d, "f"'] & f^{-1}(Z) \arrow[l] \arrow[d] \\ Y & Z \arrow[l, hook', "i"] \end{tikzcd} \Bigg\} \text{ also flat.}$$

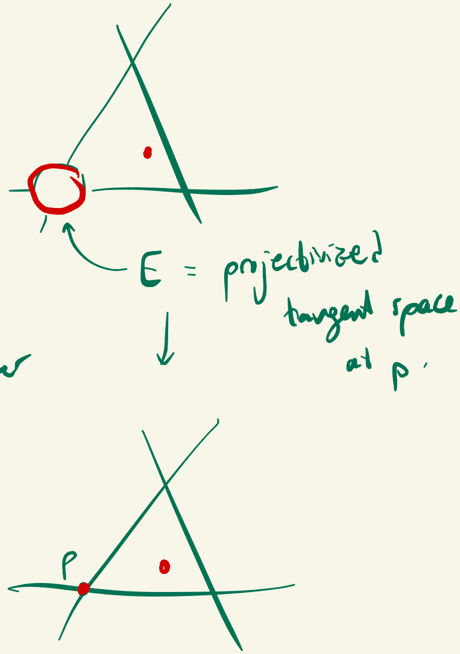
What's not flat?

① If X has a component that doesn't dominate a component of Y.
$\Rightarrow X \to Y$ is <u>not</u> flat.

(algebraically: flat modules are torsion free).

(2) Blowups.

$$\begin{array}{c} Bl_p(\mathbb{P}^2) \\ \pi \downarrow \\ \mathbb{P}^2 \end{array}$$

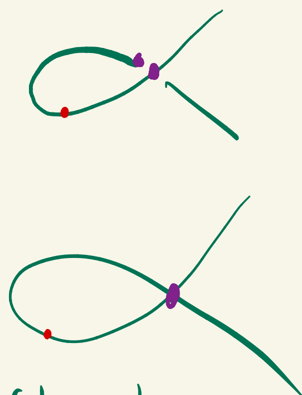


π is not flat b/c the fiber dimension isn't constant.

Blowups are never flat except in trivial cases.

(3) Normalization.

$$\begin{array}{c} \tilde{C} \\ \pi \downarrow \\ C \end{array}$$



$\Rightarrow$ π isn't flat because the fiber degree isn't constant.

This is also never flat unless the base was already normal.

Flat pullback.

If $f: X \to Y$ flat of rel dim $r \geq 0$, we define the pullback on cycles:

$$f^*: Z_k(Y) \to Z_{k+r}(X)$$

$$[Z] \xmapsto{f^*} [f^{-1}(Z)]$$

$Z \subseteq Y$ subvariety (fundamental cycle).

Theorem.

① If $Z \subseteq Y$ is any subscheme, $f^*[Z] = [f^{-1}(Z)]$.

In particular $f^*[Y] = [X]$.

② (Compatibility) Given a fiber square

$$\begin{tikzcd} X' \arrow[r, "f' \text{ (flat)}"] \arrow[d, "p'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "p"] \\ Y' \arrow[r, "f"'] & Y \end{tikzcd}$$

(proper) p', p proper, f flat

then $f^* p_* = p'_* f'^*$ on cycles.

③ f^* descends to rat'l eq. classes: $f^*: A_k(Y) \to A_{k+r}(X)$.

Day 8: Chow rings!

X = smooth variety. $\dim = n$.

Write $A^c(X) := A_{n-c}(X)$.

Fundamental Theorem of Intersection Theory:

① The Chow groups have a graded ring structure

$$A^c(X) \times A^d(X) \to A^{c+d}(X).$$

called the intersection product, satisfying

$$[A]\cdot[B] = [A \cap B]$$

$\subset$ subvarieties (fundamental cycle) with multiplicity!

if • A, B are Cohen-Macaulay (e.g. smooth)

at the generic pts of $A \cap B$, and

• $\operatorname{Codim}(A \cap B) = \operatorname{Codim}(A) + \operatorname{Codim}(B)$.

② The pullback $f^* : A^c(Y) \to A^c(X)$ exists for all
$(f : X \to Y)$

morphisms of smooth varieties and is a ring map and

satisfies: $f^*[Z] = [f^{-1}(Z)]$ if Z is Cohen-Macaulay and $\operatorname{codim}(f^{-1}Z) = \operatorname{codim}(Z)$.

subvariety of Y

(fundamental cycle!)

This makes $A(-)$ a contravariant functor:

$$\text{smooth varieties} \longrightarrow \text{graded rings}.$$

③ (Projection formula)

If $f : X \to Y$ is proper, $\alpha \in A(X)$, $\beta \in A(Y)$, then

$$f_*(f^*\beta \cdot \alpha) = \beta \cdot f_*(\alpha).$$

This says: f_* is a map of $A(Y)$-modules.

($A(X)$ is already an $A(Y)$ module via f^*).

<u>Comments</u>:

1. Very unclear what $[A]\cdot[B]$, $f^*[Z]$ are in the non-CM / non-transverse cases.

2. Being empty counts as transverse.

<u>Ex 1</u>. $\mathbb{P}^n$. $A(\mathbb{P}^n) = \bigoplus_{c=0}^{n} \mathbb{Z}\cdot[L^c]$ ← any codim-c linear space.

$$\rightsquigarrow [L^c]\cdot[L^e] = [L^{c+e}].$$

$\rightsquigarrow$ So $A(\mathbb{P}^n) \cong \dfrac{\mathbb{Z}[h]}{h^{n+1}}$, h = class of a hyperplane.

$\rightsquigarrow$ Gives Bézout's Theorem:

$$A(\mathbb{P}^2) \cong \frac{\mathbb{Z}[\ell]}{\ell^3}.$$

If A, B are curves of degrees a, b

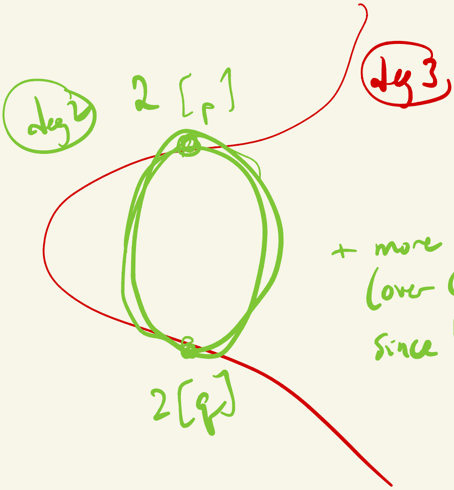
then $[A] = a\,[\text{line}]$, $[B] = b\,[\text{line}]$.

If $A \cap B$ is finite then $[A \cap B] = [A]\cdot[B]$

$= (a\,[\text{line}])(b\,[\text{line}])$

$= ab\,[\text{line}]\cdot[\text{line}]$

$= ab\,[pt].$

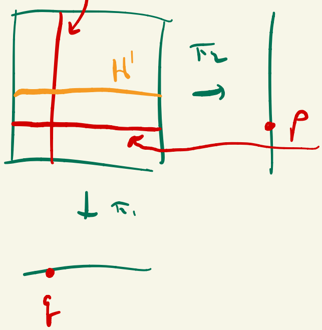


+ more (over $\mathbb{C}$)
since total is $3\cdot 2 = \underline{\underline{6}}$.

<u>Ex 2</u>. $\mathbb{P}^1 \times \mathbb{P}^1$.

$$\mathbb{P}^1 \xleftarrow{\pi_1} \mathbb{P}^1\times\mathbb{P}^1 \xrightarrow{\pi_2} \mathbb{P}^1$$

$[V] = [\pi_1^{-1}(q)] = \pi_1^*[q].$



$[H] = [\pi_2^{-1}(p)]$

$= \pi_2^*[p]$.

horiz. fiber.

Excision $\rightsquigarrow A(\mathbb{P}^1\times\mathbb{P}^1) = \bigoplus \mathbb{Z}\cdot \begin{cases} [\mathbb{P}^1\times\mathbb{P}^1] \\ [H], [V] \\ [pt] \end{cases}$

<u>Products</u>: $[H]\cdot[V] = [pt].$

$[H]^2$? $\quad [H]^2 = [H]\cdot[H']$ ← move one fiber

$= 0$ (empty).

(<u>Also</u>: π_2^* is a ring map

$[H]^2 = \pi_2^*(\underset{\text{on } \mathbb{P}^1}{[pt]^2})$.)

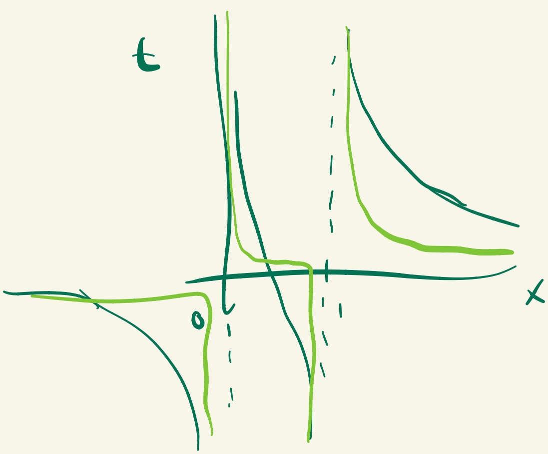
Similarly: $[V]^2 = 0$.

So, $A(\mathbb{P}^1 \times \mathbb{P}^1) \cong \dfrac{\mathbb{Z}[h,v]}{h^2,\ v^2}$ (basis $1, h, v, \underbrace{hv}_{[pt]}$)

Gives the "$\mathbb{P}^1 \times \mathbb{P}^1$ Bézout's Theorem":

Say $C \subseteq \mathbb{P}^1_{[X:Y]} \times \mathbb{P}^1_{[S:T]}$ has type (a,b) if it is given by a bihomogeneous eqn. of bi-degree $(\underset{X:Y}{a}, \underset{S:T}{b})$.

ex: in $\mathbb{A}^2$ chart



$$t = \frac{1}{x-1} + \frac{1}{x}$$

$$t\,x(x-1) = x + (x-1)$$

$$tx^2 - tx - 2x - 1 = 0$$

$$Tx^2 - Tx + 2Sx - S = 0$$

$$TX^2 - TXY + 2SXY - SY^2 = 0$$

bidegree $(\underset{X:Y}{2}, \underset{T:S}{1})$

$$[C] = 2[V] + [H]$$

If C_1 has type (a_1, b_1)

C_2 has type (a_2, b_2),

and $C_1 \cap C_2$ is finite...

$$[C_1 \cap C_2] = [C_1] \cdot [C_2] = \left(a_1 [V] + b_1 [H]\right)\left(a_2 [V] + b_2 [H]\right)$$

$$= \cancel{a_1 a_2 [V]^2} + \cdots \qquad \cancel{[H]^2}$$

$$= (a_1 b_2 + b_1 a_2) \underbrace{[H] \cdot [V]}_{[pt]}.$$

Day 9: Some counting problems.

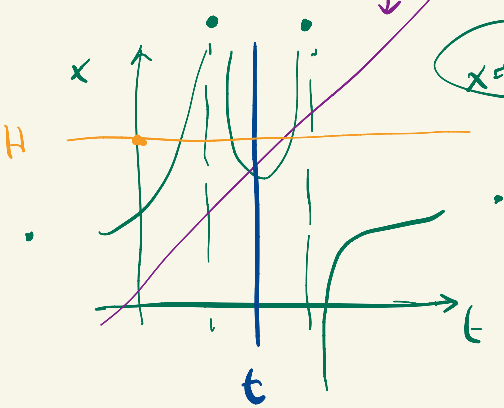
Q: Let $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ is a map of degree d.

$$\varphi([T:S]) = [F(T;S) : G(T;S)].$$

How many fixed pts does φ have?

$\rightsquigarrow$ Look @ graph of φ: $\Gamma(\varphi) \subseteq \underset{T:S}{\mathbb{P}^1} \times \underset{X:Y}{\mathbb{P}^1}$.

In $\mathbb{A}^2$ chart: diagonal $\Delta(\mathbb{P}^1) \subseteq \mathbb{P}^1 \times \mathbb{P}^1$.



fixed points of φ

$= \Gamma(\varphi) \cap \Delta(\mathbb{P}^1)$.

$\rightsquigarrow$ Use the "method of undetermined coefficients":

We know $[\Gamma(\varphi)] = a[V] + b[H]$ for some $a, b \in \mathbb{Z}$.

$\rightsquigarrow$ Figure out a, b by intersecting with V, H.

① Intersect with $[V]$:

Geometrically, $\Gamma(\varphi) \cap V = 1\,\mathrm{pt}$. $\quad$ ⊛ smooth objects intersecting.

$$[\Gamma(\varphi)] \cdot [V] = 1\,[\mathrm{pt}].$$

Algebraically: $[\Gamma(\varphi)] \cdot [V] = \big(\cancel{a[V]} + \underbrace{b[H]\big) \cdot [V]}_{[V]\cdot[H] = [\mathrm{pt}]}$ $\quad [V]^2 = 0$

$$= b\,[\mathrm{pt}]. \quad \text{So } b = 1.$$

② Intersect with $[H]$:

Geometrically $[\Gamma(\varphi) \cap H] = d\,[\mathrm{pt}]$ $\quad$ (H preimages).

Algebraically: $\big(a[V] + \cancel{b[H]}\big) \cdot [H] = a\,[\mathrm{pt}]$.

So $a = d$.

So $[\Gamma(\varphi)] = d[V] + [H]$.

Similarly $[\Delta(\mathbb{P}^1)] = [V] + [H]$ $\quad$ $\big(\Delta(\mathbb{P}^1) = \Gamma(\mathrm{id})\big)$.

$$\text{So } \underbrace{[\Gamma(\varphi) \cap \Delta(\mathbb{P}^1)]}_{\text{fundamental cycle}} = [\Gamma(\varphi)] \cdot [\Delta(\mathbb{P}^1)]$$

$$= \big(d[V] + [H]\big)\big([V] + [H]\big)$$

$$= \underbrace{(d+1)}_{\#\text{ fixed pts}} [pt].$$

ex: $d=1$: (non-identity Mobius transformation)

2 fixed pts!

Q: About conics in $\mathbb{P}^2_{[X:Y:Z]}$.

a) Given four general quadratic polynomials F, G, H, J,

$S = \text{span}(F, G, H, J)$.

How many elts of S are squares? $\longrightarrow SQ \subseteq \mathbb{P}^5$

(up to scaling $\rightsquigarrow \mathbb{P}(S)$.)

b) Given two general quadratic polynomials F, G,

$T = \text{span}(F, G)$.

How many elts of $\mathbb{P}(T)$ factor? $\longrightarrow R \subseteq \mathbb{P}^5$.

$\mathbb{P}^5$ = space of conics. $\supseteq \mathbb{P}(S) \cong \mathbb{P}^3$, $\mathbb{P}(T) \cong \mathbb{P}^1$.

$\cup$

$SQ = \{\text{squares } L^2\}$.

$R = \{\text{reducible conics } L \cdot \tilde{L}\}$. $SQ \subseteq R$.

We want: (a) $\# \, \mathbb{P}(S) \cap SQ$

(b) $\# \, \mathbb{P}(T) \cap R$.

(a). $\mathbb{P}^2_{[a:b:c]}$ = space of linear forms. $aX+bY+cZ=0$.
(dual $\mathbb{P}^2$)

$\mathbb{P}^2 \xrightarrow{\varphi} \mathbb{P}^5$ (dual) Veronese embedding.

$L \longmapsto L^2 \quad \approx \quad [a^2 : 2ab : \cdots] \in \mathbb{P}^5$.

$[a:b:c] \quad \updownarrow \qquad\qquad \updownarrow$

$(aX+bY+cZ)^2 = a^2X^2 + 2abXY + \cdots$

Image $\varphi(\mathbb{P}^2)$ is SQ

so $\dim(SQ) = 2$ (codim 3 in $\mathbb{P}^5$).

And $\dim \mathbb{P}(S) = 3$ (codim 2 in $\mathbb{P}^5$).

So $[\mathbb{P}(S) \cap SQ] = [\mathbb{P}(S)] \cdot \underbrace{[SQ]}_{}$ if finite.

(both smooth) $\qquad = \qquad \varphi_*[\mathbb{P}^2]$.

Use projection formula to "pull" back to the $\mathbb{P}^2$.

identity element.

$[\mathbb{P}(S)] \cdot \varphi_*[\mathbb{P}^2] = \varphi_*\big(\underbrace{\varphi^*[\mathbb{P}(S)] \cdot [\mathbb{P}^2]}_{\text{calculate in } \mathbb{P}^2} \big)$

$= \varphi_*\left(\varphi^*[\mathbb{P}(S)] \right)$

Amounts to saying: Calculate $[\varphi^{-1}(\mathbb{P}(S))]$ in $\mathbb{P}^2$.

(↑ in fact this is isomorphic to its image, which is $\mathbb{P}(S) \cap \varphi(\mathbb{P}^2)$.)

⇝ $\mathbb{P}(S)$ is a codim 2 linear space in $\mathbb{P}^5$.

$[\mathbb{P}(S)] = [H]^2$. H: hyperplane in $\mathbb{P}^5$.

So $\varphi^*[\mathbb{P}(S)] = \varphi^*[H]^2$ on $\underline{\mathbb{P}^2}$.

H is easier to describe: H is given by any linear condition in the coefficients of $X^2, XY, Y^2, \ldots$.

⇝ gives <u>quadratic</u> condition on $a:b:c$,

<u>i.e.</u> $\varphi^*[H] = 2[\text{line in } \mathbb{P}^2_{a:b:c}]$.

So $\varphi^*[H]^2 = (2[\text{line}])^2$

$= 4\,[pt]$ on $\mathbb{P}^2$.

⇝ $\varphi_*(4[pt]) = 4\,[pt]$ in the $\mathbb{P}^5$.

$= [\mathbb{P}(S) \cap \varphi(\mathbb{P}^2)] = [\mathbb{P}(S) \cap SQ]$.

(b) $\mathbb{P}^5$ = space of conics. $\supseteq$ $\mathbb{P}(T) \cong \mathbb{P}^1$.

$\cup$

R = reducible conics

$L \cdot \tilde{L}$.

$\rightsquigarrow$ There's a morphism $\mathbb{P}^2 \times \mathbb{P}^2 \xrightarrow{\Psi} \mathbb{P}^5$

$(L, \tilde{L}) \mapsto L \cdot \tilde{L}$.

$\underline{\text{NOT}}$ an embedding: 2-to-1 onto image!

So $\Psi_*[\mathbb{P}^2 \times \mathbb{P}^2] = \underline{2}[R]$.

We are asking for $[\mathbb{P}(T) \cap R]$.

Use projection formula to relate this to $\mathbb{P}^2 \times \mathbb{P}^2$:

$$\Psi_*\left(\Psi^*[\mathbb{P}(T)] \cdot [\mathbb{P}^2 \times \mathbb{P}^2]\right) \leftarrow = \Psi_* \Psi^* [\mathbb{P}(T)].$$

easier to calculate.

$$= [\mathbb{P}(T)] \cdot \Psi_*[\mathbb{P}^2 \times \mathbb{P}^2]$$

$$= [\mathbb{P}(T)] \cdot 2[R].$$

$\mathbb{P}(T) \cong \mathbb{P}^1$ codim $\underline{4}$ in $\mathbb{P}^5$. $[\mathbb{P}(T)] = [H]^4$.

What is $\Psi^*[H]$ in $\mathbb{P}^2 \times \mathbb{P}^2$?

Take $H = \{ F : F(p) = 0 \}$ (fix $p \in \mathbb{P}^2_{x:y:z}$).

$$\psi^{-1} H \subseteq \mathbb{P}^2 \times \mathbb{P}^2 = \left(L \times \mathbb{P}^2 \cup \mathbb{P}^2 \times L \right).$$

type $(1,1)$ on $\mathbb{P}^2 \times \mathbb{P}^2$.

$$\psi^*[H] = [H^{10}] + [H^{01}] \text{ on } \mathbb{P}^2 \times \mathbb{P}^2.$$

$$\psi^*[H]^4 = ?$$

$$A(\mathbb{P}^2 \times \mathbb{P}^2) = \mathbb{Z} \cdot \left\{ \begin{tikzcd}[column sep=tiny, row sep=small] & & H^{00} \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & & \\ & H^{10} \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & & H^{01} \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & \\ H^{20} \arrow[dr, dash, dashed] & & H^{11} \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & & H^{02} \arrow[dl, dash, dashed] \\ & H^{21} \arrow[dr, dash, dashed] & & H^{12} \arrow[dl, dash, dashed] & \\ & & H^{22} = pt & & \end{tikzcd} \right.$$

$$\begin{aligned} \psi^*[H]^4 &= ? \\ &= \left([H^{10}] + [H^{01}] \right)^4 \\ &= \cancel{[H^{10}]^4} + \cancel{4[H^{10}]^3 [H^{01}]} + 6 \underbrace{[H^{10}]^2 [H^{01}]^2}_{pt} + \cancel{0} + \cancel{0} \\ &= 6[pt]. \end{aligned}$$

Projection formula reads:

$$\psi_* \Big(\underbrace{\psi^*[H]^4 \cdot [\mathbb{P}^2 \times \mathbb{P}^2]}_{\psi_*(6[pt])} \Big) = [H]^4 \cdot \psi_*[\mathbb{P}^2 \times \mathbb{P}^2]$$

$$= [\mathbb{P}(T)] \cdot 2[R]$$

$$6[pt] = 2\,[\mathbb{P}(T) \cap R].$$

what we wanted.

So $[\mathbb{P}(T) \cap R] = \underline{\underline{3}}\,[pt]$.

SUMMARY OF STEPS

(1) Describe <u>all</u> relevant objects, express the problem as "intersection of objects (of complementary dimension)".

(2) Express their classes in terms of more familiar classes (e.g. hyperplanes). ("graph" problem)

<u>OR</u>:

Use projection formula to shift the problem to a different space. Express classes there. ("conics" problems).

(both methods work on <u>both</u> problems!)

③ (After converting to a problem involving familiar classes on familiar spaces):

Multiply, using Chow ring structure.

④ (Cleanup)

Verify over/under counting (projection formula)

Verify smoothness / transversality / multiplicity.

Day 10: More on Chow rings.

Def: Degree of a projective variety:

if $X \subseteq \mathbb{P}^n$ subvariety dim k,

we know $[X] = d\,[L_k]$ ← linear space of dim k.

We define $\deg(X) := d$.

This is also the cardinality of the intersection of X with a general complementary linear space:

$$[X \cap L^k] = [X]\cdot[L^k] = (d\,\underbrace{[L_k])\cdot[L^k]}_{[pt]}.$$

general codimension k linear space

(Note: This proves $d \geq 0$.)

So last class with $\mathbb{P}^5$: conics in $\mathbb{P}^2$

$\cup\!|$

R : reducible conics

$\cup\!|$

SQ : squares of linear forms.

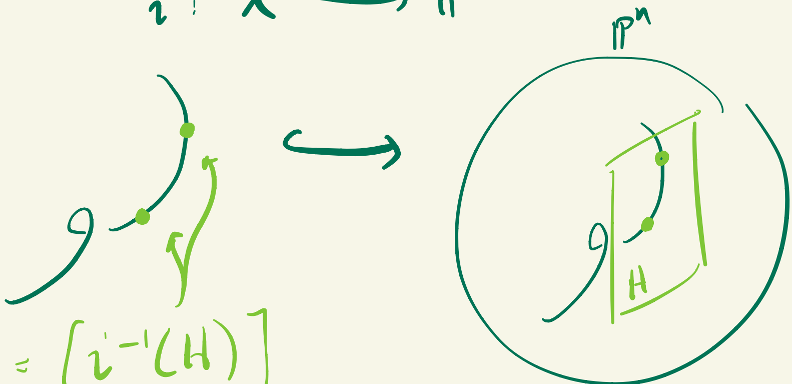
We (equivalently) showed: $\deg(R) = \left(\# R \cap \mathbb{P}(T)\right) = 3$.

$\deg(SQ) = \left(\# SQ \cap \mathbb{P}(S)\right) = 4$.

Pullbacks and Intersections

One thing we observed last class:

$$i : X \hookrightarrow \mathbb{P}^n$$



$i(X) \cap H$
$= 2$pts.

$i^*[H] = [i^{-1}(H)]$

(codim is correct).

$\in A_0(X)$

$\hookrightarrow Cl(X) = Pic(X)$.

(curve)

$[i(X) \cap H] = [i(X)] \cdot [H]$

$= i_*[X] \cdot [H]$

$= 2[pt]$.

different!

$\in A_0(\mathbb{P}^n) = \mathbb{Z}$.

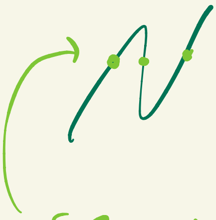
Projection Formula: $i_* i^*[H] = i_*\left(\cancel{i^*}[H] \cdot [X]\right)$

$= [H] \cdot i_*[X]$

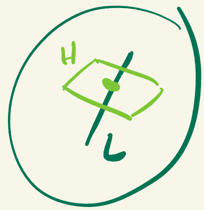
$= [H \cap i(X)]$.

Also for <u>non</u>-embeddings:

$$X \xrightarrow{f} \mathbb{P}^n$$



$\xrightarrow{f}$



$[H]\cdot[L] = [pt]$.

$f_*[X] = 3[L]$.

(3-to-1)

$i^*[H] = 3\,pts$ in $A_0(X)$.

$[i^{-1}(H)]$.

$$i_* i^*[H] = i_*\left(i^*[H]\cdot[X]\right)$$

$i_*(3\,pts)$

$$= [H]\cdot i_*[X]$$

$$= [H]\cdot 3[L]$$

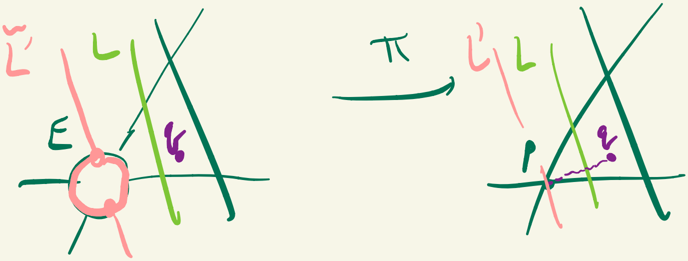
$$= 3[H\cap L] \leftarrow [H\cap L] = [H\cap i(X)]$$

$$= 3[pt].$$

In this case the projection formula is necessary to get the correct multiplicity, since f isn't an isomorphism.

⊛ We're still getting lucky to have transverse intersections.

Ex: Blowup $X = Bl_p \mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2$ blowdown map.



HW: You showed: $A(X) = \mathbb{Z} \cdot \left\{ \begin{array}{l} [X] \\ [E], [L] \quad \text{line not through } p \\ [pt] \end{array} \right.$

- $\pi^*[p] \neq [\pi^{-1}(p)]$ wrong codimension!

But $\pi^*[p] = \pi^*[q]$ for any other pt $q \in \mathbb{P}^2$

$= [\pi^{-1}(q)] = [q$, as a point in $X]$.

- Also $\pi^*[L] = \pi^*[L']$, L' = line through p.

$= [\pi^{-1}(L')]$ $\qquad \tilde{L}$ = strict transform.

$= [\tilde{L} \cup E] = [\tilde{L}] + [E]$

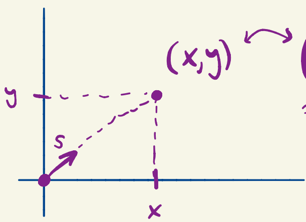
SECRET PAGE!

Why is $\pi^*[L'] = [\tilde{L}] + [E]$?? mult. 1 ?

<u>In a chart:</u> $X \xrightarrow{\pi} \mathbb{P}^2$.

$$k[x,y,s=\tfrac{y}{x}] \cong k[x,s] \xleftarrow{\pi^\#} k[x,y]$$

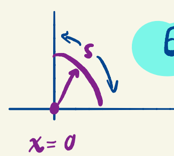
<u>Picture:</u>



$(x,y) \leftrightarrow (x, s = \text{"slope"} = \tfrac{y}{x}.)$
$= (x,s)$.

In (x,s) coordinates:

- $s=0$ $\quad (x, s=0)$ <u>x-axis.</u>
- $x=0$: $\quad (0,s)$



Exceptional divisor!

<u>Pull back</u> $f = ax+by \in k[x,y]$:

$$\pi^\#(ax+by) = ax + b\left(\tfrac{y}{x}\right)x$$

$$= \underbrace{(a+bs)}_{[\tilde{L}]} \cdot \underbrace{x}_{[E]}.$$

strict transform $\qquad$ exceptional.

Products: $[L]^2 = [pt]$ $\left(\pi^*[L^2] = pt \text{ on } \mathbb{P}^2 \right)$

$[L]\cdot[E] = 0$ since $L \cap E$ is ϕ.

$[E]^2 = ?$ Problem: not clear how to "move" E.

$\leadsto$ We know $[L] = [\hat{L}] + [E]$. $\left(= \pi^*[L] \right)$.

mult. by $[E]$.

$$\underbrace{[L]\cdot[E]}_{0} = [\hat{L}]\cdot[E] + [E]^2$$

$$0 = 1\,[pt] + [E]^2$$

$$-1\,[pt] = [E]^2$$

$\circledast$ This proves that E is not rat'l equiv. to any other effective cycle, since then $[E]^2$ would be ≥ 0.

(≥ 0)

(multiplicity of intersection scheme, i.e. length of some module)

But, $[E] = [L] - [\hat{L}]$.

Chow ring of X:

$$A(X) = \frac{\mathbb{Z}[e, \ell]}{(e \cdot \ell,\ e^2 = -\ell^2)}.$$

$(\neq A(\mathbb{P}^1 \times \mathbb{P}^1).)$

(but: rings isomorphic over $\mathbb{Q}$!

$\mathbb{Q} \otimes A(X) \cong \mathbb{Q} \otimes A(\mathbb{P}^1 \times \mathbb{P}^1).$)

Historically, the construction of the Chow ring relied on the "moving lemma".

Moving Lemma (Chow, Severi, others)

If $\alpha, \beta \in A(X)$, then there exist cycle representatives

$$\alpha = \sum n_i [A_i]$$

$$\beta = \sum m_j [B_j]$$

such that the cycles A_i, B_j intersect pairwise transversely. And, the class $\sum n_i m_j [A_i \cap B_j]$ doesn't depend on the choice of representatives.

$\Rightarrow$ Gives independent definition of intersection products.

Problems: (1) very difficult to prove.

(2) relies on the "global" geometry of X.

But: Useful in practice.

Next goal: go beyond surfaces:

$$\{\text{lines in } \mathbb{P}^3\} = \text{Grassmannian}, \ 4D.$$

$\rightsquigarrow$ The geometry will be more rich.

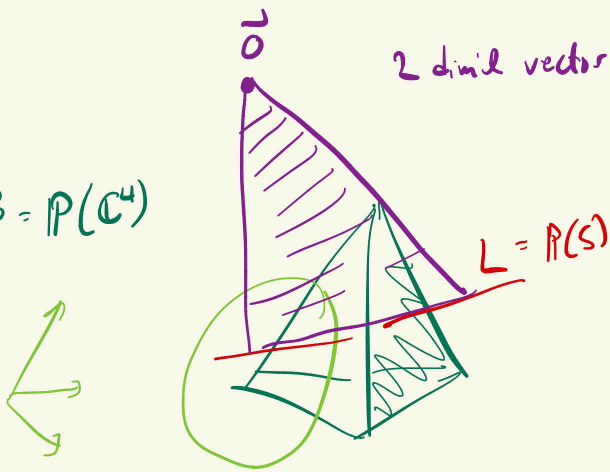
New phenomena will occur that were invisible until now.

Day 11: Coordinates on Gr(2,4).

Lines in $\mathbb{P}^3$

Affine & projective pictures:

$$\begin{array}{cc} \mathbb{C}^4 \setminus \{\vec{0}\} & \mathbb{C}^4 \\ \downarrow & \downarrow \\ \mathbb{P}^3 & \mathbb{P}^3 = \mathbb{P}(\mathbb{C}^4) \end{array}$$



How to describe $S \subseteq \mathbb{C}^4$ / $L \subseteq \mathbb{P}^3$?

$$S = \underset{\text{span}}{\text{row}} \; \overset{GL_2 \circlearrowright}{\begin{bmatrix} * & * & * & * \\ * & * & * & * \end{bmatrix}} \in \mathrm{Mat}^{\circ}_{2\times 4} \underset{\text{open}}{\subseteq} \mathrm{Mat}_{2\times 4}$$

full-rank

not unique, choice of basis

$$Gr(2,4) := \mathrm{Mat}^{0}_{2\times 4} / GL_2 \quad (\text{row ops})$$

Grassmannian of 2D subspaces of $\mathbb{C}^4$

Choosing a representative for S $\Leftrightarrow$ choosing homogenous coordinates for $p \in \mathbb{P}^n$.

$[a:b] \sim [\lambda a : \lambda b]$.

<u>Plücker embedding</u>:

Given $S = \left[\begin{smallmatrix} x & x & x & x \\ x & x & x & x \end{smallmatrix}\right]$, list all 2×2 minors of the matrix:

$g \in GL_2$ $\circlearrowleft$

$$[\ \det_{12}\ \ \det_{13}\ \ \det_{14}\ \ \det_{23}\ \ \det_{24}\ \ \det_{34}\ ] \in \underline{\mathbb{P}^5}.$$

$(\binom{4}{2} = 6.)$

(Multiplying the matrix by $g \in GL_2$ rescales all 6 minors by $\det(g)$, so we get the same point of $\mathbb{P}^5$.)

Call this map $p : Gr(2,4) \longrightarrow \mathbb{P}^5$.

$$S \longmapsto p(S) = [\ 6 \text{ dets}\ ].$$

This is called the <u>Plücker embedding</u>.

It is a closed embedding, the image is a closed subvariety of $\mathbb{P}^5$.

$\Rightarrow$ Gives one way to <u>define</u> $Gr(2,4)$ as a variety.

(shows: $Gr(2,4)$ is projective!)

Two (better) coordinate systems on $Gr(2,4)$.

① Stiefel coordinates $\longleftrightarrow$ homogeneous coords on $\mathbb{P}^3$.
(matrix coordinates)

② Affine charts $\longleftrightarrow$ affine charts on $\mathbb{P}^3$.

---

① $Gl_2 \circlearrowright R = \mathbb{C}\begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \end{bmatrix}$. $(\operatorname{Spec} R = \mathrm{Mat}_{2\times 4})$.

by row ops: $g \cdot s_{ij} = (g \cdot S)_{ij}$ entry

$= (\text{new } ij \text{ entry of } g \cdot [\vdots \; \vdots \; \vdots \; \vdots])$.

Any Gl_2-invariant ideal gives a closed subscheme of $Gr(2,4)$.

($\Leftrightarrow$ graded ideal for $\mathbb{P}^n$)

There's a functor

$$\left\{ \begin{matrix} Gl_2\text{-invariant} \\ \text{ideals of } R \end{matrix} \right\} \Big/ \sim \longrightarrow\!\!\!\!\rightarrow \left\{ \begin{matrix} \text{closed subschemes} \\ \text{of } Gr(2,4) \end{matrix} \right\}.$$

Ex 1: $p = [2:1:4:3] \in \mathbb{P}^3$.

$Z = \{\text{lines containing } p\} \subseteq Gr(2,4)$.

$= \{ S \text{ containing } [2\ 1\ 4\ 3] \}$

Ideal for Z:

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ 2 & 1 & 4 & 3 \end{bmatrix}$$

we want rank ≤ 2 $(=2)$, i.e. $S \ni [2\ 1\ 4\ 3]$.

$\rightsquigarrow I = (\text{all the } 3\times 3 \text{ minors of this matrix})$.

This ideal is GL_2 invariant.

Ex 2: $Z = \{\text{lines contained in the plane } \underbrace{X - Z = 0}\}$

$([X:Y:Z:W] \text{ in } \mathbb{P}^3)$

$$GL_2 \circlearrowright \begin{array}{c} \begin{array}{cccc} X & & Z & \end{array} \\ \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \end{bmatrix} \end{array} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} s_{11} - s_{13} = 0 \\ s_{21} - s_{23} = 0 \end{cases}$$

These two equations give a GL_2-invariant ideal.

Ex 3: Any 2×2 minor of $\begin{bmatrix} = S = \end{bmatrix}$ generates (on its own) a Gl_2-invariant ideal.

$$\det_{12} = \det(\text{cols}_{12}) = 0$$

$$I = (\det_{12}) \text{ is } Gl_2\text{-invariant}.$$

Note: We can think of this as coming from

$$\det \begin{bmatrix} \boxed{=} & S & & = \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 0 \quad (= \det_{12}).$$

which says S intersects $T = \operatorname{span}\left(\begin{matrix}[0\ 0\ 1\ 0], \\ [0\ 0\ 0\ 1]\end{matrix}\right)$.

$$(\dim(S+T) \le 3).$$

that is, $\mathbb{P}(S)$ intersects the line $\mathbb{P}(T)$.

So $I = (\det_{12})$ gives the locus:

$$Z = \{ S : \mathbb{P}(S) \text{ intersects } [0:0:*:*] \}.$$

"lines intersecting a given line".

② Affine charts

$$S = \begin{bmatrix} 2 & 1 & 5 & 5 \\ -1 & 1 & -1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1/3 & 1 & 5/3 & 0 \\ 4/3 & 0 & 2/3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 1 \end{bmatrix}.$$

Since S has full rank, at least one 2×2 minor must be nonzero.

$\hookrightarrow$ $\det_{12} \neq 0$ for this S. (Along with several others, e.g. $\det_{24}$.)

Let $\mathcal{U}_{12} = \{ S : \det_{12} \neq 0 \} \subseteq Gr(2,4)$.

$\hookrightarrow = \{ S : \mathbb{P}(S) \text{ is } \underline{\text{disjoint}} \text{ from } [0:0:*:*] \}$.

<u>Observation</u>:

Any $S \in \mathcal{U}_{12}$ has a <u>unique</u> representative by a matrix with an identity submatrix in cols 1,2.

i.e. $S = [\underbrace{A}_{\det(A)\neq 0} \mid B] \sim A^{-1}[A \mid B]$

$= [A^{-1}A \mid A^{-1}B]$

$$= \left[\begin{array}{cc|cc} 1 & 0 & x & y \\ 0 & 1 & z & w \end{array}\right].$$

This identification makes $\mathcal{U}_{12} \cong \mathbb{A}^4 \cong (\mathbb{A}^{2\times 2})$.

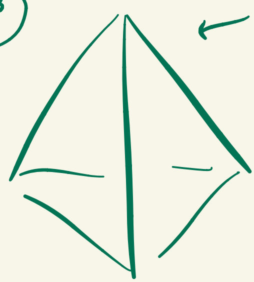
$$S \longmapsto x, y, z, w.$$

⇝ Similarly, we define $\mathcal{U}_{ij}$ for $1 \leq i < j \leq 4$. (6 choices)

Each chart is $\cong \mathbb{A}^4$.

⇝ These give the standard affine open cover of $Gr(2,4)$.

Note: $\mathbb{P}^3$



← Each $\mathcal{U}_{ij}$ is $\{$lines disjoint from the complementary coordinate line$\}$.

(6 coordinate lines $\Leftrightarrow$ 6 charts).

Ex: Descend the ideal

$$I = (s_{11} - s_{13},\ s_{21} - s_{23}) \text{ to } Gr(2,4).$$

Lines contained in the plane $\{X - Z = 0\}$.

$\rightsquigarrow$ chart $\mathcal{U}_{12}$: $\begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix}$.

plug in for s_{ij}.

$\rightsquigarrow$ $I = (\overset{s_{11} - s_{13}}{1 - a}, \overset{s_{21} - s_{23}}{0 - c}) = (1-a, -c)$.

Solutions are: $\begin{bmatrix} 1 & 0 & 1 & b \\ 0 & 1 & 0 & d \end{bmatrix}$: row span $\subseteq \{X - Z = 0\}$.

Precise bijection between Gl_2-invariant ideals of R and subschemes of $Gr(2,4)$:

Let P_2 = ideal of all 6 2x2 minors.

(locus of rank-deficient matrices $\subseteq Mat_{2\times 4}$.)

"Irrelevant ideal" for this setting.

Prop. Let $I, J \subseteq R$ be Gl_2-invariant ideals.

TFAE:

1. I, J give the same subscheme of $Gr(2,4)$.
2. For all 6 charts $\mathcal{U}_{ij}$, $I|_{\mathcal{U}_{ij}} = J|_{\mathcal{U}_{ij}}$.
3. I, J have the same saturation with respect to P_2:

$$\overline{I} := \{ f : \text{for each } ij,\ f \cdot \det_{ij}^{r} \in I \text{ for } r \gg 0 \}.$$

$$(\Leftrightarrow f \in I|_{\mathcal{U}_{ij}} \text{ for each chart})$$

Equivalently, $I \cap P_2^r = J \cap P_2^r$ for $r \gg 0$.

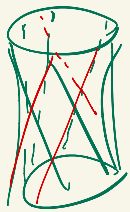
Day 12: Gr(2,4) cont'd.

Today: ① A neat computation (w/charts)

② Chow groups! $A(Gr(2,4))$.

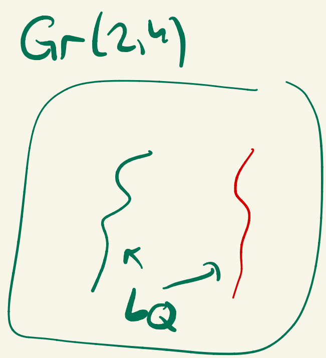
(③ end of class: revisit the GIT/Stiefel coords stuff).

Ex: $\mathbb{P}^1 \times \mathbb{P}^1 \xhookrightarrow{\text{Segre}} \mathbb{P}^3$ as a quadric surface Q



doubly-ruled surface.

two classes of lines on Q.



We're going to calculate

$Gr(2,4) \supseteq L_Q = \{ \text{lines } \ell \subseteq Q \}$. (in $\mathbb{P}^3$)

WLOG we can take $Q = \{XY = ZW\}$.

Striefel coords:

$$S = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} X & Y & Z & W \end{array}$$
$$S = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \end{bmatrix}$$

(row(S))

⊛ We want $XY - ZW = 0$ for every vector in S.

$\rightsquigarrow \vec{v} = a\,(\text{row } 1) + b\,(\text{row } 2)$

If we plug this $\vec{v}$ into "$XY - ZW = 0$"

we'll get something quadratic:

$$(\circledast) \quad a^2 f_{11} + ab\, f_{12} + b^2 f_{22} = 0$$

eqns for LQ.

$$\begin{cases} f_{11} = s_{11}s_{12} - s_{13}s_{14} & (\text{corresp. to } b=0,\ a=1) \\ f_{22} = s_{21}s_{22} - s_{23}s_{24} \\ f_{12} = \text{"permanents"} = (s_{11}s_{22} + s_{12}s_{21}) - (s_{13}s_{24} + s_{14}s_{23}) \end{cases}$$

For the expression ⊛ to vanish for all a, b,

$\Longleftrightarrow f_{11}, f_{12}, f_{22}$ must be 0.

$I = (f_{11}, f_{12}, f_{22})$ is the ideal of LQ.

$\Rightarrow$ Chart $\mathcal{U}_{12}$: $\begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix}$

f_{11} : $0 = a \cdot b$

f_{22} : $0 = c d$

f_{12} : $1 = ad + bc$

Two components:

$\underline{a = 0}$: $1 = bc$, $d = 0$ $\rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & t \\ 0 & 1 & 1/t & 0 \end{bmatrix}$

disjoint components!

$\underline{b = 0}$: $1 = ad$, $c = 0$ $\rightsquigarrow \begin{bmatrix} 1 & 0 & s & 0 \\ 0 & 1 & 0 & 1/s \end{bmatrix}$

<u>Check</u>: they are disjoint in all 6 charts.
(also smooth!)

Def: A *stratification* of a variety X is a decomposition

$$X = \bigsqcup X_i^0$$

into *pairwise disjoint*, *locally closed* subvarieties

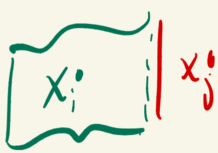
(open subset of closed subset)

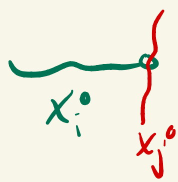
The X_i^0 are called *open strata* or *cells*.

The closures $X_i := \overline{X_i^0}$ are called *closed strata*.

Such that $\forall i$, the closure $\overline{X_i^0}$ is a union of $X_j^{(0)}$'s.

Equivalently: If $\overline{X_i^0}$ *intersects* X_j^0, then it *contains* X_j.





OK ✓.

Not OK .

affine space!

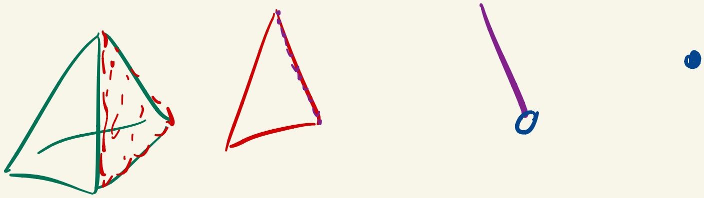
We say the stratification is ⊛ *affine* if each $X_i^0 \cong \mathbb{A}^{n_i}$

quasi-affine if each $X_i^0 \cong \mathcal{U} \underset{\text{open}}{\subseteq} \mathbb{A}^{n_i}$.

(e.g. $X_i^0 \cong (\mathbb{G}_m)^{a_i} \times \mathbb{A}^{b_i}$)

Ex:

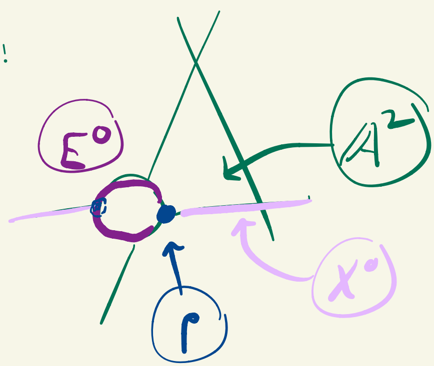
$$\begin{array}{ccccccccc}
\mathbb{P}^3 & = & \mathbb{A}^3 & \sqcup & \mathbb{A}^2 & \sqcup & \mathbb{A}^1 & \sqcup & \mathbb{A}^0 = pt \\
{[X:Y:Z:W]} & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & [x:y:z:1] & & [x:y:1:0] & & [x:1:0:0] & & [1:0:0:0].
\end{array}$$



$\mathbb{P}^r \times \mathbb{P}^s$: $\bigsqcup (\text{cells of } \mathbb{P}^r) \times (\text{cells of } \mathbb{P}^s)$.

$Bl_p \mathbb{P}^2$:

$Bl_{p_n}(\mathbb{P}^n)$



(⊛ Affine stratification ⇝ CW-complex!)

Theorem (Totaro 2014)

If X has an affine stratification with open strata X_i^0, closed strata X_i

then $A(X) \cong \bigoplus_i \mathbb{Z}\cdot[X_i]$.

(Chow groups but doesn't say how the ring structure works.)

Proof sketch:

① $A(X)$ is generated by the classes $[X_i]$.

"easy": Use excision, induct on # of cells.

X_i^0 cell of max dimension. open

$$A(\overset{\text{closed}}{X \setminus X_i^0}) \longrightarrow A(X) \rightarrow \overset{\text{open}}{A(X_i^0)} \rightarrow 0$$

$= A(\mathbb{A}^{n_i}) = \mathbb{Z}\cdot[\mathbb{A}^{n_i}]$.

affine space.

(2) $A(X)$ is free on the classes $[X_i]$.

(no relations.) hard. "higher Chow groups".

Relations in $A_i(X)$ come from $A_{i+1}(X)$.

but are 0 if all cells are $\mathbb{A}^k$'s. $\blacksquare$

$\Rightarrow$ We will never need (2).

We will always see it by other means

(using intersection pairings.)

$\Rightarrow$ But it's useful to have in mind.

Day 13: Chow ring of $Gr(2,4)$.

* We'll build an affine stratification of $Gr(2,4)$.

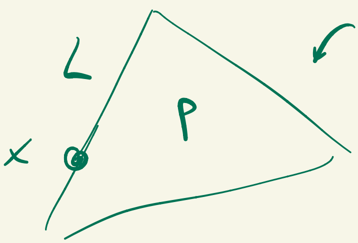
We have seen that there are open charts

$$\mathcal{U}_{12} = \{ \text{lines disjoint from } [0:0:*:*] \}$$

$$\cong \left\{ \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix} \right\}. \cong \mathbb{A}^4.$$

To describe the other strata, we introduce a <u>complete</u> flag in $\mathbb{C}^4$:

$\mathbb{P}(\mathcal{F})$:



$\mathbb{P}(\mathcal{F}) : x \in L \subseteq P.$

We're going to stratify $Gr(2,4)$ according to how $S \subseteq \mathbb{C}^4$ intersects $\mathcal{F}$.

($\mathbb{P}(S) \subseteq \mathbb{P}^3$ -- $\mathbb{P}(\mathcal{F})$)

Open strata : Schubert cells

⊛ Closed strata : Schubert cycles/varieties

classes in $A(Gr(2,4))$: Schubert classes.

Closed strata:

($X^{\square}(\mathcal{F})$, etc. $=$)

$$X^{\emptyset} = Gr(2,4).$$

⊛ singular, 3-dim'l. $\quad X^{\square} = \{ S : \mathbb{P}(S) \text{ intersects } L \}.$

$\cong \mathbb{P}^2 \quad X^{\square\square} = \{ S : \mathbb{P}(S) \text{ contains } x \}.$

$\cong \mathbb{P}^2 \quad X^{\boxminus} = \{ S : \mathbb{P}(S) \subseteq P \}.$

$\cong \mathbb{P}^1 \quad X^{\boxplus} = \{ S : x \in \mathbb{P}(S) \subseteq P \}.$

pt. $\quad X^{\boxplus} = \{ L \} = \{ S : \text{“} x \in \mathbb{P}(S) \subseteq L \text{''} \}.$

Fact: We have containments: $\left(Gr(k,n) : \; k \updownarrow \overset{\leftarrow n-k \rightarrow}{\boxed{\quad\quad\quad}} \right)$.

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & & & X^{(2)} \arrow[dr, phantom, "\supseteq"] & & & \\
X^{\emptyset} \arrow[r, phantom, "\supseteq"] & X^{(1)} \arrow[urr, phantom, "\supseteq"] \arrow[drr, phantom, "\supseteq"] & & & X^{(2,1)} \arrow[r, phantom, "\supseteq"] & X^{(2,2)} \\
 & & & X^{(1,1)} \arrow[ur, phantom, "\supseteq"] & & &
\end{tikzcd}
$$

$X^{(2,1)} = \underset{\text{codim } 2}{X^{(2)} \cap X^{(1,1)}}$

codim 3

but this is not transverse.

To get the open strata: $X_i^{\circ} = X_i \setminus X_j$'s contained in X_i.

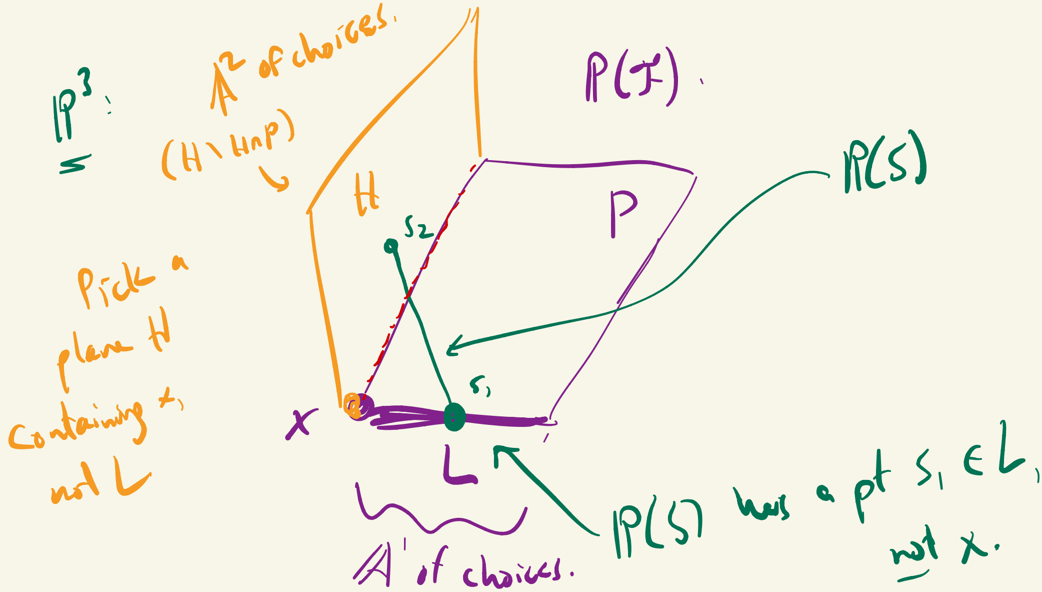
Fact: The open strata are all affine spaces.

Proof: Most of the strata are easy to check.

Also $(X^{\emptyset})^{\circ} = \mathcal{U}$ open chart $\cong \mathbb{A}^4$.

Let's show $\underbrace{X^{(1)} \setminus \left(X^{(2)} \cup X^{(1,1)} \right)}_{(X^{(1)})^{\circ}} \cong \mathbb{A}^3$.

$(X^{(1)})^{\circ} = \{ S : \mathbb{P}(S)$ intersects L but doesn't contain x and isn't contained in $P \}$.



$\mathbb{P}(S)$ intersects H and isn't contained in it (since $s_1 \notin H$) or in P (by hypothesis)

$\rightsquigarrow$ gives $s_2 \in H$, $s_2 \notin P$.

So, $(X^{\square})^{\circ} \cong \mathbb{A}^1 \times \mathbb{A}^2 = \mathbb{A}^3$. $\blacksquare$

There's also an important algebraic description of the open cells:

Let $\mathcal{F} = \langle e_4 \rangle \subset \langle e_3, e_4 \rangle \subset \langle e_2, e_3, e_4 \rangle \subset \mathbb{C}^4$.

"backwards flag".

So $(X^{\emptyset})^0$ = open chart

$$= \{ S : \mathbb{P}(S) \text{ disjoint from } \mathbb{P}(\langle e_3, e_4\rangle) \}$$

$$[0:0:*:*]$$

$$= \left\{ \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix} \right\} = \mathbb{A}^4.$$

The smaller cells are:

$$(X^{\square})^0 = \left\{ \begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix} \right\}$$

$$(X^{\yng(2)})^0 = \left\{ \begin{bmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}$$

$$(X^{\yng(1,1)})^0 = \left\{ \begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix} \right\}$$

$$(X^{\yng(2,1)})^0 = \left\{ \begin{bmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right.$$

$$X^{\yng(2,2)} = \left\{ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\} = \langle e_3, e_4 \rangle.$$

⊗ All the RREF's of matrices!

(Notice: the partition shape appears in the highlighted boxes 0 0).

Ex: For $(X^1)^\circ$: S intersects $\langle e_3, e_4 \rangle \rightsquigarrow [0:0:c:d] \in S$.

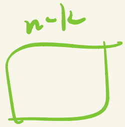
but $S \not\supset \langle e_4 \rangle$ so $c \neq 0$.

so $[0:0:1:d] \in S$. (2^{nd} row of matrix!)

Continue in this way. $\blacksquare$

Corollary: $A(Gr(2,4)) = \bigoplus \mathbb{Z} \cdot \left\{ \begin{array}{ll} [X^{\emptyset}] & \\ [X^{\ydiagram{1}}] & \text{codim } 1 \\ [X^{\ydiagram{1,1}}] \quad [X^{\ydiagram{2}}] & 2 \\ [X^{\ydiagram{2,1}}] & 3 \\ [X^{\ydiagram{2,2}}] \text{ pt.} & 4. \end{array} \right.$

For $Gr(k,n)$:

X^λ: $\lambda = \ydiagram{4,3,1} \subseteq_k$ 

$|\lambda| = \operatorname{codim} X^\lambda$.

(All possible ways S can intersect a complete flag.)

Some easy intersection products!

$$[X^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}] \cdot [X^{\square\square}] = 0.$$

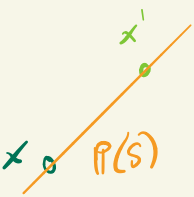
general choice of 2^{nd} flag $\mathcal{F}'$.

$\{$lines containing $x\} \cap \{$lines contained in $P'\}$.

disjoint if $x \notin P'$.

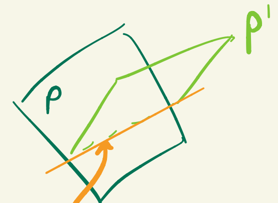
$$[X^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}]^2 = 1\,[pt].$$

$\{$lines containing $x\} \cap \{$lines containing $x'\}$



$$[X^{\square\square}]^2 = 1\,[pt].$$

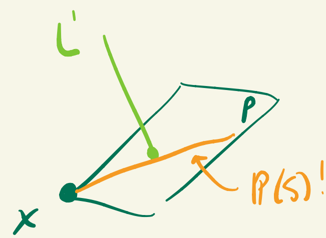
$\{$lines $\subseteq P\} \cap \{$lines $\subseteq P'\}$



1 choice for $\mathbb{P}(S)$.

$$[X^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}] \cdot [X^{\square}] = 1\,[pt].$$

$\{x \in \mathbb{P}(S) \subseteq P\} \cap \{$lines intersecting $L'\}$



(Day 14 : $A(Gr(2,4))$ cont'd.)

So far we've calculated:

$$\left.\begin{array}{l} \text{codim } 2 \cdot \text{codim } 2 \\ \text{codim } 3 \cdot \text{codim } 1 \end{array}\right\} \text{complementary dimension.}$$

Corollary: The Schubert classes are non-torsion in $A(Gr(2,4))$.

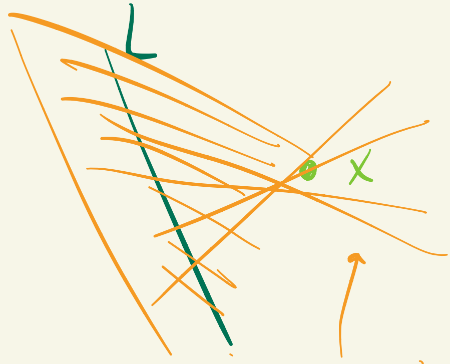
PF: ex: $[X^{\yng(2,1)}] \cdot [X^{\square}] = 1\,[pt]$

We know $[pt]$ is not torsion (since $Gr(2,4)$ is proper.)

$\Rightarrow [X^{\yng(2,1)}], [X^{\square}]$ also not torsion. $\blacksquare$

Next products:

codim 1 · codim 2.



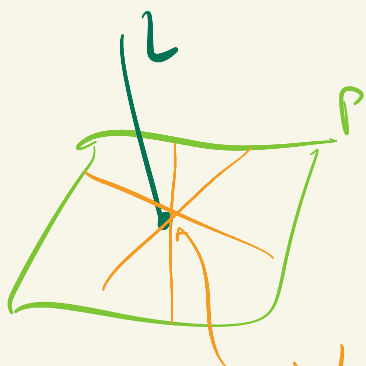
possible $\mathbb{P}(S)$'s.

$$[X^{\square}] \cdot [X^{\square\square}]$$

$$\{S : \mathbb{P}(S) \text{ intersects } L\} \cap \{S : x \in \mathbb{P}(S)\}$$

$$= \{S : x \in \mathbb{P}(S) \subseteq \operatorname{span}(x, L)\}$$

$$= [X^{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}}] \text{ for this new flag.}$$



point $L \cap P$.

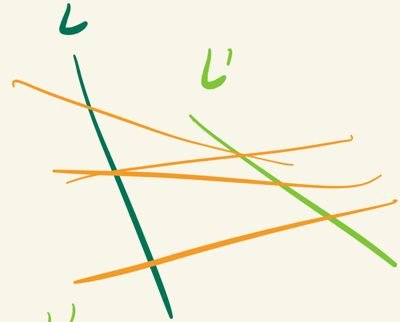
$$[X^{\square}] \cdot [X^{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}]$$

S: intersects L

$\subseteq P$

$$= \{S : (L \cap P) \in \mathbb{P}(S) \subseteq P\}.$$

codim 1 · codim 1:



$$[X^{\square}]^2 = [X^{\square}] \cdot [X^{\square}]$$

lines $\mathbb{P}(S)$ intersecting L, L'.

$\Rightarrow$ doesn't give a smaller Schubert variety.

$\Rightarrow$ Also not a union of Schubert varieties.

In fact it's irreducible for general L, L' (called Richardson variety).

Ex: $L = (0:0:*:*)$ $\qquad L' = [*:*:0:0]$

$X^{\square}(L) = \{\det_{12} = 0\}$ $\qquad X^{\square}(L') = \{\det_{34} = 0\}$.

Check in each chart that this is irred.

In the chart $\mathcal{U}_{13}$:

$$\begin{bmatrix} 1 & a & 0 & b \\ 0 & c & 1 & d \end{bmatrix} \qquad \begin{aligned} \det_{12} &= c = 0 \\ \det_{34} &= -b = 0. \end{aligned}$$

$\blacksquare$

We'll calculate $[X^{\square}]^2$ using the method of <u>undetermined coefficients</u>:

We know $[X^{\square}]^2 = a\,[X^{\square\square}] + b\,[X^{\substack{\square\\\square}}]$.

$\rightsquigarrow$ We'll multiply by each <u>complementary class</u> to pick out each coefficient.

(We can do this because our generators are <u>self-dual</u>: for each generator $\alpha \in A_k(Gr)$, $\exists!\ \beta \in A_{n-k}(Gr)$ such that:

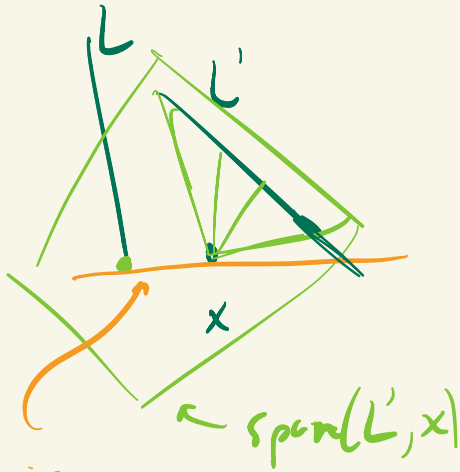
$$\alpha \cdot \beta = 1\,[pt],$$

<u>and</u> $\alpha' \cdot \beta = 0$ for all other generators $\alpha' \in A_k(Gr)$.)

① Multiply by $[X^{\square\square}]$:

$$\underbrace{[X^{\square}]^2 \cdot [X^{\square\square}]}_{\text{triple intersection:}} = a \cdot \underbrace{[X^{\square\square}]^2}_{=1[pt]}.$$

$\mathbb{P}(S)$: • intersecting L, L'
• $x \in \mathbb{P}(S)$



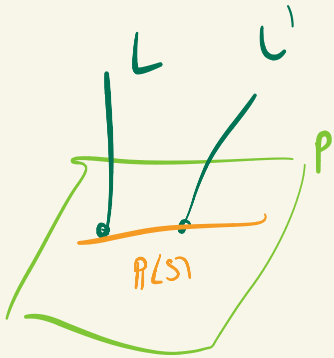
1 choice of $\underline{\underline{S}}$.

$\Rightarrow 1\,[pt]$.

$\Rightarrow a = 1$.

② $[X^{\square}]^2 \cdot [X^{\boxminus}]$:

$= 1\,[pt]$.
(similarly).



$\rightsquigarrow$ So $a = b = 1$,

$$[X^{\square}]^2 = [X^{\square\square}] + [X^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}].$$

Completes $A(Gr(2,4))$!

---

① <u>Question</u> from day 1 of class:

How many lines intersect 4 given (general) lines in $\mathbb{P}^3$?

$\hookrightarrow L_1, L_2, L_3, L_4$

$$\rightsquigarrow Z = X^{\square}(L_1) \cap X^{\square}(L_2) \cap X^{\square}(L_3) \cap X^{\square}(L_4).$$

Assuming transversality,

$$[Z] = [X^{\square}]^4.$$

$$= \left(X^{\square\square} + X^{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} \right)\left(X^{\square\square} + X^{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} \right)$$

$$= 2\,[pt] \in A(Gr(2,4)).$$

Ans: $\underline{\underline{2}}$!

(2) $C \subseteq \mathbb{P}^3$ smooth curve of deg d.

a) What is the class of
$Z = \{\text{lines intersecting } C\}$?

b) How many lines intersect 4 general curves of degrees $d_1, \dots, d_4$?

a) $\dim(Z) = 3$:
- 1 dim of choice of $x \in C$
- 2 dim's of directions of lines through x.

To make the dimension count precise, we use an <u>incidence correspondence</u>:

$$\begin{tikzcd} & Gr(2,4)\times C \supseteq \Phi = \{(\mathbb{P}(S), x) : \overset{(x\in C)}{x\in \mathbb{P}(S)}\} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \boxed{?} \quad Z & & C \end{tikzcd}$$

π_2 (surjective). C dim 1.

Fiber of π_2: $\pi_2^{-1}(x) = \{\text{lines through } x\}$

$\otimes$ 2 dim'l. $= X^{\square}(x)$.

$\Rightarrow \dim \Phi$.

($\Rightarrow$ Also Φ is irreducible!

<u>Thm</u>: If $f: X \to Y$ proper, <u>surjective</u> map of schemes, Y <u>irreducible</u>, all fibers $f^{-1}(y)$ also <u>irreducible</u> & all the same dimension.

Then X is irreducible!)

Fiber of π_1: L line $\in Z$,

$$\pi_1^{-1}(L) = \underbrace{L \cap C}$$

$\Rightarrow$ finite (generally 1 pt).

$\Rightarrow \dim \Phi = \dim Z$.

$\Rightarrow$ $\boxed{\dim Z = 3}$.

So: $[Z] = a\,[X^{\square}]$.

To find a: $[Z]\cdot[X^{\boxplus}] = a\,[pt]$.

lines $\mathbb{P}(S)$: $\mathbb{P}(S)$ intersects C,

$x \in \mathbb{P}(S) \subseteq P$

d choices of $\mathbb{P}(S)$.

$C \cap P = d$ points. $(= \deg(C))$

So $[Z] = d[X^{\square}]$.

(b) Given 4 general curves $C_i \subseteq \mathbb{P}^3$, $\deg(C_i) = d_i$,

how many lines intersect all 4?

$$\rightsquigarrow [Z_1 \cap Z_2 \cap Z_3 \cap Z_4]$$

$$= (d_1[X^{\square}]) \cdot \cdots (d_4[X^{\square}])$$

$$= \prod_i d_i \cdot (X^{\square})^4$$

$$= \underline{2 \prod_i d_i} \ [pt].$$

Day 15: Various enumerative problems.

Def: A <u>homogeneous space</u> is a variety X with a transitive action by a group variety G.

ex: $\mathbb{P}^n \circlearrowleft GL_{n+1}$. $\qquad \mathbb{P}^r \times \mathbb{P}^s \circlearrowleft GL_{r+1} \times GL_{s+1}$.

$Gr(k, \mathbb{C}^n) \circlearrowleft GL_n$

E elliptic curve $\circlearrowleft$ itself.

<u>Lemma/obs:</u> If $G = GL_n$ or product of GL_n's and $Z \subseteq X$ subvariety, then $[gZ] = [Z]$.

<u>Sketch pf:</u> GL_n is rational.

So we can explicitly give the rational equivalence. ■

<u>Kleiman's Theorem.</u> Let X be a homogeneous space w/ group variety G.

Let $Y, Z \subseteq X$ be subvarieties of codim c, d.

Then there's an open subset $\mathcal{U} \subseteq G$ s.t. for all $g \in \mathcal{U}$,

$Y \cap gZ$ has codimension $c+d$ (empty if $c+d < 0$)

and is generically reduced (in particular, Y and gZ are generically smooth along $Y \cap gZ$).

Therefore $[Y \cap gZ] = [Y] \cdot [gZ]$.

($= [Z]$ if $G = GL_{n+1}$, etc)

(If Y, Z are smooth, $Y \cap gZ$ can also be taken to be smooth.)

"Moving lemma for homogeneous spaces".

<u>Ex</u>. If $\mathcal{F}$ = complete flag, (in $\mathbb{P}^3$)

$X^{\boxplus}(\mathcal{F}) = \{ \text{lines } \mathbb{P}(S) : x \in \mathbb{P}(S) \subseteq P \} \subseteq Gr(2,4)$.

For $g \in GL_4$, $g \cdot X^{\boxplus}(\mathcal{F}) = X^{\boxplus}(g^{-1}\mathcal{F})$.

$\boxplus$ Just a different choice of flag!

Thus if $Y \subseteq Gr(2,4)$ is any subvariety,

$Z = X^{\boxplus}(\mathcal{F})$, then by Kleiman's Thm,

$Y \cap gZ$ ($= Y \cap Z'$, general choice of $\mathcal{F}$)

is guaranteed to be generically reduced & to represent the class $[Y] \cdot [X^{\boxplus}]$.

$\Rightarrow$ Justifies our calculation of $A(Gr(2,4))$!

Ex: Q : quadric $\{XY - ZW = 0\} \subseteq \mathbb{P}^3$.

$Z = \{\text{lines in } Q\} \subseteq Gr(2,4)$.

We calculated Z and found $\dim Z = 1$.

So $[Z] = k \cdot [X^{\boxplus}]$.

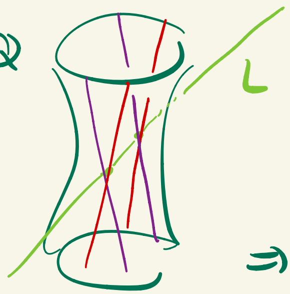
To find k: $[Z] \cdot [X^{\square}] = k\,[pt]$.

$$= [Z \cap X^{\square}(\mathcal{F})] \text{ for general choice of } \mathcal{F}.$$

(Kleiman's Thm)

$\mathbb{P}(S)$ intersecting L (general choice of L).

Q



L (intersects Q in 2 reduced pts) (← Kleiman!)

$\Rightarrow$ 4 lines.

$\Rightarrow$ So, $k = 4$.

So $[Z] = 4\,[X^{\boxplus}]$.

(In fact each "ruling" of Q gave a component of Z of class $2[X^{\boxplus}]$.)

Next examples:

Two relevant facts about curves:

(Adjunction formula) ① A smooth curve in $\mathbb{P}^2$ of deg d has genus $g = \binom{d-1}{2}$.

More generally a singular curve has "arithmetic genus" $\binom{d-1}{2}$

$$\binom{d-1}{2} = \underset{(g(\tilde{C})\text{ normalization})}{\text{true genus}} + \underset{(\text{counted w/ mult})}{\#\text{ singularities}}.$$

ex:  nodal cubic, deg 3. $\binom{3-1}{2}$ = true genus + 1

0 = true genus.

☆ ② (Riemann-Hurwitz formula)

Ch 4 - Hartshorne.

If $f: C \to C'$ is a map of proper smooth curves, $\deg(f) = d$, then the # of ramif. pts is

$$2g(C) - 2 - d(2g(C') - 2).$$

Ex: $S \subseteq \mathbb{P}^3$ smooth surface of deg d, not containing any lines (so $d > 2$).

$T(S) = \{\text{tangent lines to } S\} \subseteq Gr(2,4)$.

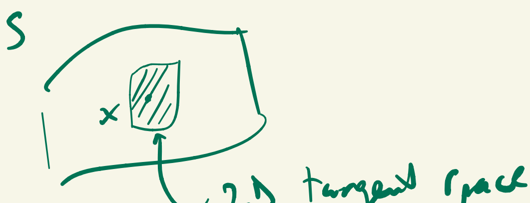
What's the class of $T(S)$?

① Dim count:

$Gr(2,4) \times \mathbb{P}^3 \supseteq \{(l,x) : l \text{ tangent to } S \text{ at } x \in S\} = \Phi$

$$\begin{tikzcd} & \Phi \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ T(S) & & S \end{tikzcd}$$

- fiber of π_2: $\cong \mathbb{P}^1$,



$\Rightarrow \mathbb{P}^1$ of tangent lines.

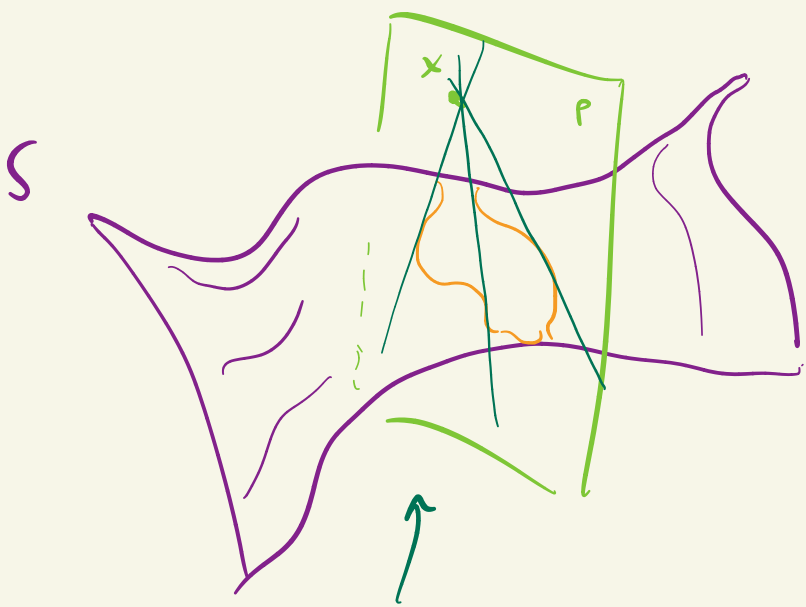
$\Rightarrow$ so $\dim \Phi = \dim(S) + 1 = 3$.

• Π_1 is finite since each ℓ can only touch S @ finitely-many points.

$\Rightarrow \dim T(S) = \dim \Phi = 3$.

② So $[T(S)] = k\,[X^{\square}]$.

$\Rightarrow$ Intersect with $X^{\mathcal{F}}(\mathcal{F})$, for general $\mathcal{F}$, to find k.

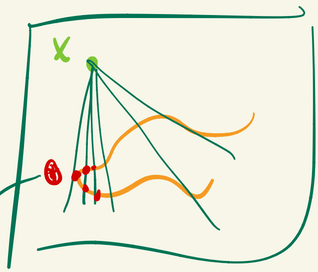


⊛ By Kleiman, $P \cap S$ is a smooth curve of deg d. ($d = \deg(S)$)

How many lines through x are tangent

to $C = P \cap S$?

Picture in $\mathbb{P}^2$ is:



Projection from x:

$\pi_x : \mathbb{P}^2 \setminus \{x\} \to \mathbb{P}^1$.

$\Rightarrow \pi_x : C \to \mathbb{P}^1$.

A line ℓ through x is tangent to C

$\Leftrightarrow$ $\ell \cap C$ is a ramification point.

$\rightsquigarrow$ we know $g(C) = \binom{d-1}{2}$ (Formula ① above)

$\rightsquigarrow$ By Riemann-Hurwitz, the # of ramif. pts:

$$2\binom{d-1}{2} - 2 - d(0-2)$$

$$= \dots = \underline{d(d-1)}.$$

$\rightsquigarrow$ So $[T(S)] = d(d-1)[X^0]$. //

Q: How many lines are simultaneously tangent to 4 general quintic surfaces?

Ans: (By Kleiman's Thm)

$$\left[T(S_1) \cap \cdots \cap T(S_4)\right] = \left(5 \cdot 4 \left[X^{\square}\right]\right)^4$$

$$= 5^4 \cdot 4^4 \cdot 2[pt]$$

$$= 320{,}000\ [pt].$$

Day 16. Enumeration & Kleiman's Theorem.

Example: C = smooth curve in $\mathbb{P}^3$,
deg d, genus g, not contained in a plane.
($d \geq 3$).

Z = { chords to C } $\subseteq Gr(2,4)$.

Find the class of Z.

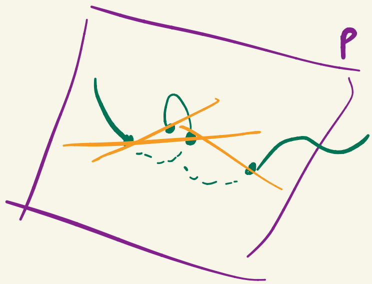
dim count: dim $Z = 2$ (check with an incidence corresp.)

$\rightsquigarrow$ So $[Z] = a[X^{\square\square}] + b[X^{\boxminus}]$.

① Find b (easier).

$[Z]\cdot[X^{\boxminus}] = b\,[pt]$.

↑ lines contained in a plane P



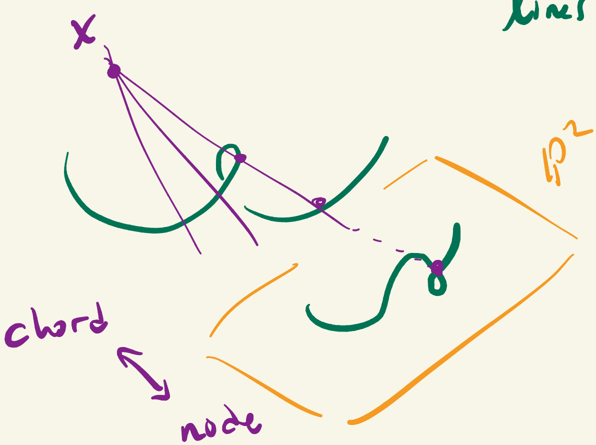
$C\cap P$: d reduced pts.

$\Rightarrow$ $\binom{d}{2}$ choices of bisecant contained in P.

$= b$.

② Find a : $[Z]\cdot[X^{\boxbar}] = a\,[pt]$.

↑ lines containing a given $x\in\mathbb{P}^3$.



Think about projection from x!

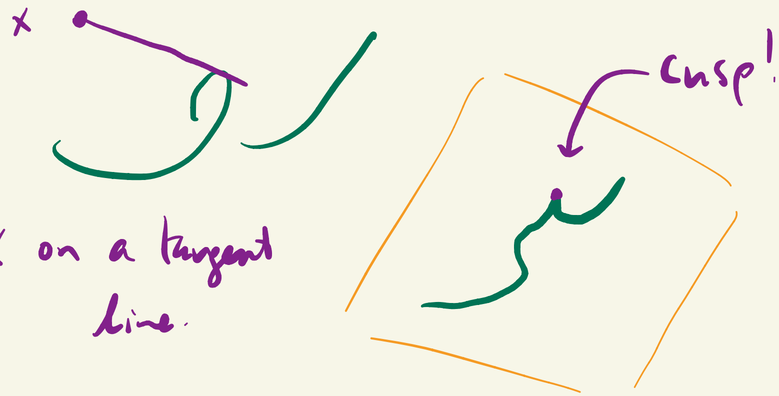
$\pi : \mathbb{P}^3\setminus\{x\}\to\mathbb{P}^2$.

no a = # chords through x

= # of simple nodes in $\pi(C)$.

For this to be correct, we need to ensure $\pi(C)$ has no other singularities.

ex! a cusp in $\pi(C)$

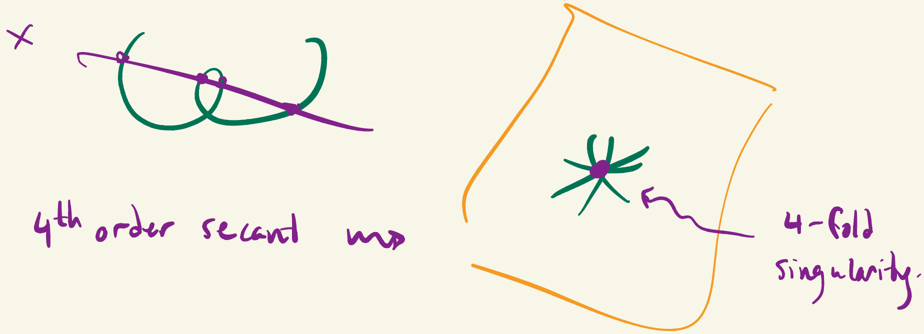


x on a tangent line.

$\Rightarrow$ But, union of all tangent lines to C is a surface in $\mathbb{P}^3$.

$\Rightarrow$ Take $x \notin$ this surface $\Rightarrow$ $\pi(C)$ has no cusps.

The other type of singularity to avoid is a multisecant, which would give a higher order singularity:



$\Rightarrow$ $\exists$ only finitely-many of these.

(Check Hartshorne IV §2. Ex 2.3)

<u>Recall</u>: For a singular curve in $\mathbb{P}^2$:

$$\binom{d-1}{2} = \underset{(g(\tilde{C})\ \text{normalization})}{\text{true genus}} + \underset{(\text{counted w/ mult})}{\#\ \text{singularities}}.$$

$$\binom{d-1}{2} = g + \#\ \text{nodes}.$$

So, $\#\,\text{nodes} = a = \binom{d-1}{2} - g.$

$$\rightsquigarrow [Z] = \left(\binom{d-1}{2} - g\right)\left[X^{\square\square}\right] + \binom{d}{2}\left[X^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\right].$$

Question: C_1, C_2 general curves of deg d_1, d_2, genus g_1, g_2.

How many common chords do they have?

$\rightsquigarrow Z_1 \cap Z_2$. $\leftarrow$ $Z_1 \cap gZ_2$ transverse by Kleiman. (corresp. to $C_1, g^{-1}C_2$.)

So, $[Z_1 \cap Z_2] = [Z_1]\cdot[Z_2]$

$$= \left(\left(\binom{d_1-1}{2} - g_1\right) \square + \binom{d_1}{2}\theta\right)\left(\cdots \square + \cdots \theta\right)$$

$$= \left(\binom{d_1-1}{2} - g_1\right)\left(\binom{d_2-1}{2} - g_2\right) + \binom{d_1}{2}\binom{d_2}{2} \quad [pt].$$

Kleiman's Theorem: X = homogeneous space w/ group variety G.

char 0 only!

a) $Y, Z \subseteq X$ subvarieties of codims c, d.

Then $\exists\, \mathcal{U} \subseteq G$ dense open s.t. for all $g \in \mathcal{U}$,

$Y \cap gZ$ is generically reduced + codim $c+d$.

b) $f: Y \to X$ morphism, Y = variety.

$Z \subseteq X$ subvariety codim d.

Then $\exists\, \mathcal{U} \subseteq G$ dense open s.t. $f^{-1}(gZ)$ is generically reduced & codim d, for all $g \in \mathcal{U}$.

⊛ (a) follows from (b) for $f = i : Y \hookrightarrow X$.

⇝ Consequence of generic smoothness (char 0 only)

+ easy dimension counting.

Thm: For any map $V \xrightarrow{f} W$ of varieties, V smooth,

then $\exists\, \mathcal{U}$ dense open in W s.t.

$f^{-1}(\mathcal{U}) \to \mathcal{U}$ is a smooth morphism

of rel dimension $= \dim V - \dim W$.

(⇒ all fibers are smooth & pure of the given dimension.)

Proof of Kleiman: ((b) only).

1. dim count.
2. smoothness argument.

1. $$Y \times Z \times G \supseteq \{(y, z, g) : f(y) = g \cdot z\} \quad \left(= \text{"}\Gamma(f) \cap g\mathcal{Z}\text{"} \right) = \Gamma.$$

$$\begin{tikzcd} & Y \times Z \times G \arrow[dl, "\pi_{12}"'] \arrow[dr, "\pi_3"] & \\ Y \times Z & & G \end{tikzcd}$$

Fibers of π_{12}: $\pi_{12}^{-1}((y,z)) = \{g : f(y) = g\cdot z\}$.

$\cong$ stabilizer of a point under the action of $G \curvearrowright X$.

$\circledast$

fiber dimension of π_{12} is $\dim G - \dim X$.

$G \to X$

$g \mapsto g\cdot x_0$ surjective, (since G acts transitively)

so the fiber (stabilizer of x_0) has $\dim = \dim G - \dim X$.

$\Rightarrow$ So,

$$\dim \Gamma = \dim Y + \dim Z + \underbrace{\dim G - \dim X}_{(\dim(\text{fiber of } \pi_{12}).)}$$

(Not necessarily integral: stabilizers could have >1 connected compos.)

$\Rightarrow$ also shows Γ is reduced.

Fibers of $\pi_3 : \Gamma \to G$.

$$\pi_3^{-1}(g) = \Gamma(f) \cap gZ = \{(y,z) : f(y) = g\cdot z\}.$$

$$\cong f^{-1}(gZ).$$

$\Rightarrow$ This fiber may not have the same dimension for all g.

We want it to have dim:

$$\dim f^{-1}(gZ) = \dim Y - \underbrace{(\dim X - \dim Z)}_{\operatorname{codim}_X(Z)}$$

$$= \dim \Gamma - \dim G.$$

② Smoothness.

Look first @ $(\Gamma \setminus \operatorname{Sing}\Gamma) \to G$.

By construction, this is smooth so

by generic smoothness $\exists\, \mathcal{U}_1 \subseteq G$ dense open,

s.t. fibers in $\Gamma \setminus \operatorname{Sing}\Gamma$ are

smooth & pure of rel. dim $= \dim\Gamma - \dim G$.

$\rightsquigarrow$ We just need to bound the size

of $(\text{fiber}) \cap \operatorname{Sing}\Gamma$.

Look at $\operatorname{Sing}\Gamma \to G$.

General fiber has dim

$$\dim(\operatorname{Sing}\Gamma) - \dim G$$

(dense open $U_2 \subseteq G$)

$$<$$

$$\dim\Gamma - \dim G.$$

(or is empty (if $\operatorname{Sing}\Gamma$ doesn't map dominantly to G).)

this would imply that the general fiber of $\Gamma \to G$ is smooth.

This is not always the case:

every fiber is singular.

So, for $g \in \mathcal{U}_1 \cap \mathcal{U}_2$,

$$\dim\left(f^{-1}(gZ) \cap \mathrm{Sing}(\Gamma) \right) < \dim f^{-1}(gZ).$$

So $f^{-1}(gZ)$ is <u>generically</u> smooth. $\blacksquare$

<u>Comments:</u>

1 Why doesn't this lead directly to a general "moving lemma"?

1) general varieties X may have <u>no</u> automorphisms.

2) particular cycles $Z \subseteq X$ may not move (e.g. exc. divisor $E \subseteq Bl_p \mathbb{P}^2$)

3) There's no way to parametrize all possible

rational equivalences (esp. non-effective ones).

No route to dim-counting / genericity.

(2) char p?

ex: $\mathbb{P}^1 \to \mathbb{P}^1 \circlearrowleft GL_2$.

$t \mapsto t^p$.

Every fiber is nonreduced.

In char p, a weaker version of Kleiman's Thm holds if G acts transitively on points + tangent vectors.

(Rmk: This doesn't apply to $Gr(k,n)$ $2 \le k \le n-2$.)

Questions we can't answer w/ Kleiman:

① $\mathbb{P}^3 \supseteq S$ cubic surface.

$Gr(2,4) \supseteq Z = \{ \text{lines} \subseteq S \}$.

Later we'll see by dimension count that

Z is finite for general S.

$\rightsquigarrow Z = k \cdot [pt]$.

(I told you previously:
$k = 27 \ldots$)

$\rightsquigarrow Z$ isn't readily describable as an
intersection of $\dim > 0$ loci.

② $\mathbb{P}^5 = \{\text{conics in } \mathbb{P}^2\}$. $\circlearrowleft PGL_6$

$\circlearrowleft PGL_3$ (induced action).

$\cup$

$[C]$ fixed conic.

$Z := \overline{\{C' : C' \text{ smooth} + \text{tangent to } C\}}$.

$= \{C' : C' \cap C \text{ has a pt of mult} \geq 2\}$.

dim count: $\dim Z = 4$.

So, $[Z] = k[H]$ in $\mathbb{P}^5$, some $k \in \mathbb{Z}$.

To find k: compute $[Z \cap L]$

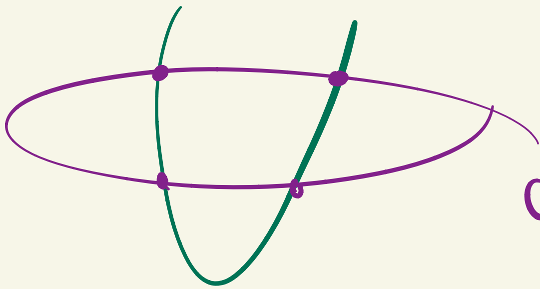
general line in $\mathbb{P}^5$.

$L = \{aF + bG\} \subseteq \mathbb{P}^5$

Look at $C \overset{i}{\hookrightarrow} \mathbb{P}^2$

$\cong \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$

$aF + bG$ = eqn of 2nd conic.



$C' = \{aF + bG = 0\}$.

$C \cong \mathbb{P}^1_{[T:S]}$.

$aF + bG$ gives a quartic eqn. on $\mathbb{P}^1 \cong C$.

We want it to have a double root,

i.e. discriminant $(aF + bG) = 0$.

$\Rightarrow$ this is a degree $2 \cdot 4 - 2 = 6$ polynomial in a, b.

$\Rightarrow$ So, 6 <u>values</u> where discr $= 0$.

So $[Z] = 6\,[H]$.

Say $C_1, \dots, C_5$ are general conics.

$Z_1, \dots, Z_5$ loci of tangent conics.

$$\Rightarrow [Z_1] \cdot \dots \cdot [Z_5] = 6^5\,[pt] = 7776\,[pt].$$

$\Rightarrow$ Problem: $Z_1 \cap \dots \cap Z_5$ is NOT

a transverse intersection.

It will have the form

$$\underbrace{A}_{\text{finite set}} \cup \underbrace{B}_{\dim > 0} = A \cup SQ.$$

Problem is

$$Z \supseteq \{ \text{squares } L^2 \text{ of linear forms} \}$$

$$= SQ \subseteq \mathbb{P}^5.$$

2 dimensional!

$PGL_3 \circlearrowright \mathbb{P}^2$

moving conics.

But the induced action

$PGL_3 \circlearrowright \mathbb{P}^5$ is <u>not</u>

transitive.

e.g. SQ is PGL_3-invariant.

And applying PGL_6 has no meaning

in terms of the loci Z.

Day 18: Wrap-up on transversality
+
Vector bundles!

What does an intersection product tell us?

Say $Z_1, \dots, Z_k \subseteq X$ are subvarieties.

Say we've computed $[Z_1] \cdot \dots \cdot [Z_k] = \alpha \in A(X)$.

Some class.

What can we conclude about $Z_1 \cap \dots \cap Z_k$?

(1) α is a "lower bound" for $Z_1 \cap \dots \cap Z_k$.

- of course $Z_1 \cap \dots \cap Z_k$ could have dimension <u>too large</u>.
- it can never have dimension too <u>small</u> (unless the intersection is empty).
- if the codimension is correct, the multiplicities are "$\geq \alpha$":

$$\alpha = \sum_{W \subseteq Z_1 \cap \cdots \cap Z_k} k_W [W],$$

where $1 \leq k_W \leq \mathrm{mult}(W, Z_1 \cap \cdots \cap Z_k)$.

(can only be < in non-CM case)

② If $\alpha \neq 0$, $Z_1 \cap \cdots \cap Z_k$ can't be empty!

($\Rightarrow$ ex: $\exists$ conic tangent to 5 conics).

③ If $\alpha = 0$. $Z_1 \cap \cdots \cap Z_k$ is either empty

or not dimensionally transverse.

e.g. $X^{\square\square} \cdot X^{\substack{\square\\\square}} = 0$ in $A(Gr(2,4))$.

$\nearrow$ $\quad\quad$ $\uparrow$

lines $\ni x$ $\quad$ lines $\subseteq P$.

general intersection is ϕ.

special ones give $\underbrace{X^{\boxplus}(x, P)}_{\text{dim } 1}$
$(x \in P)$

(too big!).

(4) If α is a non-effective class.

(not representable by an effective cycle).

e.g. $\alpha = -1\,[pt]$.

then $Z_1 \cap \cdots \cap Z_k$ must be nonempty

and dim too large.

For dim > 0, it's hard to know which classes are effective.

e.g. $\overline{M_{0,n}}$ = moduli space of stable curves genus 0.

$\overline{M_{g,n}}$

this is *open* for $n \geq 7$.

But on $X = Gr(2,4)$ $(Gr(k, \mathbb{C}^n))$:

$\alpha \in A(X)$ is effective

$\iff$ α is a ≥ 0 linear comb. of Schubert classes.

Proof: Kleiman's Thm + dual Schubert classes. $\blacksquare$

Vector bundles (& Degeneracy loci)

<u>Def</u>: $V \xrightarrow{\pi} X$ is a <u>vector bdle</u> of rank r over X if $\exists$ open cover $\{U_i\}$ of X, & compatible isoms

(H Exer II 5.18)

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow[\varphi_i]{\sim} & U_i \times \mathbb{A}^r \\ \pi \downarrow & & \pi_1 \downarrow \\ U_i & & U_i \end{array}$$

and s.t. $\varphi_j \circ \varphi_i^{-1} : (U_i \cap U_j) \times \mathbb{A}^r \circlearrowleft$

is linear on each fiber.

$$\left(\begin{array}{ccc} U_i \cap U_j & \to & GL_r \\ u \in & \mapsto & T_u \end{array} \right).$$

Ex: $Gr(k,V)$ has two tautological bundles S, Q, defined as follows:

S:

$$\{(v,S) : v \in S\}.$$

$$\begin{tikzcd}
\underbrace{V \times Gr(k,V)}_{\text{trivial vector bdle}} \arrow[dr, "\pi_2"] & \supseteq & S \arrow[dl, "\pi_2"] \\
& Gr(k,V) &
\end{tikzcd}$$

where $S = \{(v,S) : v \in S\}$.

Obs: If $(v_1, S), (v_2, S)$ are $\in$ same fiber then (v_1+v_2, S) is also $\in$ fiber.

$\Rightarrow$ S is a subbundle of $V \times Gr$,

rank $= k = \dim(S)$.

fibers are $\pi_2^{-1}(S) = \{v : v \in S\} = S$.

To trivialize: say $\mathcal{U}_{12} = \{\det_{12} \neq 0\}$.
$(Gr(2,4))$

$$\mathbb{C}^4 \times \mathcal{U} \supseteq S|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{A}^2_{s_1, s_2}.$$

$$[x\ y\ z\ w] \qquad \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix}$$

pairs

$$\left(\begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix}, \vec{v} \in S \right)$$

$$\circledast \quad \vec{v} = \underbrace{[s_1\ s_2]}_{\mathbb{A}^2_{s_1, s_2}} \cdot \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix}$$

$$= [s_1 \quad s_2 \quad as_1 + cs_2 \quad bs_1 + ds_2]$$

To change to another chart:

$$\text{"} \quad \vec{v} = \vec{s} \cdot S \quad \text{"}$$

$$= \underbrace{(\vec{s} g)}_{\text{linear change of coordinates on}} \cdot \underbrace{(g^{-1} S)}_{GL_2 \text{ change of}}$$

s_1, s_2. basis for S

Similarly $\exists$ tautological quotient bundle Q
(fibers are $\mathbb{C}^4/S$).

It can't be written as an incidence correspondence ($Q \ncong$ trivial bundle).
(have to use charts).

$\Rightarrow$ Tautological SES on $Gr(k,V)$:

$$0 \to S \to \underbrace{V \times Gr(k,V)}_{\text{trivial bundle}} \to Q \to 0.$$

[$\mathbb{P}^n$: $0 \to \mathcal{O}(-1) \to V \times \mathbb{P}(V) \to Q \to 0$.]

$\Rightarrow$ Much easier to work globally rather than in charts — just as long as all recipes are natural.

ex: Fix $v \in V$, $(\neq 0)$, gives a "constant section"

$$\sigma : Gr \to V \times Gr.$$

$\Rightarrow$ this also induces a section

$$\bar{\sigma} : Gr \to Q$$

$\bar{\sigma}$ = "always $\bar{v} \in \mathbb{C}^n/S$"

v doesn't vary

but S varies.

$\Rightarrow$ Where is $\bar{\sigma} = 0$?

(vanishing locus)

$$\mathbb{V}(\bar{\sigma}) = \{ S \in Gr : \bar{\sigma}(S) = \vec{0} \}.$$

$\Updownarrow$ $\bar{v} = \bar{0}$ in $\mathbb{C}^n/S$.

$\Updownarrow$ $v \in S$.

$$= X^{\square}(v).$$

all S containing v.

⊛ **Notice**: $\mathbb{V}(\bar{\sigma})$ is codimension $\underline{2}$.

In general we expect

$\mathbb{V}(\sigma : X \to E)$ to have

codimension $= \operatorname{rank}(E)$.

Locally: $\sigma = \left.\begin{bmatrix} \\ \\ \\ \end{bmatrix}\right\}$ $\operatorname{rank}(E)$ coefficient vector.

$\Rightarrow$ r-tuple of regular functions.

Day 19: Chern & Segre classes

These are classic examples of degeneracy loci.

Important Fact:

Prop: (comm alg).

Let M = $a \times b$ matrix of ring elts $x_{ij} \in R$.

I_d = (ideal of $(d+1)\times(d+1)$ minors of M) (rank $\leq d$ locus).

Then the codim of R/I_d is $\leq$ $(a-d)(b-d)$.

"$=$" in the case $R = k[x_{ij}]$.

Pf sketch: (polynomial ring case).

Invert the top left $d \times d$ minor $\mathcal{U} = \{\frac{1}{\det_d}\} \subseteq \mathbb{A}^{a\times b}$.

$$\begin{array}{c c} a & \underbrace{\begin{bmatrix} \boxed{d\times d} & \\ & \end{bmatrix}}_{b} \end{array} \sim \begin{bmatrix} \boxed{I} & 0 \\ 0 & \underbrace{*}_{} \end{bmatrix}$$

these $(a-d)\times(b-d)$ entries must vanish.

So codim $\leq (a-d)(b-d)$ (Krull). ∎

$\Rightarrow$ We'll generalize to vector bundles.

$E \xrightarrow{\pi} X$ is a rank r vector bundle.

$s_1, \dots, s_k : X \to E$ sections. locally: $\begin{array}{c} r \updownarrow \begin{bmatrix} | & & | \\ \vec{s}_1 & \cdots & \vec{s}_k \\ | & & | \end{bmatrix} \\ \longleftarrow k \longrightarrow \end{array}$.

① If $1 \le k \le r$: the (Chern) degeneracy locus of $\vec{s} = (\vec{s}_1, \dots, \vec{s}_k)$

is: $CD(\vec{s}) = \left\{ x \in X : \vec{s}_1(x), \dots, \vec{s}_k(x) \text{ are lin. dep. in } E_x \right\}$.

⊛ scheme structure: maximal minors.

⊛ If $k = r+1-i$, expected codim is i.

(example: $k=1$: $CD(\vec{s}_1) = \mathbb{V}(\vec{s}_1)$, expected codim r).

② If $k \ge r$: the (Segre) degeneracy locus is

$SD(\vec{s}) = \left\{ x \in X : \vec{s}_1(x), \dots, \vec{s}_k(x) \text{ doesn't span } E_x \right\}$.

⊛ scheme structure: maximal minors.

⊛ if $k = r-1+i$: expected codim is i.

Thm: X = smooth.

① If $k = r+1-i$ and $\operatorname{codim}(CD(\vec{s})) = i$, then the class $[CD(\vec{s})]$ only depends on E and it's called the i^{th} Chern class $c_i(E)$.

② If $k = r-1+i$ and $\operatorname{codim}(SD(\vec{s})) = i$, ...

... $[SD(\vec{s})] =: (-1)^i s_i(E)$, the ith Segre class.

⊛ of course the loci can be larger. Never smaller (by Prop.) (unless ϕ).

(Later we'll see that
$c_i(E^*) = (-1)^i c_i(E)$.
$s_i(E^*) = (-1)^i s_i(E)$.)

Ex: $E = \mathcal{O}(1) \oplus \mathcal{O}(2)$ on $\mathbb{P}^2$. rank 2.

section $s = \begin{bmatrix} L \\ Q \end{bmatrix} = \begin{bmatrix} X+Y \\ Y^2 - XZ \end{bmatrix}$.

One section:

$k=1$.

Codim 2

$$CD(\vec{s}) = \left\{ x \in \mathbb{P}^2 : \vec{s}(x) = \begin{bmatrix} L(x) \\ Q(x) \end{bmatrix} = \vec{0} \right\}.$$

$$\Rightarrow L = Q = 0$$

$$= 2 \text{ points.}$$

$$\Rightarrow \quad c_2(\mathcal{O}(1) \oplus \mathcal{O}(2)) = 2\,[pt].$$

Two sections:

codim 1.

$c_1 = -s_1$.

$$CD(\vec{s}) = \left\{ x \in \mathbb{P}^2 : \det \begin{bmatrix} L_1 & L_2 \\ Q_1 & Q_2 \end{bmatrix} = 0 \right\}.$$

$$= \underbrace{L_1 Q_2 - L_2 Q_1}_{\text{cubic poly. on } \mathbb{P}^2} = 0$$

$$= \{\text{cubic curve}\}.$$

$$\Rightarrow \quad c_1(\mathcal{O}(1) \oplus \mathcal{O}(2)) = 3\,[\text{line}].$$

$$= -s_1(\mathcal{O}(1) \oplus \mathcal{O}(2)).$$

Three sections:

s_2.

codim 2.

$$SD(\vec{s}) = \left\{ x \in \mathbb{P}^2 : \operatorname{rk} \underbrace{\begin{bmatrix} L_1 & L_2 & L_3 \\ Q_1 & Q_2 & Q_3 \end{bmatrix}}_{\text{matrix of } \left[\begin{smallmatrix} \deg 1 \\ \deg 2 \end{smallmatrix}\right] \text{ forms}} \leq 1 \right\}.$$

Exercise. Find a nice choice of matrix.

It's 7 points. ($s_2(E)$).

Rmk: This is also "the locus where a general map $\mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to \begin{matrix} \mathcal{O}(1) \\ \oplus \\ \mathcal{O}(2) \end{matrix}$ drops rank."

($\mathcal{O}_X \xrightarrow{s} E$ same as $s \in H^0(E)$.)

Ex: $X = Gr(2,4)$.

Recall: $0 \to \mathcal{S} \to \underbrace{\mathbb{C}^4 \times X}_{\text{trivial bundle}} \to \mathcal{Q} \to 0$.

Fibers: $0 \to S \to \mathbb{C}^4 \to \mathbb{C}^4/S \to 0$.

$\quad$ rk 2. $\qquad\qquad$ rk 2.

$E = \mathrm{Sym}^2(\mathcal{S}^*)$

$\mathcal{S}^*$: bundle of linear forms on $\mathcal{S}$.
rk 2. $\qquad [0 \to \mathcal{Q}^* \to \mathbb{C}^{4*} \to \mathcal{S}^* \to 0]$.

$\mathrm{Sym}^2(\mathcal{S}^*)$: bundle of quadratic forms on $\mathcal{S}$.
$\hookrightarrow$ rk $= 3$. (locally "s^2, st, t^2".)

Let's make this concrete: say $F = XY - ZW$ on $\mathbb{P}^3$.
$\in H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \cong k^{10}$.

(quadratic functions in X, Y, Z, W) $\quad = \mathrm{Sym}^2(\mathbb{C}^{4*})$.

$\Rightarrow$ F gives a section of the trivial bundle

$$\mathrm{Sym}^2(\mathbb{C}^{4*}) \times Gr.$$

By functoriality/naturality of $\mathrm{Sym}^2(-)$, the map

$$\mathbb{C}^{4*} \times Gr \xrightarrow{res} \mathcal{S}^* \quad \text{(restriction of functionals)}$$

induces

$$\mathrm{Sym}^2(\mathbb{C}^{4*}) \times Gr \xrightarrow{res} \mathrm{Sym}^2(\mathcal{S}^*).$$

$$F = XY - ZW. \longmapsto \text{“} F|_S \text{ for each } S\text{”}$$

(as S varies.)

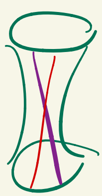
Try this out.

1 section: $\vec{s}$ = "$F|_S$" section. "$XY-ZW|_S$".

$$CD(\vec{s}) = \{ S \in Gr : F|_S \equiv 0 \}.$$

codim 3.

($\mathrm{Sym}^2(\mathcal{S}^*)$ is rank 3.)



$\updownarrow$

$XY-ZW$ vanishes on S.

$\updownarrow$

$\mathbb{P}(S) \subseteq$ quadric $Q = [XY-ZW]$.

= {lines contained in the quadric}.

$\leftarrow$ 2 1-parameter families of lines.

$$c_3(\mathrm{Sym}^2(\mathcal{S}^*)) = 4[X^{\square}].$$ (worked out earlier).

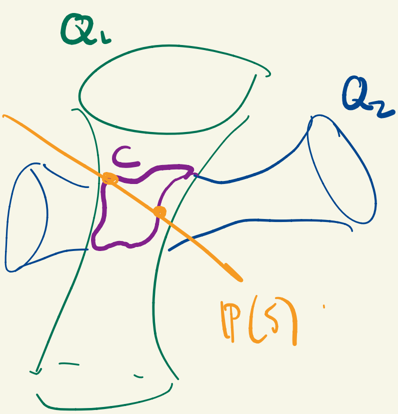
(Day 20)

2 sections: s_1, s_2 quadrics F, G.

Q_1, Q_2

codim 2.

(c_2) $\quad CD(s_1, s_2) = \{ S \in Gr : F|_S, G|_S$ are linearly dependent $\}$.



$\updownarrow$

two quadratic fns on $\mathbb{P}(S) \cong \mathbb{P}^1$.

lin. dep $\Leftrightarrow$ same roots on $\mathbb{P}^1$

($\Leftrightarrow$ proportional).

i.e. $Q_1 \cap \mathbb{P}(S) = Q_2 \cap \mathbb{P}(S)$

$=$ same 2 pts on $\mathbb{P}(S)$.

$= Q_1 \cap Q_2 \cap \mathbb{P}(S)$.

Note: $Q_1 \cap Q_2 =$ some smooth curve C of deg 4, genus 1

$= \{ S : S$ is a chord of $C \}$.

$$c_2(\mathrm{Sym}^2 S^*) = \left(\binom{4-1}{2} - 1 \right) \sigma_2 + \binom{4}{2} \sigma_{1,1}$$

$$= 2\;\square\!\square + 6\;\begin{smallmatrix}\square\\ \square\end{smallmatrix}.$$

3 sections.

codim 1.

c_1

$(= -S_1)$

$$CD(s_1, s_2, s_3) =$$

$$\{ S : F|_S, G|_S, H|_S \text{ lin. dep. on } S \}.$$

Idea: this says $\exists\, a, b, c$

$$\underbrace{aF + bG + cH}_{\text{gives some other quadric } Q' \text{ s.t. } \mathbb{P}(S) \subseteq Q'} \equiv 0 \text{ on } S.$$

gives some <u>other</u> quadric Q' s.t. $\mathbb{P}(S) \subseteq Q'$.

$$= \bigcup_{\substack{\text{quadrics } Q', \\ Q' = \{aF + bG + cH = 0\}}} \{ S : \mathbb{P}(S) \subseteq Q' \}.$$

$\mathbb{P}^2_{a,b,c}$ of choices of Q'

1 divisor for each Q'.

$\Rightarrow$ some locus of dim 3 (codim 1 in $Gr(2,4)$).

⊛ "union" means "projection":

$$\begin{array}{c} Gr(2,4) \times \mathbb{P}^2_{a,b,c} \\ \cup\!| \\ \{(S, Q') : \mathbb{P}(S) \subseteq Q'\} \\ \downarrow \pi_1 \\ CD(s_1, s_2, s_3) \end{array}$$

☆ **Exer**: $c_1(\mathrm{Sym}^2(\mathcal{S}^*)) = 3[X^{\square}]$.

$(= -s_1(\mathrm{Sym}^2(\mathcal{S}^*)).)$

(Intersect with $X^{\boxplus}$.)

4 sections! $(5, 6)$

$SD(s_1, \dots, s_4) = \{S \in Gr :$ the 4 given quadrics don't span $\mathrm{Sym}^2(S^*)\}$.

Segre:	s_2	s_3	s_4
#sections	4	5	6

Delicate geometry (oy vey!)

$s_2(\mathrm{Sym}^2\mathcal{S}^*) = 3[X^{\boxminus}] + 7[X^{\square\square}]$.

$s_3(\qquad) = -10[X^{\boxplus}]$.

$s_4(\qquad) = 10\,[X^{\boxplus}] = 10\,[pt]$.

⊛ I computed these using some tricks!

<u>Question</u>: Let $F_1, F_2, \dots, F_6$ are arbitrary quadratic polys in X, Y, Z, W.

Does there exist a line $L \subseteq \mathbb{P}^3$ s.t. $\{F_i|_L\}$ doesn't give all of the quadratic fns on L?

<u>Yes</u>. ALWAYS. $\exists$ <u>at least</u> 10 such L. ($\uparrow s_4(\mathrm{Sym}^2 \mathcal{S}^*)$.)

<u>Question</u>: Let F be <u>any</u> quadratic poly. in $XYZW$.

Does $\mathbb{V}(F)$ contain a line?

<u>Yes</u> ALWAYS because $c_3(\mathrm{Sym}^2 \mathcal{S}^*) \neq 0$.

Question: How many lines lie on a cubic surface? (in $\mathbb{P}^3$).

$$F \in H^0(\mathcal{O}_{\mathbb{P}^3}(3))$$

$$\begin{array}{c} \uparrow \\ \mathrm{Sym}^3(\mathbb{C}^{4*}) \\ \Big\downarrow \text{ restriction.} \\ \mathrm{Sym}^3(S^*) \end{array}$$

"$F|_S$" $\quad \mathrm{Sym}^3(S^*)$ cubic fns on S.

$\uparrow$

rank 4 (locally s^3, s^2t, st^2, t^3)

$\mathbb{P}(S) \subseteq$ cubic sf.

$\Leftarrow F|_S \equiv 0$.

$\Rightarrow$ What is the vanishing locus of 1 section,

a.k.a. $c_4(\mathrm{Sym}^3 S^*)$

and why is it $27\,[pt]$??

Remark: There's an equivalence

$$\text{vec bdles}/X \xleftrightarrow{\sim} \text{locally free sheaves of } \mathcal{O}_X\text{-modules}.$$

Beware:

good terminology

"injective map of modules + injective on each fiber"

"injective map of vec bdles" $\not\Leftrightarrow$ "injective map of loc. free sheaves."

"generically injective map of vec bdles"

misleading terminology

Ex: $X = \mathbb{P}^2$

$$\mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \xrightarrow{\varphi} \begin{matrix}\mathcal{O}(1) \\ \oplus \\ \mathcal{O}(2)\end{matrix} \quad \text{by} \quad \begin{array}{cc} s_1 & s_2 \end{array} \begin{bmatrix} X & Y \\ YZ & X^2 \end{bmatrix}$$

$$\mathbb{P}^2 \times \mathbb{A}^2 \xrightarrow{\varphi} E \quad \varphi \text{ dominant!}$$

Geometrically: • isomorphism on fibers unless

$\det = X^3 - Y^2 Z = 0$.

(cuspidal cubic).

$\left(\rightsquigarrow c_1(\mathcal{O}(1) \oplus \mathcal{O}(2)) = 3[\text{line}] \right)$

• rank $\underset{\cdot}{0}$ if $\underbrace{X = Y}$ $= YZ = X^2 = 0$.

$[0:0:1] \in \mathbb{P}^2$.

Algebraically: $\mathbb{A}^2_{x,y}$ chart: $R = k[x,y]$.

$$R^2 \xrightarrow{\varphi} R^2 \text{ by } \begin{bmatrix} x & y \\ y & x^2 \end{bmatrix}.$$

• Injective as map of modules.

because $\det \varphi = x^3 - y^2$ NZD on R.

$\Rightarrow$ Localization is exact, so φ is injective over all local rings + @ generic pt $k(x,y)$.

- Look @ fiber $(x,y) = (1,1)$.

i.e. $\otimes \dfrac{k[x,y]}{(x-1,y-1)}$:

$k^2 \longrightarrow k^2$ by $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ which has rank 1.

(kernel is $\mathrm{Tor}_1\left(\dfrac{k[x,y]}{(x-1,y-1)}, \text{coker } \alpha\right)$.)

Look @ fiber $(0,0)$:

$k^2 \to k^2$ by $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (rank 0).

Look @ fiber $(1,0) \notin$ cuspidal cubic:

$k^2 \to k^2$ by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ rank 2.

Day 21: Properties of Chern & Segre classes.

Def: $E \xrightarrow{\pi} X$ vec bdle rank r.

The total Chern class is

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_r(E).$$

Likewise the total Segre class is

$$s(E) = 1 + s_1(E) + s_2(E) + \cdots + s_{\dim(X)}(E).$$

Ex: $E = \mathrm{Sym}^2 \mathcal{S}^*$ on $Gr(2,4)$.

$$s(E) = 1 - \underbrace{3\,\ydiagram{1}}_{s_1} + \underbrace{\left(3\,\ydiagram{1,1} + 7\,\ydiagram{1,1}\right)}_{s_2} - \underbrace{10\,\ydiagram{2,1}}_{s_3} + \underbrace{16\,\ydiagram{2,2}}_{\substack{s_4 \\ (pt)}}.$$

Easy exercise: $c(\mathcal{S}^*) = 1 + \underbrace{\ydiagram{1}}_{c_1} + \underbrace{\ydiagram{1,1}}_{c_2}.$

Theorem (Formal properties of $c(E)$, $s(E)$)

(1) Functoriality: $f: X \to Y$, E = vec bdle on Y.

$$c(f^*E) = f^* c(E).$$
$$s(f^*E) = f^* s(E).$$

(2) "Normalization": For line bundles $\mathcal{L} = \mathcal{O}(D)$,

$$c_1(\mathcal{L}) = [D].$$

(3) Whitney sum formula:

If $0 \to A \to B \to C \to 0$ is a SES of vec bdls,

then $c(A) \cdot c(C) = c(B)$.

$$\left(\sum_{j+k=i} c_j(A)\, c_k(C) = c_i(B) \right)$$

Same for $s(-)$.

Special case: If $E = \bigoplus L_i$ line bundles,

by induction

$$c(E) = \prod_i c(L_i)$$

$$= \prod_i \left(1 + c_1(L_i)\right)$$

$$= 1 + \underbrace{\sum_i c_1(L_i)}_{} + \underbrace{\sum_{i<j} c_1(L_i)\, c_1(L_j)}_{} + \cdots$$

elementary symmetric polynomials in $c_1(L_j)$.

⊛ This holds also when E is filtered with line bundle quotients L_i.

④ For any E, $s(E) = \dfrac{1}{c(E)}$.

Ex: If $E = \bigoplus L_i$.

$$c(E) = \prod \left(1 + c_1(L_i)\right).$$

So, $$s(E) = \prod \frac{1}{1+c_1(L_i)}$$

$$= \prod_i \left(1 - c_1(L_i) + c_1(L_i)^2 - c_1(L_i)^3 + \cdots\right)$$

$$= 1 - \underbrace{\sum c_1(L_i)} + \underbrace{\sum c_1(L_i)^2 + \sum_{i<j} c_1(L_i)c_1(L_j)}$$

ith (complete) homogeneous symmetric polynomials in $c_1(L_j)$.

" $\sum$ all monomials of total degree i ".
in the $c_1(L_j)$'s.

<u>Questions</u>: ① Is $\mathrm{Sym}^2(S^*)$ on $Gr(2,4)$ the pullback of a vec. bdle from a lower-dimensional variety?

<u>Ans</u>: <u>No</u>. Because s_4 would have to vanish.

But $s_4(E) = 10\,[pt]$.

② Does Q contain any subbundles?

i.e. $0 \to \mathcal{L} \to Q \to \mathcal{L}' \to 0$

or $Q = \mathcal{L} \oplus \mathcal{L}'$.

<u>Ans</u>: (Easy exercise): $c(Q) = 1 + \square + \square\square$.

and this <u>can't</u> be factored as

$$c(\mathcal{L})\,c(\mathcal{L}') = (1 + a\,\square)(1 + b\,\square)$$

$$= 1 + (a+b)\,\square + ab\,\underbrace{\square^2}_{=\left(\begin{smallmatrix}\square\\ \square\end{smallmatrix} + \square\square\right)}$$

So, <u>No</u>.

Splitting principle:

(Whitney): "To prove a polynomial relation involving Chern/Segre classes, it suffices to prove it for sums of line bundles."

(* really: vec bdles filtered by l.b.s).

Theorem: $E \xrightarrow{\pi} X$.

There exists a morphism $f: Y \to X$ such that

1. $f^*: A(X) \to A(Y)$ is injective.

2. f^*E is filtered by line bundles.

$$\left.\begin{array}{l} 0 \to E' \to f^*E \to L_1 \to 0 \\ 0 \to E'' \to E' \to L_2 \to 0 \end{array}\right\} \begin{array}{l} c(f^*E) \\ = \prod (1+c_1(L_i)) \end{array}$$

etc.

IDEA: Prove the relation in $A(Y)$.

Then by 1 it holds in $A(X)$!

Ex: $c_i(E^*) = (-1)^i c_i(E)$.

Proof: If $E = \bigoplus L_i$, then $E^* = \bigoplus_i L_i^*$

(also if E is filtered by L_i, E^* filtered by L_i^*.)

Factor $c(E)$, $c(E^*)$ and compare:

$$c(E) = \prod (1 + c_1(L_i)).$$

$$c(E^*) = \prod (1 - c_1(L_i))$$

because $c_1(L^*) = -c_1(L)$ (by normalization property.)

So,

$$c_i(E^*) = (-1)^i \underbrace{\left(i^{\text{th}} \text{ elem. symm. fn in } c_1(L_j)\text{'s} \right)}_{c_i(E)}.$$

By the splitting principle, the relation holds for <u>all</u> E.

Ex 2: $E \to X$ rk 2.

Claim: $c(\mathrm{Sym}^2 E) = 1 + 3c_1(E) + 2c_1(E)^2 + 4c_2(E) + 4c_1(E)c_2(E)$.

(only rk 2)

Proof: If $E = \mathcal{L}_1 \oplus \mathcal{L}_2$,

then $\mathrm{Sym}^2(E) = \mathcal{L}_1^{\otimes 2} \oplus \mathcal{L}_1 \otimes \mathcal{L}_2 \oplus \mathcal{L}_2^{\otimes 2}$

(Similarly, if $0 \to \mathcal{L}_1 \to E \to \mathcal{L}_2 \to 0$,

then $\mathrm{Sym}^2 E$ is filtered by these bundles.

$$0 \to E' \to \mathrm{Sym}^2 E \to \mathrm{Sym}^2 L_2 \to 0 \quad \left(\mathrm{Sym}^2 L_2 = L_2^{\otimes 2}\right)$$

$$0 \to L_1^{\otimes 2} \to E' \to L_1 \otimes L_2 \to 0.$$
)

Compute Chern classes.

Set $\alpha = c_1(\mathcal{L}_1)$, $\beta = c_1(\mathcal{L}_2)$.

$$c(E) = (1+\alpha)(1+\beta) = 1 + \underbrace{(\alpha+\beta)}_{c_1(E)} + \underbrace{\alpha\beta}_{c_2(E)}.$$

$$\mathrm{Sym}^2(E) = \mathcal{L}_1^{\otimes 2} \oplus \mathcal{L}_1 \otimes \mathcal{L}_2 \oplus \mathcal{L}_2^{\otimes 2}$$

$$c(\mathcal{L}_1^{\otimes 2}) = (1 + 2\alpha)$$
$$c(\mathcal{L}_1 \otimes \mathcal{L}_2) = (1 + (\alpha+\beta))$$
$$c(\mathcal{L}_2^{\otimes 2}) = (1 + 2\beta).$$

So $c(\mathrm{Sym}^2 E) = (1+2\alpha)(1+\alpha+\beta)(1+2\beta)$.

⊛ Symmetric in α, β.

$\Rightarrow$ = some poly in $\underbrace{(\alpha+\beta)}_{c_1(E)}$, $\underbrace{\alpha\beta}_{c_2(E)}$.

Expand out:

$$= 1 + 3\underbrace{(\alpha+\beta)}_{c_1(E)} + \underbrace{2(\alpha+\beta)^2 + 4\alpha\beta}_{2c_1(E)^2 + 4c_2(E)} + 4\alpha\beta(\alpha+\beta)$$

$$c_1(E) + 2c_1(E)^2 + 4c_2(E) + 4c_2(E)c_1(E)$$

$\Rightarrow$ By the splitting principle, the relation holds for all E of rank 2.

Lines on a cubic surface.

$E = \mathrm{Sym}^3 S^*$ on $Gr(2,4)$.

We want $c_4(E)$ = vanishing of a section.

$\left(= \{ S \in Gr(2,4) : F|_S \equiv 0 \}.\right)$

Exer: $c(S^*) = 1 + \square + \boxminus$.

pretend this factors as $= (1+\alpha)(1+\beta)$.

$$\alpha + \beta = \square$$
$$\alpha\beta = \boxminus.$$

Factors of $\mathrm{Sym}^3(S^*)$?

(pretend $S^* = L_1 \oplus L_2$)

$$L_1^{\otimes 3} \rightsquigarrow (1+3\alpha)$$
$$L_1^{\otimes 2} \otimes L_2 \rightsquigarrow (1+2\alpha+\beta)$$
$$L_1 \otimes L_2^{\otimes 2} \rightsquigarrow (1+\alpha+2\beta)$$
$$L_2^{\otimes 3} \longrightarrow (1+3\beta)$$

$\Rightarrow$ product is symmetric in α, β.

$\Rightarrow$ quartic term is $c_4 = 3\alpha \cdot (2\alpha+\beta)(\alpha+2\beta)\, 3\beta$

$$= 9\alpha\beta \cdot \left(2(\alpha+\beta)^2 + \alpha\beta\right)$$

$$9 \underbrace{\boxminus}_{c_2(S^*)} \cdot \left(2\, \square^2 + \boxminus\right)$$

$$= 9\,\boxminus \cdot \left(2\,\square^2 + \boxminus\right)$$

$$= 9\,\boxminus\left(3\,\boxminus - 2\,\square\square\right) \qquad \left(\boxminus \cdot \square\square = 0\right).$$

$$= 27\,\underbrace{\boxplus}_{[pt]}.$$

Cor: Every cubic surface contains a line. (because $c_4 \neq 0$.)

Cor: A general cubic surface contains exactly 27 lines.

Day 22: Chern & Segre cont'd.

Remark 1. Other degeneracy loci

$E, F \to X$ vec bdles ranks e, f.

$$\begin{tikzcd} T: E \arrow[r] \arrow[d] & F \arrow[dl] \\ X & \end{tikzcd}$$ map of vec bdles.

For each k, let $D_k(T) := \{ x \in X : \operatorname{rank}(T_x : E_x \to F_x) \leq k \}$.

- $(k+1) \times (k+1)$ minors (locally).
- expected codim $= (e-k)(f-k)$.

Thm: If codim is correct,

$[D_k(T)] =$ a particular (universal) polynomial in $c_i(E), c_j(F)$.

Thom-Porteous formula.

$\Rightarrow$ Recovers Chern/Segre classes!

locus where:

Segre classes: $\mathcal{O}_X^{\oplus k} \to E$ not surjective

Chern classes: $\mathcal{O}_X^{\oplus k} \to E$ not injective.

Remark 2: It's straightforward to show

$$[CD(\vec{s}_i)] = [CD(\vec{s}_i')]$$

by explicitly giving a rational equivalence

$$t\vec{s}_1 + (1-t)\vec{s}_1'$$
$$t\vec{s}_2 + (1-t)\vec{s}_2'$$
$$\vdots$$

⊛ Assuming codimensions are correct.

Problem: What if E has no global sections?

How to even define $c_i(E)$, $s_i(E)$?

Ans: Find def. in terms of natural Chow operations that specializes to the degeneracy locus one if enough sections exist.

First step: (easiest case)

Def: X scheme (dim n), $\mathcal{L}$ line bdle on X.

If s is a rational section of $\mathcal{L}$ ($s \in H^0(\mathcal{L}|_U)$, $U \subseteq X$ open, dense)

(locally $s = \frac{f}{g}$)

It still makes sense to define $\mathrm{div}(s) \in A_{n-1}(X)$ for a rational section.

(zeros - poles).

The <u>first chern class</u> <u>of $\mathcal{L}$</u> is $c_1(\mathcal{L}) := [\mathrm{div}(s)]$ for <u>any</u> rat'l section s.

⊕ Agrees with "vanishing of a section" <u>if</u> $\mathcal{L}$ has global sections.

<u>Even better</u>: We can define homomorphisms

$$A_k(X) \longrightarrow A_{k-1}(X) \quad \text{"}\alpha \mapsto c_1(\mathcal{L}) \cap \alpha\text{"}$$

(Fulton Prop. 2.5)

$$[Z] \longmapsto c_1(\mathcal{L}|_Z)$$

<u>If</u> $\mathcal{L}$ has a global section s s.t. $\mathbb{V}(s) \cap Z$ is transverse, <u>then</u> $c_1(\mathcal{L}|_Z) = [\mathbb{V}(s) \cap Z]$.

$\Rightarrow$ This effectively <u>defines</u> the first case of an intersection product.

Projective Bundles

If $E \xrightarrow{\pi} X$ vector bdle of rank r.

$\mathbb{P}(E) \xrightarrow{\pi} X$ <u>projectivization</u> of E. Fibers are $\mathbb{P}(E_x)$.

Set-theoretically: $\mathbb{P}(E) = \{ (x, L) : x \in X, \ L \underset{\text{line}}{\subseteq} E_x \}$.

Schema theoretically: $\big(E = \underline{\mathrm{Spec}}\ \mathrm{Sym}^{\bullet}(E^*)$

$\mathbb{P}(E) = \underline{\mathrm{Proj}}\ \mathrm{Sym}^{\bullet}(E^*) . \big)$

<u>Ex</u>: $\mathcal{S}$ on $Gr(2,4)$.

$$\mathcal{S} \subseteq \underbrace{Gr \times \mathbb{C}^4}_{\text{trivial bdle.}} \quad \overset{\text{associated projective bundle}}{\rightsquigarrow} \quad \mathbb{P}(\mathcal{S}) \subseteq Gr \times \underset{\updownarrow}{\mathbb{P}(\mathbb{C}^4)} \ (\mathbb{P}^3)$$

$\mathcal{S} = \{ (S, v) : v \in S \}$.

"vector/subspace correspondence" on $\mathbb{C}^4$.

$\mathbb{P}(\mathcal{S}) = \{ (S, \ell) : \ell \underset{\text{line}}{\subseteq} S \subseteq \mathbb{C}^4 \}$

$\updownarrow$

$\{ (\mathbb{P}(S), x) : x \in \mathbb{P}(S) \subseteq \mathbb{P}^3 \}$.

"point-line corresp. on $\mathbb{P}^3$".

On $\mathbb{P}(E)$: A point of $\mathbb{P}(E)$ is a pair $(x \in X,\ \underset{\text{line}}{L} \subseteq E_x)$.

$$\begin{array}{ll} \mathbb{P}(E) & x, L \\ \pi \downarrow & \\ X & x \end{array}$$

So, there is a tautological SES on $\mathbb{P}(E)$

$$0 \to \mathcal{O}_E(-1) \to \pi^* E \to Q_E \to 0.$$

Fiber @ (x, L) is:

$$0 \to L \to E_x \to E_x / L \to 0.$$

Very similar to (on $\mathbb{P}(\mathbb{C}^n)$):

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{\oplus n} \to Q \to 0$$

Fiber at $L \in \mathbb{P}^{n-1}$:

$$0 \to L \to \mathbb{C}^n \to \mathbb{C}^n / L \to 0.$$

Comment: Nothing special about projective bdles!
Many other varieties have natural tautological families.

1 $\left(\mathbb{P}(\mathbb{C}^k)\right)^n \underset{\text{open}}{\supseteq} \mathcal{U} = \{ (L_1, \dots, L_n) : L_1 + \dots + L_n = \mathbb{C}^k \}$.

$n > k$ "spanning line arrangements".

On $\mathcal{U}$ there are n line bundles $\mathcal{L}_1, \dots, \mathcal{L}_n$

$$\mathcal{L}_i = \pi_i^* \mathcal{O}(-1)|_{\mathcal{U}}.$$

and there's a surjection

$$0 \to \underbrace{K}_{} \to \bigoplus_{i=1}^{n} \mathcal{L}_i \to \underbrace{\mathbb{C}^k \times \mathcal{U}}_{\text{trivial bdle.}} \to 0$$

Some "kernel bundle".

$$\left(\bigoplus \mathcal{L}_i \mapsto \sum \mathcal{L}_i = \mathbb{C}^k \right)$$

Since by definition $\mathrm{span}(L_1, \dots, L_n) = \mathbb{C}^k$ on $\mathcal{U}$.

(2) Flag bundles: $E \xrightarrow{\pi} X$ rank r

$Fl(E) \xrightarrow{\pi} X$ flag bundle.

Fibers are flag varieties $Fl(E_x) = \{ \mathcal{F} \in E_x \text{ complete flag} \}$.

$\{ \mathcal{F} : F_1 \subseteq F_2 \subseteq \cdots \subseteq F_r = E_x \}$.

So on $Fl(E)$, $\pi^* E$ has a complete flag of subbundles

$$\underset{\text{line bundle.}}{\mathcal{F}_1} \subseteq \cdots \subseteq \underbrace{\mathcal{F}_{r-2} \subseteq \mathcal{F}_{r-1}} \subseteq \pi^* E$$

$\mathcal{F}_{r-1} / \mathcal{F}_{r-2} = E_{r-1}$ line bdle.

③ C = smooth proper curve genus g.

$\mathrm{Jac}^d(C) = \{ L : L \text{ is a deg-}d \text{ line bdle on } C \}$.

⊛ <u>nontrivial</u>: $\mathrm{Jac}^d(C)$ is representable by a scheme.
(i.e. exists)

On $\mathrm{Jac}^d(C) \times C$, there should be a <u>universal</u> line bundle (called the <u>Poincaré bundle</u>) $\mathcal{L}$ s.t.

@ $(L, x \in C)$: $\mathcal{L}|_{(L,x)} = L_x$

L = deg d l.b.

Day 23: Projective bundles cont'd.

(⊛ no more Fund. Thm of I.Th.)

X = scheme

$E \to X$ vec bdle rank r

$\mathbb{P}(E) \xrightarrow{\pi} X$. flat rel dim $r-1$, proper.

On $\mathbb{P}(E)$:

$$0 \to \mathcal{O}_E(-1) \to \pi^* E \to Q_E \to 0$$

@ (x, L): $\quad 0 \to L \to E_x \to E_x/L \to 0$.

$\Rightarrow \mathcal{O}_E(1)$ controls all the geometry:

(res. to L)

$$0 \to Q_E^* \to \pi^* E^* \to \mathcal{O}_E(1) \to 0$$

Suppose E^* (on X) has a section $\varphi \in H^0(E^*)$.

(So for each x: $\varphi_x : E_x \to \mathbb{C}$.)

- Lift φ to $\pi^*\varphi \in H^0(\pi^* E^*)$
- Map to $\overline{\pi^*\varphi} \in H^0(\mathcal{O}_E(1))$.

Compare: $\mathbb{V}(\varphi) = \{ x \in X : \varphi_x \equiv 0 \} \subseteq X.$

$$\mathbb{V}(\pi^*\varphi) = \{ (x,L) \in \mathbb{P}(E) : \varphi_x \equiv 0 \} \subseteq \mathbb{P}(E)$$

$$= \pi^{-1}\, \mathbb{V}(\varphi).$$

$$\mathbb{V}(\overline{\pi^*\varphi}) = \{ (x,L) : \overline{\varphi_x} \equiv 0 \in L^* \}.$$

$$\Leftrightarrow \overline{\varphi_x} \in L^*$$

$$\Leftrightarrow \varphi_x|_L = 0 \quad \text{where } \varphi_x : E_x \to \mathbb{C}.$$

⊛ codim 1 in $\mathbb{P}(E)$.

$$\Leftrightarrow = \{ (x,L) : L \subseteq \ker \varphi_x \subseteq E_x \} \subseteq \mathbb{P}(E).$$

⊛ twisting hyperplane in each $\mathbb{P}(E_x)$.

Note: this contains the whole fiber if $x \in \mathbb{V}(\varphi)$ since $\varphi_x \equiv 0$ on E_x for those fibers.

Def: For $i \geq 0$, the i^{th} Segre homomorphism

$$A_k(X) \longrightarrow A_{k-i}(X)$$

$$\text{“} \alpha \longmapsto s_i(E) \cap \alpha \text{”}.$$

Recall: “$c_1(\mathcal{L}) \cap \alpha$” $:= c_1(\mathcal{L}|_\alpha)$”

$$s_i(E) \cap \alpha := \pi_* \left(c_1(\mathcal{O}_E(1))^{r-1+i} \cap \pi^*\alpha \right).$$

preserves dimension.

lose $r-1+i$ dims

gains $r-1$ dims. $\in A_{k+r-1}(\mathbb{P}(E))$.

<u>Note</u>: The ith <u>Segre</u> <u>class</u> is $s_i(E) \cap [X]$.

<u>Claim</u>: This agrees with $SD(\varphi^{(1)}, \ldots, \varphi^{(r-1+i)})$ degeneracy locus <u>if</u> the dimension is correct.

<u>Proof</u>: Sound out definition using $\alpha = [X]$.

$$s_i(E) \cap [X] := \pi_*\left(c_1 \mathcal{O}_E(1)^{r-1+i} \cap \underbrace{\pi^*[X]}_{[\mathbb{P}(E)]}\right).$$

By def of "$c_1(\mathcal{L}) \cap \_$", this can be represented by intersecting vanishing loci of sections if they exist (and cut down the dimension as expected.)

$$\pi_*\left(\bigcap_i \mathbb{V}\left(\overline{\pi^*\varphi(i)}\right)\right).$$

(Otherwise we'd need to use <u>rational sections</u> $\rightsquigarrow$ more complicated.)

$$= \pi_X \left\{ (x, L) : L \subseteq \ker \varphi_x^{(1)} \text{ and } \subseteq \ker \varphi_x^{(2)} \text{ etc.} \ldots \right\}$$

$$L \subseteq \bigcap_i \ker \varphi_x^{(i)}$$

$$= \left\{ x \in X : \text{there } \underline{\text{exists}} \ L \text{ in } \bigcap_i \ker \varphi_x^{(i)} \right\}.$$

$\varphi_x^{(1)}, \ldots, \varphi_x^{(r-l+1)}$ don't span E_x^*.

$$= SD\left(\varphi^{(1)}, \ldots, \varphi^{(r-l+i)} \right). \qquad \blacksquare$$

What about the same formula with $c_1 \mathcal{O}_E(1)^k$, $k < r$?

$(r-l+i,$
$i \leq 0)$

Theorem. ① For $i<0$ and all α,

$$\pi_*\left(c_1\mathcal{O}_E(1)^{r-1+i} \cap \pi^*\alpha\right) = 0.$$

② For all α, $(i=0)$.

$$\pi_*\left(c_1\mathcal{O}_E(1)^{r-1} \cap \pi^*\alpha\right) = \alpha.$$

Proof: · Enough to prove for $\alpha = [Z]$, $Z \subseteq X$ subvariety.

· By restricting E, $\mathbb{P}(E)$ to Z, we effectively replace X by Z.

$\Rightarrow$ i.e. assume $X = Z$.

variety dim k.

① $A_k(X) \to A_{k-i}(X)$

larger than $\dim(X)$!

$\Rightarrow \equiv 0$ automatically.

(2) $A_k(X) \to A_k(X)$.

$\cong \mathbb{Z} \cdot [X]$.

We know the formula must give some integer multiple:

$$\pi_*\left(c_1 \mathcal{O}_E(1)^{r-1} \cap \pi^*[X] \right) = n \cdot [X]$$

for some $n \in \mathbb{Z}$.

$\Rightarrow$ We need $n = 1$.

Let $U \xhookrightarrow{i} X$ be any open set.

Restrict whole bundle to U.

$$\begin{tikzcd}
E,\ \mathbb{P}(E) \arrow[d] \arrow[dr, phantom, "\square"] & E|_U,\ \mathbb{P}(E|_U) \arrow[l, hook'] \arrow[d] \\
X & U \arrow[l, hook']
\end{tikzcd}$$

$$\mathcal{O}_E(1)\big|_{\pi^{-1}\mathcal{U}} = \mathcal{O}_{E|_\mathcal{U}}(1), \text{ etc.}$$

$\Rightarrow$ This commutes with the whole formula (left-hand side).

$\Rightarrow$ and, $i^*[X] = [\mathcal{U}]$, so we get the same $n \in \mathbb{Z}$ (right-hand side).

$\Rightarrow$ Take $\mathcal{U}$ = <u>trivializing</u> open set,

$$E|_\mathcal{U} \cong \mathcal{U} \times \mathbb{A}^r$$

$$\mathbb{P}(E)\big|_\mathcal{U} \cong \mathcal{U} \times \mathbb{P}^{r-1}.$$

Now $\underline{c_1 \mathcal{O}(1)^{r-1}} \cap \pi^*[X]$

$\Rightarrow$ intersect $r-1$ hyperplanes in $\mathbb{P}^{r-1}$!

$$= U \times \{pt\}.$$

$$\pi_*(\qquad) = 1 \cdot [U].$$ ∎

Note: Sometimes say "$s_0(E) = 1$" in light of this theorem.

Corollary: $\pi^* : A(X) \to A(\mathbb{P}(E))$ is injective.

Proof: Partial inverse is

$$\alpha \mapsto \pi^*\alpha \mapsto \pi_*\left(c_1\, \mathcal{O}_E(1)^{r-1} \cap \_\right).$$ ∎

__Corollary:__ Let $E \to X$ vec bdle rk r.

There exists $\pi : Y \to X$ proper, flat, such that:

① $\pi^* : A(X) \to A(Y)$ is injective

__and__

② $\pi^* E$ is filtered by line bundles.

__Proof:__ Let $Y = Fl(E)$ flag bundle.

This is a __tower__ of projective bundles:

$$
\begin{array}{ll}
\vdots & \\
\downarrow & \\
\mathbb{P}(Q_E) : & 0 \to \mathcal{L}_2 \to \pi^* Q_E \to Q' \to 0 \\
\pi \downarrow & \\
\mathbb{P}(E) & 0 \to \underbrace{\mathcal{O}_E(-1)}_{\mathcal{L}_1} \to \pi^* E \to Q_E \to 0 \\
\pi \downarrow & \\
X &
\end{array}
$$

__Note:__ On $\mathbb{P}(Q_E)$, $\mathcal{L}_2$ gives a rank 2 subbundle of $\pi^* E$:

$\mathcal{L}_1 \subseteq V_2 \subseteq \pi^* E$

When finished, $\pi^* E$ is completely filtered by line bundles. $\blacksquare$

Day 24: Chow groups of Projective bundles + Vector bundles.

$E \xrightarrow{\pi} X$ vec bdle rk r.

$\mathbb{P}(E) \xrightarrow{\pi} X$ proj bdle, rel dim $r-1$.

Multiple ways to produce classes in $\mathbb{P}(E)$:

$$c_1 \mathcal{O}_E(1)^i \cap \pi^* \alpha \quad : \quad \alpha \in A_j(X)$$

$$\rightsquigarrow A_{j+(r-1)-i}(\mathbb{P}(E)).$$

If $0 \le i \le r-1$, this class represents a "twisting linear space (codim i) over α".

Theorem.

① $\pi^*: A_{k-r}(X) \to A_k(E)$ is an isomorphism for all k.

② $$\Theta_E : \bigoplus_{i=0}^{r-1} A_{k-(r-1)+i}(X) \longrightarrow A_k(\mathbb{P}(E))$$

is an isomorphism $\forall k$.

$$(\oplus \alpha_i) \longmapsto \sum c_1 \mathcal{O}_E(1)^i \cap \pi^* \alpha_i.$$

Rmk: Up to reindexing / shifting, $A(\mathbb{P}(E)) \cong A(X)^{\oplus r-1}$.

Later we'll see $A(\mathbb{P}(E))$ is a free module over $A(X)$.

Proof: 4 things:

1. π^* surjective ✓
2. Θ_E surjective ✓
3. Θ_E injective ✓
4. π^* injective. ✓

(3) Extend the corollary from last class.

Suppose $\Theta_E(\oplus \alpha_i) = 0$.

$$\sum_{i=0}^{r-1} c_1 \mathcal{O}_E(1)^i \cap \pi^* \alpha_i = 0.$$

Apply $c_1 \mathcal{O}_E(1)$ one more time, then π_*:

$$\pi_* \left(\sum_{i=0}^{r-1} \cdots^{i+1} \cap \pi^* \alpha_i \right)$$

↳ All lower terms vanish, left with

$= \alpha_{r-1}$.

So $\alpha_{r-1} = 0$. Rinse and repeat. ✓

---

① $\pi^* : A(X) \to A(E)$ surjective.

Excision: if $Z \subseteq X$ closed,

$\mathcal{U} = X \setminus Z$ open

then $E|_Z \subseteq E$ closed,

$E|_{\mathcal{U}} \subseteq E$ open.

$$\begin{tikzcd}
A(E|_Z) \arrow[r] & A(E) \arrow[r] & A(E|_{\mathcal{U}}) \arrow[r] & 0 \\
A(Z) \arrow[r] \arrow[u, "\pi^*"] & A(X) \arrow[r] \arrow[u, "\pi^*"] & A(\mathcal{U}) \arrow[r] \arrow[u, "\pi^*"] & 0
\end{tikzcd}$$

Diagram chase: if cols 1, 3 surj

$\Longrightarrow$ col 2 surj.

By Noetherian induction, can assume #1 surj.

Enough to show for $\mathcal{U}$.

Take $\mathcal{U}$ to be trivializing,

so: reduce to $E = \mathcal{U} \times \mathbb{A}^r$ = trivial bundle.

By composing $\mathcal{U} \times \mathbb{A}^r \to \mathcal{U} \times \mathbb{A}^{r-1} \to \cdots \to \mathcal{U} \times \mathbb{A}^1$
$$\downarrow$$
$$\mathcal{U}$$

(each $\mathcal{U} \times \mathbb{A}^i \to \mathcal{U} \times \mathbb{A}^{i-1}$ is a rank 1 bundle.)

Enough to show when $r = 1$. (for all varieties X).

$$A_{k-1}(X) \to A_k(X \times \mathbb{A}^1).$$

Let $Z \subseteq X \times \mathbb{A}^1$ subvariety of dim k.

$\pi(Z) \subseteq X$. (dim k or $k-1$).

<ins>case 1</ins>: $\dim \pi(Z) = k-1 < \dim Z$.

Then $Z = \pi^{-1}(\pi(Z))$.

Then $[Z] = \pi^*([\pi(Z)])$.

Case 2. $\dim \pi(Z) = \dim Z = k$.

Work on $\overline{\pi(Z)} \subseteq X$ and forget about X.

$$\underbrace{A_{k-1}\left(\overline{\pi(Z)}\right)}_{} \to \underbrace{A_k\left(\overline{\pi(Z)} \times \mathbb{A}^1\right)}_{}.$$

$$Cl\left(\overline{\pi(Z)}\right) \to Cl\left(\overline{\pi(Z)} \times \mathbb{A}^1\right).$$

Hartshorne: this is an isomorphism.

$\Rightarrow$ surjective. ✓

② θ_E surjective.

Excision: can pass to open set U.

Not necessary (nor useful) to trivialize completely (can't reduce to rank 1 either way since $\nexists\ \mathbb{P}^n \twoheadrightarrow \mathbb{P}^{n-1}$.)

Instead trivialize partly to where

$$E \cong E' \oplus \mathbb{C}.$$

trivial summand.

$\mathbb{P}(E' \oplus \mathbb{C})$ is called the projective completion of E'.

pts are $(x, \langle e, t \rangle)$

$e \in E'_x$, scalar $t \in \mathbb{C}$.

$$Z = \{t = 0\} \underset{\text{closed}}{\subseteq} \mathbb{P}(E' \oplus \mathbb{C}) \cong \mathbb{P}(E').$$

"hyperplane @ ∞ in each fiber".

$$\mathcal{U} = \{t \neq 0\} \underset{\text{open}}{\subseteq} \mathbb{P}(E' \oplus \mathbb{C})$$

$\hookrightarrow (x, \langle e, t \rangle) \rightsquigarrow$ divide by t: $(x, \langle \frac{1}{t} e, 1 \rangle)$

$$[e:t] \rightsquigarrow \left[\tfrac{1}{t}e : 1\right].$$

$$\circlearrowleft \cong E'_x.$$

So, $\mathcal{U} \cong E'$.

(Analogous to $\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{P}^{n-1}$.)

<u>Note:</u> Since Z is a hyperplane in each fiber, $[Z] = c_1(\underbrace{\mathcal{O}_{\mathbb{P}(E'\oplus\mathbb{C})}(1)}_{})$.

(represents <u>any</u> twisting hyperplane.)

Modified excision:

$$\begin{tikzcd}
\overset{\text{(smaller)}}{A_*(\mathbb{P}(E'))} \arrow[r, "i_*"] & \overset{\text{(bigger)}}{A_*(\mathbb{P}(E'\oplus\mathbb{C}))} \arrow[r, "j^*"] & A_*(E') \\
& A_{*-r}(X) \arrow[ul, dashed, "\pi^*"] \arrow[u, "\pi^*"] \arrow[ur, "\pi^*"'] &
\end{tikzcd}$$

α $\quad j^*\alpha = \pi^*\beta_0$ $\quad \beta_0$

Note: $i_*\pi^*\beta = c_1\mathcal{O}(1) \cap \pi^*\beta$

(Portion of $\pi^*\beta$ supported on hyperplane @ ∞.)

Let $\alpha \in A_k(\mathbb{P}(E'\oplus\mathbb{C}))$.

By surjectivity on E',

$j^*\alpha = \pi_E^*\beta_0$ for some $\beta_0 \in A(X)$.

By excision, $\alpha - \pi^* \beta_0$ comes

from $A_k(\mathbb{P}(E'))$.

So,

$$\alpha - \pi^*\beta_0 = i_*\left(\sum_{i=0}^{r-2} c_1 \mathcal{O}_{E'}(1)^i \cap \pi^*\beta_i\right)$$

for some $\beta_i \in A(X)$.

(Inductive hypothesis on $A(\mathbb{P}(E'))$.)

By **Note**, i_* of this has one extra

$c_1 \mathcal{O}(1)$ factor:

$$= \sum_{i=0}^{r-2} c_1(\quad)^{i+1} \cap \pi^*\beta_i .$$

So, $\alpha = \underbrace{\pi^*\beta_0}_{\substack{i=0 \\ \text{term}}} + \underbrace{\sum_{i=1}^{r-1} c_1(\ )^i \cap \pi^*\beta_i}_{\text{terms } i=1,\dots,r-1}.$

(4) π^* injective on $A(X) \to A(E)$.

Embed $E \xhookrightarrow[\text{open}]{} \mathbb{P}(E \oplus \mathbb{C})$:

Diagram chase:

$$\begin{tikzcd} A_k(\mathbb{P}(E)) \arrow[r] & A_k(\mathbb{P}(E\oplus\mathbb{C})) \arrow[r] & A_k(E) \\ & A_{k-r}(X) \arrow[u, "\pi^*"] \arrow[ur] & \end{tikzcd}$$

γ $\quad \pi^*\alpha \quad$ $\pi_E^*\alpha = 0$.

α

If $\pi_E^*\alpha = 0$, $\pi^*\alpha = i_*\gamma$

$\Rightarrow \pi^*\alpha = \sum_{i=0}^{r-1} c_1(\ )^{i+1} \cap \pi^*\beta_i$.

A relation! Contradicts θ_E being injective. ■

Day 25: The top Chern class & our first intersection product!

$E \xrightarrow{\pi} X$ vec bdle rank r

σ zero section.

Note: π is flat,

σ is proper.

Theorem: For all k, $\pi^*: A_{k-r}(X) \to A_k(E)$ is an isomorphism.

$[Z] \mapsto [\pi^{-1} Z]$

$\circlearrowright E|_Z$.

We're going to think of this as a moving lemma for vector bundles:

(☆) Every cycle class $\alpha \in A_k(E)$ is rat'lly equiv. to a sum $\sum_i [\pi^{-1}(Z_i)]$ for various $Z_i \subseteq X$.

Def: The **Gysin map** $A_k(E) \to A_{k-r}(X)$ is

$$s^* := (\pi^*)^{-1}.$$

Fact: ⊛ is not obvious. That theorem took work!

But **some** rat'l equiv's on E **are** obvious.

Ex 1: Let $Z \subseteq E$ subvariety dim k.

Let $t \in k^*$.

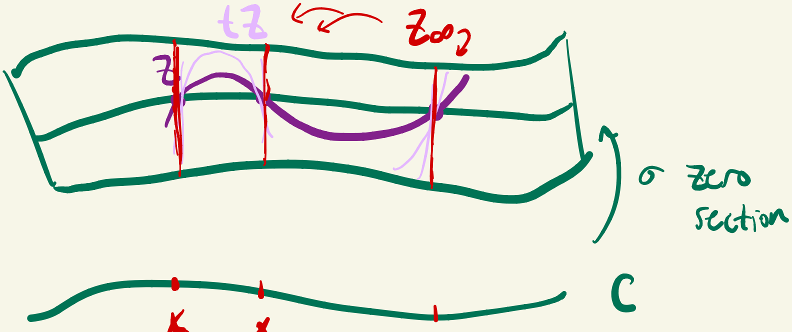
Set $$tZ := \{ (x, t e_x) : (x, e_x) \in Z \}.$$

multiply by a global scalar.

In fact $Z_\infty := \lim_{t \to \infty} tZ$ makes sense and

is rationally equivalent to Z.

E line bdle
$\downarrow \pi$
C



In this example,

$$Z_\infty = \pi^{-1}(x_1) \cup \pi^{-1}(x_2) \cup \pi^{-1}(x_3)$$

So,

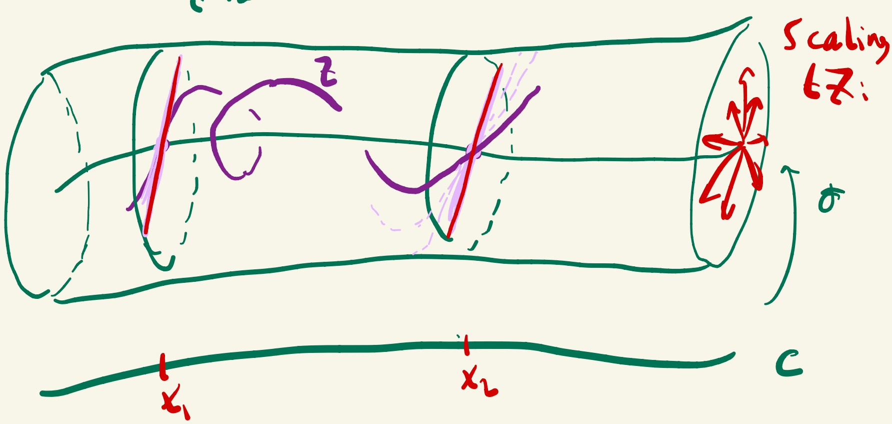
$$s^*[Z] = s^*[Z_\infty] = s^*\left(\pi^{-1}(x_1)\right) + \cdots + s^*\left(\pi^{-1}(x_3)\right)$$

rat'lly equiv.

$$= [x_1] + [x_2] + [x_3].$$

<u>Ex 2</u>: In general $\lim_{t\to\infty} tZ$ is not a union of fibers.

E rk 2
$\downarrow$
C curve



Now Z_∞ = line in $\pi^{-1}(x_1)$ $\cup$ line in $\pi^{-1}(x_2)$.

$$[Z_\infty] \neq \pi^*[x_1] + \pi^*[x_2]$$

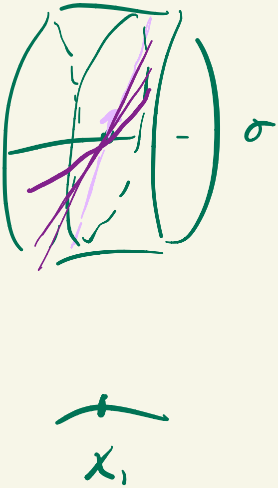
- In fact $s^*[Z] = 0$ since $A_{-1}(C) = 0$.

- If E had a global section $\sigma' : X \to E$, we could use σ' to shift Z:

$$\sigma' + Z := \{ (x, \sigma'(x) + e_x) : (x, e_x) \in Z \}$$

This would probably move Z off the zero section.

Rmk: Which lines did we get in $\pi^{-1}(x_i)$?



Tangent space:

$\mathbb{G}_m \circlearrowright$

$$T_{\sigma(x_1)} E \cong \underbrace{T_{x_1} C}_{\text{"horiz"}} \oplus \underbrace{T_{\sigma(x_1)}(\pi^{-1}(x_1))}_{\substack{\text{"vertical"} \\ \text{(fiber)}}}$$

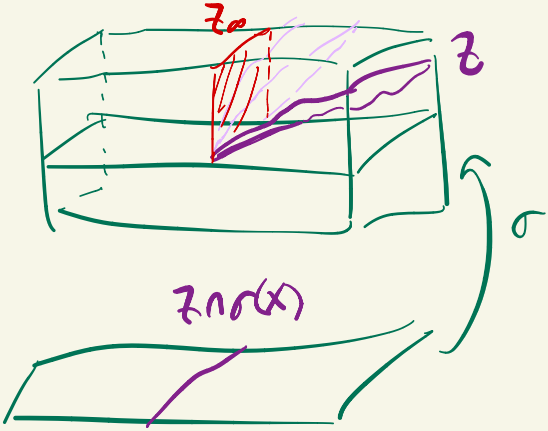
$T_{\sigma(x_1)} Z$ is a line in $T_{\sigma(x)} E$.

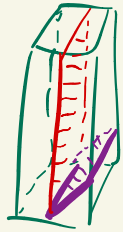
In this example,

$T_{\sigma(x_1)}(Z_\infty)$ is the projection of $T_{\sigma(x_1)}(Z)$ into the fiber.

Ex 3:

E line bundle
$\downarrow$
X surface





$$\Rightarrow T_{\sigma(x)} Z_\infty = \underbrace{T_x\big(Z \cap \sigma(X)\big)}_{X \text{ directions}} \oplus \underbrace{T_{\sigma(x)}\big(\pi^{-1}(x)\big)}_{\text{entire fiber.}}$$

General statement: $\underbrace{S}_{\text{linear subspace}} \subseteq V \oplus W \circlearrowleft \mathbb{G}_m$ (W only).

Then $\lim_{t\to\infty} tS = (S \cap V) \oplus \pi_2(S)$.

(Exer).

split.

So: $$T_{\sigma(x)}(Z_\infty) = \underbrace{\big(T_{\sigma(x)} Z \cap T_{\sigma(x)} \sigma(X)\big)}_{\text{tangent space along } Z \cap \sigma(X)} \oplus \underbrace{\pi_2\big(T_{\sigma(x)} Z\big)}_{\text{"outward" directions in } E.}$$

Recall def:

$A, B \subseteq X$ subvarieties, $x \in A \cap B$.

We say A, B intersect transversely @ x if

A, B, X are smooth at x and $\underbrace{T_x A + T_x B = T_x X}$.

(i.e. $T_x A \cap T_x B$ is as small as possible.)

Proposition: $Z \subseteq E$ subvariety, dim k.

Suppose Z intersects $\sigma(X)$ (zero section) transversely.

Then $s^*[Z] = [Z \cap \sigma(X)]$.

Proof: (1) Replace Z by tZ.

Doesn't change $Z \cap \sigma(X)$, $[Z]$, transversality.

Take limit as $t \to \infty$.

Now $Z_\infty \subseteq \pi^{-1}(Z \cap \sigma(X))$

some homogeneous subvariety.

($\mathbb{G}_m$-invariant)

② We claim $Z_\infty = \pi^{-1}(Z \cap \sigma(X))$.

Examine tangent spaces:

$$T_z(Z_\infty) \subseteq T_z X \oplus T_z(\pi^{-1}(z)).$$

split subspace.

By transversality, $T_z(Z_\infty) + \underbrace{T_z X}_{(T_z \sigma(X))} = T_z E$.

So by split-ness,

$$T_z(Z_\infty) \supseteq T_z(\pi^{-1}(z)).$$

So by $\mathbb{G}_m$ invariance of Z_∞, $Z_\infty \supseteq \pi^{-1}(z)$.

True for all $z \in Z_\infty \cap \sigma(X)$.

So, $Z_\infty = \pi^{-1}(Z \cap \sigma(X))$.

So, by definiton of the Gysin map,

$$s^*[Z] = s^*[Z_\infty] = [Z \cap \sigma(X)]. \quad \blacksquare$$

(*) Our first intersection product!

"Intersection with the zero section"

(*) By abuse of notation, s^* is also sometimes called "σ^*", where $\sigma : \underbrace{X \to E}_{\text{not flat}}$ zero section.

(!) "pullback along the zero section".

<u>Def</u>: The top <u>Chern homomorphism</u>

$$c_r(E) \cap \_\_ : A_k(X) \to A_{k-r}(X)$$

is by definition $\alpha \mapsto \underbrace{s^*}_{} \circ \underbrace{\sigma_*}_{}(\alpha)$

move off, intersect with $\sigma(X)$.

embed inside the zero section.

The top Chern class is $c_r(E) \cap [X]$.
(aka. Euler class)

Prop: If E has a global section $s : X \to E$ such that $\mathbb{V}(s)$ has codim r, then $c_r(E) \cap [X] = [\mathbb{V}(s)]$.

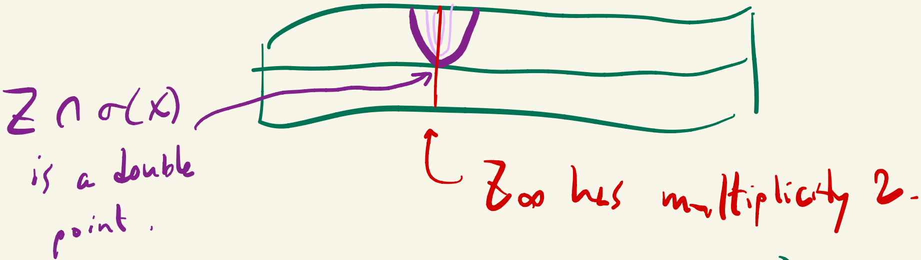
Proof: $\sigma(X)$ is rat'lly equivalent to $s(X)$.

So,

$$
\begin{aligned}
s^* \circ \sigma_* [X] &= s^* [\sigma(X)] \\
&\quad \text{by rat'l equiv.} \\
&= s^* [s(X)] \\
&\quad \text{by transversality} \\
&= [s(X) \cap \sigma(X)] \\
&= [\mathbb{V}(s)].
\end{aligned}
$$

(Extra detail: also true with multiplicity:

$Z_\infty = \lim_{t \to \infty} tZ$ has same multiplicity as $Z \cap \sigma(X)$ does:



$$[Z_\infty] = 2\,[\pi^{-1}(z)].$$)

∎

Day 26

① Other Chern classes $c_i(E)$?

There's a formula like for $s_i(E)$

It reduces to c_r.

$$\begin{array}{ccc} E & & X \times \mathbb{P}^{r-i} \\ \pi\downarrow & & \pi_1 \downarrow \\ X & & X \end{array}$$

} define $\hat{E} = \pi_1^* E \otimes \pi_2^* \mathcal{O}(1)$.

Def: $c_i(E) \cap \_ : A_k(X) \to A_{k-i}(X)$

$$c_i(E) \cap \alpha := \underbrace{\pi_{1*}}_{\text{preserves dims.}} \Big(\underbrace{c_r(\hat{E})}_{\text{lose } r \text{ dims}} \cap \underbrace{\pi_1^* \alpha}_{\text{gain } r-i \text{ dims}} \Big)$$

(HW): If E has appropriate sections $\sigma_1, \dots, \sigma_{r-i+1}$

$$c_i(E) \cap [X] = CD(\sigma_1, \dots, \sigma_{r-i+1})$$

② Formal properties

$$0 \to A \to B \to C \to 0 \quad \text{SES}.$$

$$\left.\begin{array}{l} c(A)\,c(C) = c(B) \\ s(A)\,s(C) = s(B) \\ \text{For any } E,\ s(E) = \dfrac{1}{c(E)}. \end{array}\right\} \text{HW.}$$

Cones & Normal bundles

Def: A *cone* C over a scheme X is Spec of a sheaf of *graded* $\mathcal{O}_X$-algebras.

i.e. $C \xrightarrow{\pi} X$

$\mathbb{G}_m \circlearrowright$ (same as grading).

Ex: ① Affine cone over any projective variety:

$X = \operatorname{Proj} R$ ← graded ring.

$\tilde{X} = \operatorname{Spec} R.$

$\tilde{X}$ (in $\mathbb{A}^{n+1}$) $\qquad$ X (in $\mathbb{P}^n$)

$\circlearrowleft$
$\mathbb{G}_m$

"$\mathbb{G}_m$ invariant affine variety".

(2) vector bundles

$$\begin{array}{c} E \\ \downarrow \\ X \end{array} : \quad E = \underset{\text{over } X}{\underline{\operatorname{Spec}}}\ \underbrace{\operatorname{Sym}^{\bullet}(E^*)}_{\text{symmetric alg.}}.$$

graded!

locally $R[e_1, \dots, e_r]$.

(3) Last class

$$\begin{array}{c} E \\ \downarrow \text{ vec bdle} \\ X \end{array} \supseteq Z \text{ subvariety.} \quad \operatorname{Sym}^{\bullet}(E^*)/\mathcal{J}$$

ideal in $\operatorname{Sym}^{\bullet}(E^*)$.

We defined $Z_\infty := \lim_{t \to \infty} tZ$.

This is a cone!

Given by "lowest degree terms" in f.

(④ Normal cones!)

X = smooth variety.
$\cup$
Z = smooth subvariety.

$$0 \to T_Z \hookrightarrow T_X|_Z \twoheadrightarrow N_{Z/X} \to 0.$$

tangent directions away from Z.

<u>Rmk</u>: $c(T_X|_Z) = c(T_Z) \cdot c(N_{Z/X})$.

Many consequences of this formula!

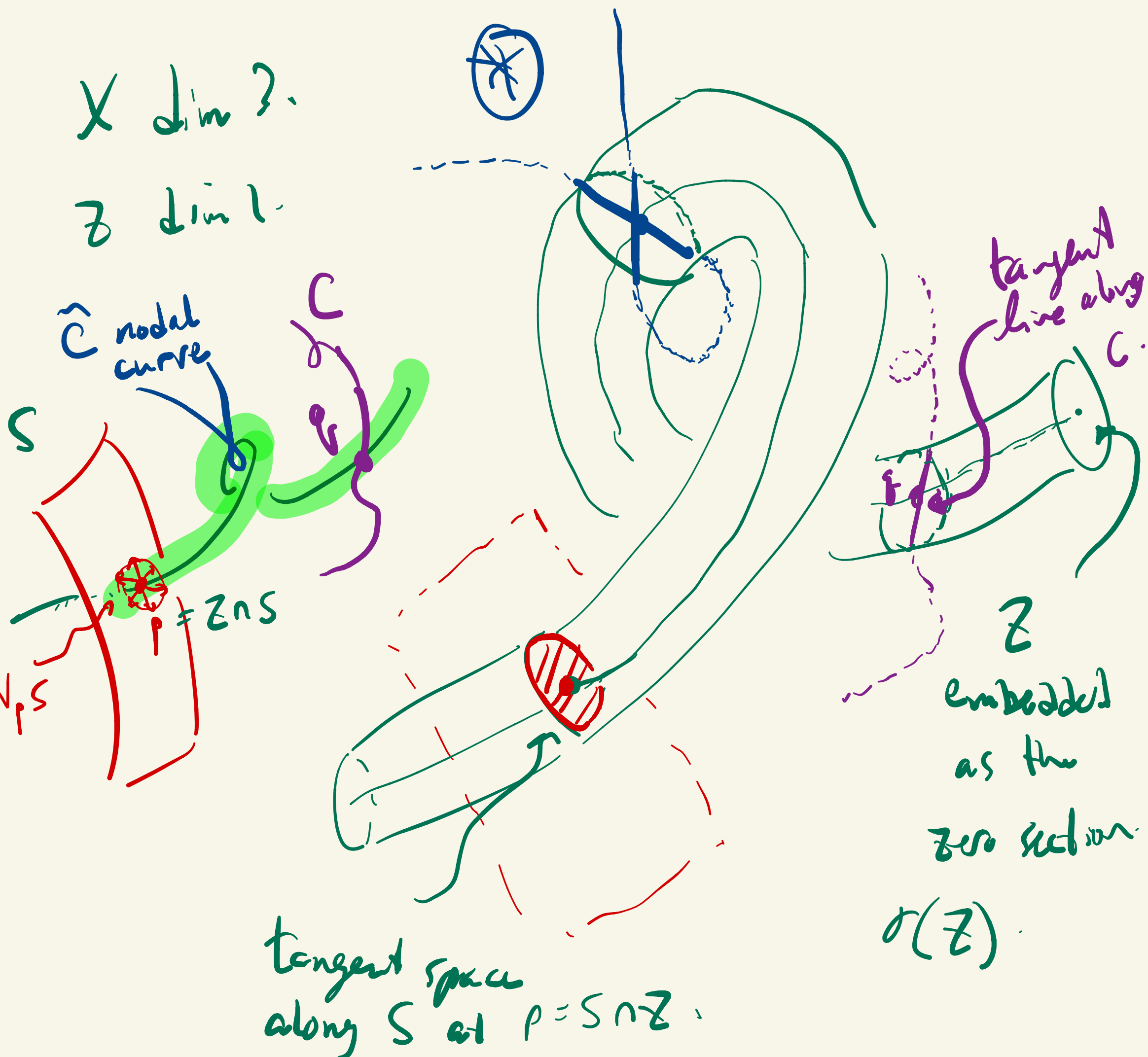
Key idea: $Z \hookrightarrow X$. dim n.

$$\begin{array}{c} N_{Z/X} \\ \pi \downarrow \uparrow \sigma \\ Z \end{array}$$ dim n.

This is like a tubular neighborhood around Z in X.

X dim 3.

Z dim 1.



(*): In this case we get a tangent "cone" showing $\check{C} \cap Z$ to first order.

Def: X scheme.
Z subscheme.

The <u>normal cone</u> <u>to Z in X</u> $C_Z(X)$.

is $\underline{\mathrm{Spec}}\, \mathcal{A}$ where

$$\underline{\mathrm{Spec}}\, \mathcal{A} \downarrow X \qquad \mathcal{A} = \bigoplus_{d \geq 0} I_Z^d / I_Z^{d+1}$$

Fact: If X, Z are smooth,

$$I_Z^d / I_Z^{d+1} \cong \mathrm{Sym}^d\left(I_Z / I_Z^2\right)$$

So $\mathcal{A} = \mathrm{Sym}^\bullet\left(I_Z / I_Z^2\right)$.

I_Z / I_Z^2 is the conormal bundle

$$N^*_{Z/X},$$

"functions on X, to first order near Z".

So $\mathcal{A} = \mathrm{Sym}^\bullet\left(N^*_{Z/X}\right)$

and $C_Z(X) = N_{Z/X}$.

Prop: $X \supseteq Z$ smooth varieties.

$\supseteq W$ arbitrary.

$Z \cap W$ scheme theoretic intersection.

Then the normal cone to $Z \cap W$ in W

embeds in $N_{Z/X}$:

$$\begin{tikzcd} C_{Z\cap W}(W) \arrow[r, hook] \arrow[d] & N_{Z/X} \arrow[d] \\ Z \cap W \arrow[r, hook] & Z \end{tikzcd}$$

Pf: $N_{Z/X} = \underline{\mathrm{Spec}}\ \mathrm{Sym}^{\bullet}(\underbrace{N_{Z/X}^{*}}_{I_Z/I_Z^2})$.

Since $W \hookrightarrow X$, we have

$$\mathcal{O}_X \twoheadrightarrow \mathcal{O}_W.$$

$$\mathcal{I}_Z \longmapsto \underbrace{\mathcal{I}_Z + \mathcal{I}_W \big/ \mathcal{I}_W}_{}.$$

So $\mathcal{I}_Z \twoheadrightarrow \mathcal{I}_{Z\cap W}$

ideal of Z in X. ideal of $Z \cap W$ in W.

$$\underbrace{\bigoplus_{d\geq 0} \mathcal{I}_Z^d \big/ \mathcal{I}_Z^{d+1}}_{= \operatorname{Sym}^\bullet(\mathcal{I}_Z/\mathcal{I}_Z^2)} \twoheadrightarrow \underbrace{\bigoplus_{d\geq 0} \mathcal{I}_{Z\cap W}^d \big/ \mathcal{I}_{Z\cap W}^{d+1}}_{\mathcal{A}}.$$

Gives $N_{Z/X} \hookleftarrow C_{Z\cap W}(W)$. $\blacksquare$

Day 27: Intersection products & Deformation to the normal bundle

Def: $X \supseteq Z$ smooth, codim d.

W arbitrary, pure dim k.

tubular nbhd.

$$\begin{tikzcd} N_{Z/X} \arrow[d, shift left, "\pi"] & N_{Z/X}\big|_{Z\cap W} \arrow[d] & \supseteq C_{Z\cap W}(W) \\ Z \arrow[u, shift left, "\sigma"] & Z\cap W \arrow[l, hook'] & \end{tikzcd}$$

rank d.

normal cone to W.

local picture of W.

local picture of Z = $\sigma(Z)$ (zero section).

Def: The intersection product of Z by W in X

is $(Z \cdot_X W) := \underbrace{s^*}_{\text{Gysin map}} [C_{Z\cap W}(W)]$

Gysin map: perturb, then intersect with $\sigma(Z)$ (zero section).

Key cases:

① W, Z intersect transversely. $T_pZ + T_pW = T_pX$
for all $p \in Z \cap W$.

$\Rightarrow$ In this case $C_{Z\cap W}(W) = N_{Z/X}|_{Z\cap W}$.
fills up the space.

So $s^*[\qquad] = [Z \cap W]$.

(Also: if $Z \cap W$ is empty: get $\underline{0}$.)

② $W = Z$: self-intersection.

☆ $C_{Z\cap W}(W) = Z \xrightarrow{\sigma} N_{Z/X}$

no extra normal directions, just get
the zero section again.

$$(Z \cdot_X Z) = s^*(\sigma(Z))$$
$$= s^* \sigma_* [Z] \overset{\text{def!!}}{=} c_d(N_{Z/X}) \cap [Z].$$

Similarly, if $W \subseteq Z$,

$$(Z \cdot_X W) = s^* \sigma_* [W] = c_d(N_{Z/X}) \cap [W].$$

"perturb W off of Z out into X".

③ W, $Z \cap W$ smooth but not transverse.

Say $Z \cap W \hookrightarrow W$ codim d'.

So, $$C_{Z\cap W}(W) = \underset{\text{rank } d'}{N_{Z\cap W/W}} \hookrightarrow \underset{\text{rank } d}{N_{Z/X}|_{Z\cap W}}.$$

Then $$(Z \cdot_X W) = c_{d-d'}\left(\text{quotient bundle } N_{Z/X}|_{Z\cap W} \big/ N_{Z\cap W/W}\right) \cap [Z \cap W].$$

This is called the <u>excess normal bundle</u>.

(fiber is $\cong \dfrac{T_pX}{T_pZ + T_pW}$.)

(<u>Proof</u>: Multiplicativity of top Chern class.)

④ Our intended uses:

X = smooth, dim n.

A, B = arbitrary cycles, dims a, b.

Then $X \xhookrightarrow{\Delta} X \times X$ codim n, smooth!

$A \times B \hookrightarrow X \times X$ dim $a+b$. (ugly).

Note: $\Delta(X) \cap (A \times B) = A \cap B$ essentially by definition.

So $\Delta(X) \underset{X \times X}{\cdot} (A \times B) \in A_{a+b-n}(A \cap B)$ $\left(\xrightarrow{i_*} A_{a+b-n}(X) \right)$.

Thm This makes $A(X)$ a graded, commutative ring.

Similarly: $X \xrightarrow{f} Y$ of smooth varieties. $d = \dim X - \dim Y$.

$$\begin{array}{c} Y \\ \cup\!\! \mid \\ Z \end{array}$$ subvariety dim k.

Graph of f: $X \xrightarrow{\Gamma} X \times Y$ as $\Gamma(f)$. Smooth!

codim = dim Y.

$$X \times Z \subseteq X \times Y.$$

By defn, $\Gamma(f) \cap (X \times Z) = f^{-1}(Z)$.

$\rightsquigarrow$ $\Gamma(f) \underset{X \times Y}{\cdot} (X \times Z) \in A_{k-d}(f^{-1}(Z)) \left(\xrightarrow{i_*} A_{k-d}(X) \right)$.

For this to work, we need $(Z \underset{X}{\cdot} W)$ to descend to rat'l equivalence in the W argument. (Keep Z fixed.)

<u>Thm</u>: The map

$$\overset{\text{(cycles)}}{Z_k(X)} \to A_{k-d}(Z)$$

$$W \mapsto (Z \underset{X}{\cdot} W)$$

descends to $A_k(X)$.

Think of this in two steps:

$$Z_k(X) \xrightarrow{\text{DNB}} Z_k(N_{Z/X}) \xrightarrow[\text{(Gysin)}]{s^*} A_{k-d}(Z)$$

$$W \longmapsto C_{Z\cap W}(W) \longmapsto s^*[C_{Z\cap W}(W)].$$

$$=: (Z \cdot_X W).$$

⊛

Enough to show that the first step descends to $A_k(X)$, since we know s^* does.

DNB = "Deformation to the Normal Bundle".
(Cone).

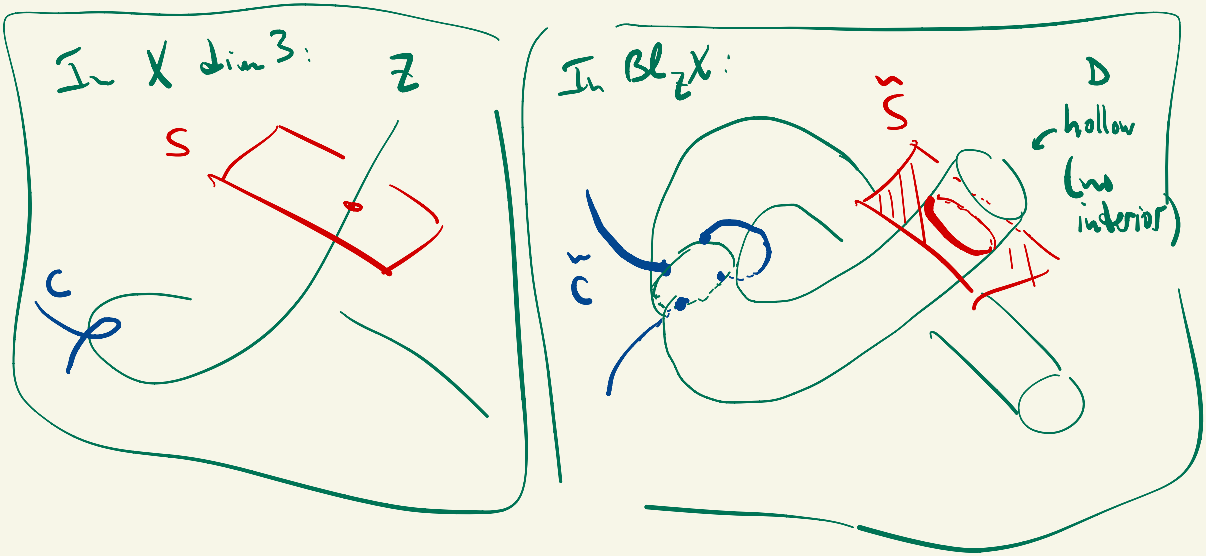
Note on blowups:

$$\begin{array}{ccc} D & \hookrightarrow & \tilde{X} = Bl_Z(X) \\ \downarrow & & \downarrow \\ Z & \hookrightarrow & X \end{array}$$

smooth. (for simplicity)

The exceptional divisor D is the projectivized normal bundle!

⊛ $D = \mathbb{P}(N_{Z/X})$.



This is the exterior view of last class's picture of $N_{Z/X}$ (tubular neighborhood) (interior view).

Deformation to the normal bundle

$$\begin{array}{ccc} Z \times \mathbb{P}^1 & \hookrightarrow & X \times \mathbb{P}^1 \\ & & \cup\!\!\!| \\ & & Z \times \{\infty\}. \end{array}$$

Note: $N_{Z \times \{\infty\}/X \times \mathbb{P}^1} = N_{Z/X} \oplus \underbrace{\mathbb{C}}_{\text{trivial (from } \mathbb{P}^1 \text{ factor)}}$.

$$M = M(Z, X) := Bl_{Z \times \{0\}}(X \times \mathbb{P}^1).$$

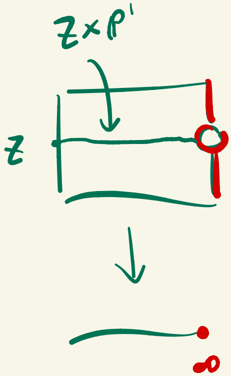
⊛ Exceptional divisor is

$$\mathbb{P}\left(N_{Z\times\{\infty\}/X\times\mathbb{P}^1}\right) = \mathbb{P}\left(N_{Z/X}\oplus\mathbb{C}\right).$$

The projective completion of $N_{Z/X}$.

Note: $\begin{array}{c} M \\ \pi\downarrow \\ \mathbb{P}^1 \end{array}$ is still flat.

(Since M is irreducible).



$\leftarrow$ fiber M_∞ becomes reducible, (but M is irreducible)

$$M_\infty = \underbrace{Bl_Z(X)}_{\text{exterior}} \cup \underbrace{\mathbb{P}(N_{Z/X}\oplus\mathbb{C})}_{\text{interior}}.$$

exc. div.

Z (as zero section of $N_{Z/X}$).

hyperplane at ∞.

joined along $\mathbb{P}(N_{Z/X})$.

<u>Note</u>: By universal property of blowups,

$$\underbrace{Bl_{Z\times\{\infty\}}(Z\times\mathbb{P}^1)}_{} \subseteq Bl_{Z\times\{\infty\}}(X\times\mathbb{P}^1)$$

$$= Z\times\mathbb{P}^1 \xhookrightarrow{\otimes} M.$$

because $Z\times\{\infty\}$ is a Cartier divisor on $Z\times\mathbb{P}^1$.
$(t=\infty)$.

for $t\neq\infty$, this is just $Z\times\{t\} \xhookrightarrow{i} X\times\{t\}$.

for $t=\infty$, this is $Z\times\{\infty\} \xhookrightarrow{\sigma} N_{Z/X} \subseteq M_\infty$.

$\Rightarrow$ $M(Z,X)$ deforms (degenerates) the inclusion $Z \xhookrightarrow{i} X$ into the inclusion $Z \xhookrightarrow{\sigma} N_{Z/X}$ as the zero section.

(Day 28!)

Continuing: $Z \overset{i}{\hookrightarrow} X$.

$$M = M(Z,X) := Bl_{Z\times\{0\}}(X\times\mathbb{P}^1).$$
$$\downarrow$$
$$\mathbb{P}^1$$

$Z\times\mathbb{P}^1 \longrightarrow M$ mostly as $Z\hookrightarrow X$.

$t=\infty$: as $Z \overset{0}{\hookrightarrow} N_{Z/X} \subseteq M_\infty$.

One last lemma needed:

(pullback)
Gysin map for Cartier divisors

$D \overset{i}{\hookrightarrow} X$ Cartier divisor.

We have $c_1\mathcal{O}(D)\cap \_\_ : A_k(X)\to A_{k-1}(X)$.

We need something stronger:

"Gysin map":

$$i^* : A_k(X) \to A_{k-1}(\underline{D}).$$

(Fulton Prop. 2.6): $[W] \mapsto \begin{cases} [W \cap D] & \text{if } W \not\subseteq D \\ c_1(\mathcal{O}(D)|_W) & \text{if } W \subseteq D. \end{cases}$

<u>Main observation about i^*</u>:

If $\mathcal{O}(D)|_D$ is trivial, then the 2nd part of the def is 0.

<u>that is</u>, $i^* i_*[W] = 0$ for $W \subseteq D$.

<u>Ex</u>:

$$\begin{tikzcd} X \arrow[d, "\pi"'] & \supseteq D = \pi^{-1}(p) \arrow[d] \\ \mathbb{P}^1 & \arrow[l, hook'] p \end{tikzcd}$$

so $\mathcal{O}(D) = \pi^* \mathcal{O}(1)$.

$\mathcal{O}(D)|_D = \pi^* \underbrace{\mathcal{O}(1)|_p}_{\substack{\text{vec bdle on a pt!} \\ \Rightarrow \text{trivial.}}}$

<u>Note</u>: $\mathcal{O}_D(D)$ is the normal bdle to D! $(N_{D/X})$

This is saying D has a trivial normal bdle if $D = \pi^{-1}(p)$ from $X \xrightarrow{\pi} C$ curve.

Proposition:

$$Z_k(X) \xrightarrow{\text{DNB}} Z_k(N_{Z/X})$$

$$W \longmapsto C_{Z\cap W}(W)$$

This map descends to $A_k(X) \to A_k(N_{Z/X})$.

Proof: Excision:

$$\begin{tikzcd} M_\infty \arrow[r, hook, "\text{closed}", "(t=\infty)"'] & M(Z,X) & X \times \mathbb{A}^1 \arrow[l, hook', "\text{open}"', "(t\neq\infty)"] \end{tikzcd}$$

where $M_\infty = \underset{\text{exterior}}{Bl_Z(X)} \cup \underset{\text{interior}}{\mathbb{P}(N_{Z/X} \oplus \mathcal{O})}$.

Gives

$$\begin{tikzcd} A_{k+1}(M_\infty) \arrow[r, "i_*"] & A_{k+1}(M(Z,X)) \arrow[r, "j^*"] \arrow[d, "i^*"'] & A_{k+1}(X \times \mathbb{A}^1) \arrow[r] \arrow[dl, dashed] & 0 \\ & A_k(M_\infty) & A_k(X) \arrow[u, "\pi^*"'] \arrow[l, dashed] & \end{tikzcd}$$

(Gysin map for M_∞)

Note: M_∞ is a Cartier divisor : $t=\infty$.

And since $M_\infty = \pi^{-1}(\infty)$ from $M(Z,X) \to \mathbb{P}^1$,

i^* vanishes on $i_*(A_{k+1}(M_\infty))$.

$\Rightarrow$ Gives well-defined map

$$A_{k+1}(X \times \mathbb{A}^1) \longrightarrow A_k(M_\infty)$$

$$S \longmapsto i^*(\overline{S}).$$

Extensive:

$$A_k(X) \xrightarrow{\pi^*} A_{k+1}(X \times \mathbb{A}^1) \to A_k(M_\infty)$$

$$W \longmapsto \pi^{-1}(W) = W \times \mathbb{A}^1 \longmapsto \underbrace{\begin{pmatrix} \text{closure of} \\ W \times \mathbb{A}^1 \end{pmatrix}}_{M(Z \cap W, W)} \cap M_\infty$$

by naturality of blowups

$$\left(Bl_{(Z \cap W) \times \{\infty\}}(W \times \mathbb{P}^1) \subseteq Bl_{Z \times \{\infty\}}(Z \times \mathbb{P}^1) \right)$$

and $M(Z \cap W, W) \cap M_\infty$

$$= \underset{\text{(exterior)}}{Bl_{Z \cap W}(W)} \cup \underset{\text{(interior)}}{\mathbb{P}(C_{Z \cap W}(W) \oplus \mathbb{C})}.$$

Lastly, $N_{Z/X} \subseteq M_\infty$ is an open subset.

(complement of $Bl_Z(X)$).

So, $A_k(M_\infty) \to A_k(N_{Z/X})$.

$M_\infty(Z \cap W, W) \longmapsto C_{Z \cap W}(W)$. ∎

Cor: Existence & functoriality of Chow rings of smooth varieties!

(graded by codim).

Cor (Projective bundle formula).

$E \xrightarrow{\pi} X$ vec bdle.

$\mathbb{P}(E) \to X$ proj bundle.

Ring structure of $A(\mathbb{P}(E))$ is:

- We have $A(X) \xrightarrow{\pi^*} A(\mathbb{P}(E))$.

• As an $A(X)$-algebra,

$$A(\mathbb{P}(E)) \cong \underline{A(X)[\zeta]}$$

$c_1 \mathcal{O}_E(1)$.

$\Rightarrow$ We need this earlier: $c_1 \mathcal{O}_E(1)^i \frown \pi^* \alpha$, etc.

one relation, homog of deg r.

$$\circledast \quad (\zeta^r + c_1(E)\zeta^{r-1} + \cdots + c_{r-1}(E)\zeta + c_r(E)\cdot)$$

Proof: On $\mathbb{P}(E)$, we have

$$0 \to \mathcal{O}_E(-1) \to \pi^* E \to Q_E \to 0.$$

So,
$$\frac{c(\pi^* E)}{c(\mathcal{O}_E(-1))} = c(Q_E).$$

$\pi^* c(E)$

$$= \frac{1 + \pi^* c_1(E) + \cdots + \pi^* c_r(E)}{1 - \zeta} = \underbrace{c(Q_E)}_{\text{rank } r-1 \text{ vec bdl.}}$$

$\Rightarrow$ deg r part of this equation vanishes.

$$= \left(1 + \cdots + \pi^* c_r(E)\right)\left(1 + \zeta + \zeta^2 + \cdots\right) = c(Q_E).$$

$\downarrow$ deg r part:

$$\boxed{\zeta^r + \zeta^{r-1} c_1(E) + \cdots \zeta c_{r-1}(E) + c_r(E) = 0.}$$

By additive description, this gives the correct Chow groups. (No other relations or the groups would be too small.)

Similar idea: Grassmann bundles:

$E \to X$ vec bdle.

$\underline{Gr}(k, E) \xrightarrow{\pi} X$ Grassmann bundle. || Fiber: $Gr(k, E_x)$.

On $\underline{Gr}$, we have

$$0 \to \underbrace{S_E}_{\text{rank } k \text{ subbundle}} \to \pi^* E \to Q_E \to 0.$$

So, $A(\underline{Gr})$ is described by relations from:

$$\circledast \quad \frac{c(\pi^* E)}{c(\mathcal{S}_E)} = \underbrace{c(Q_E)}_{\text{rank } r-k}.$$

$\Vert \Rightarrow$ deg $r-k+1, \dots, r$ parts vanish.

$$A(\underline{Gr}) \simeq \underbrace{A(X)[c_1, \dots, c_k]}_{\text{relations } \circledast}. \qquad c_i(\mathcal{S}_E).$$

Day 29: Excess intersections.

X = smooth variety.
$\cup$
Y_i = subvarieties, regularly embedded.

Our definition of intersection product gives:

$$(Y_1 \underset{X}{\cdot} \cdots \underset{X}{\cdot} Y_r) \in A(\cap Y_i).$$

If $\underset{\text{(open)}}{Z} \subseteq \cap Y_i$ is a connected component. (open)

$\Rightarrow$ By applying $A(\cap Y_i) \xrightarrow{j^*} A(Z)$

we get a class in $A(Z)$, denoted

$$(Y_1 \cdot \cdots \cdot Y_r)^Z,$$

called the <u>equivalence of Z</u> in $(Y_1 \cdot \cdots \cdot Y_r)$.

Thm

$$\underbrace{(Y_1 \underset{X}{\cdot} \cdots \underset{X}{\cdot} Y_r)}_{\text{global product.}} = \sum_{\substack{Z \subseteq \cap Y_i \\ \text{conn. comp.}}} \underbrace{(Y_1 \cdot \cdots \cdot Y_r)^Z}_{\text{"local" contribution from each } Z.}.$$

By default $(Y_1 \cdot \cdots \cdot Y_r)$ is computed via

$$X \xhookrightarrow{\Delta} X^r$$

$$(Y_1 \cdot \cdots \cdot Y_r) = \Delta(X) \underset{X^r}{\cdot} (Y_1 \times \cdots \times Y_r) .$$

If Z is smooth, the equivalence of Z can be computed in $\mathbb{P}(N_{Z/X} \oplus \mathbb{1})$:

(projective completion)

<u>How?</u> multiply the classes

$$\left[\mathbb{P}(C_Z(Y_i) \oplus \mathbb{1})\right] \in A(\mathbb{P}(N \oplus \mathbb{1}))$$

projective completion of $C_Z(Y_i)$ (closure).

$\Rightarrow$ Amounts to "perturbing the Y_i <u>locally</u> at Z, fixing the global picture".

Ex: $\mathbb{P}^3 \supseteq S_1, S_2, S_3$ sm surfaces, degs s_1, s_2, s_3.

Suppose $S_1 \cap S_2 \cap S_3 = \underbrace{C}_{\substack{\text{sm curve} \\ \text{deg } d_1 \\ \text{genus } g}} \cup \underbrace{X}_{\substack{\text{finite} \\ \text{set of} \\ \text{pts.}}}$

What is $\deg(X)$?

Idea:

$$\underbrace{S_1 \cdot S_2 \cdot S_3}_{\substack{s_1 s_2 s_3 \\ (\text{Bézout}, \mathbb{P}^3)}} = \underbrace{(S_1 \cdot S_2 \cdot S_3)^C}_{\substack{\text{Calculate} \\ \text{locally.}}} + \underbrace{\deg(X)}_{\text{rest.}}$$

Since C is smooth, pass to $\mathbb{P}(\underbrace{N_{C/\mathbb{P}^3} \oplus \mathbb{1}}_{\text{rank } 3})$.

Step 1. Total Chern class of

SES: $0 \to T_C \to T_{\mathbb{P}^3}|_C \to N_{C/\mathbb{P}^3} \to 0.$

On a curve, just have c_1, so

$$c_1(N_{C/\mathbb{P}^3} \oplus \cancel{\mathbb{1}}) = c_1(N_{C/\mathbb{P}^3})$$

From $c(N_{C/\mathbb{P}^3}) = \dfrac{c(T_{\mathbb{P}^3}|_C)}{c(T_C)}$,

get $$c_1(N) = \underbrace{c_1(T_{\mathbb{P}^3}|_C)}_{\substack{\det(T_{\mathbb{P}^3}) \\ = \mathcal{O}(4).}} - \underbrace{c_1(T_C)}_{2-2g.}$$

$$= [4d - (2-2g)]\,(\text{pts})$$

(technically a 0-cycle in $A_0(C)$, not just an integer.)

Chow ring:

$$A(\mathbb{P}(N \oplus \mathbb{1})) = \frac{A(C)[\zeta]}{\zeta^3 + \zeta^2 \underbrace{c_1(N \oplus \mathbb{1})}_{4d-2+2g.} + \cancel{\ldots}}$$

($c_2 = 0$ on a curve)

Step 2: For each S_i, since S_i is smooth,

$$C_C(S_i) = N_{C/S_i}.$$

SES of normal bdles:

$$0 \to N_{C/S_i} \to N_{C/\mathbb{P}^3} \to N_{S_i/\mathbb{P}^3}\big|_C \to 0.$$

$$N_{S_i/\mathbb{P}^3}\big|_C \to \mathcal{O}_{\mathbb{P}^3}(S_i)\big|_C.$$

$\mathbb{P}(N_{C/S_i} \oplus \mathbb{1})$ is a divisor in $\mathbb{P}(N_{C/\mathbb{P}^3} \oplus \mathbb{1})$.

Fact: It comes from the line bundle $\mathcal{O}_N(1) \otimes N_{S_i/\mathbb{P}^3}$ on $\mathbb{P}(N_{C/\mathbb{P}^3} \oplus \mathbb{1})$.

Reason: $\mathcal{O}_N(-1) \longrightarrow \pi^* N_{C/\mathbb{P}^3}$

$$0 \to N_{C/S_i} \to N_{C/\mathbb{P}^3} \twoheadrightarrow N_{S_i/\mathbb{P}^3} \to 0$$

Get a map $\mathcal{O}_N(-1) \to N_{S_i/\mathbb{P}^3}$

($\hookrightarrow$ $0 \to \mathcal{O}_N(1) \otimes N_{S_i/\mathbb{P}^3}$ section.)

which vanishes at $\Leftrightarrow$ $\ell = N_{C/S_i}$.

$(x, \ell) \in \mathbb{P}(N \oplus \mathbb{1})$

$\Rightarrow$ N_{C/S_i} cut out by a section of $\mathcal{O}_N(1) \otimes N_{S_i/\mathbb{P}^3}$.

So,

$$[\mathbb{P}(N_{C/S_i} \oplus \mathbb{1})] = c_1\left(\mathcal{O}_N(1) \otimes N_{S_i/\mathbb{P}^3}\right)$$

(*) Local class of S_i surface along C.

$$= c_1(\mathcal{O}_N(1)) + c_1(N_{S_i/\mathbb{P}^3})$$

$$= \underbrace{c_1(\mathcal{O}_N(1))}_{\zeta} + \underbrace{c_1(N_{S_i/\mathbb{P}^3})}_{\mathcal{O}(S_i)|_C}$$

$$= \zeta + d \cdot s_i\, [\pi^* \text{pts}].$$

Step 3: Multiply:

$$\prod_{i=1}^{3} \left(\zeta + d s_i [\pi^* \text{pts}]\right)$$

$$= \zeta^3 + \zeta^2 d (s_1 + s_2 + s_3) \pi^* [\text{pts}]$$

(no higher terms since $[\text{pt}]^2 = 0$.)

$$= \gamma^2 \Big[\underbrace{-(4d-2+2g) + d(s_1+s_2+s_3)} \Big] \pi^*(pts).$$

(<u>Note</u>: $\gamma^2 \cdot \pi^*[pts]$ gives 0-cycles on $\mathbb{P}(N \oplus 1)$.)

<u>Therefore</u> in $\mathbb{P}^3$:

$$\underbrace{s_1 s_2 s_3}_{\text{global product}} = \underbrace{(s_1 \cdot s_2 \cdot s_3)^c}_{\text{just calculated!}} + \underbrace{\deg(X)}_{\text{rest}}.$$

So,

$$\deg(X) = s_1 s_2 s_3 + 4d - 2 + 2g - d(s_1+s_2+s_3).$$

<u>Ex</u>: $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ as twisted cubic. ($d=3$, $g=0$).

Cut out by 3 quadrics.

$$0 = 2\cdot 2\cdot 2 + 4\cdot 3 - 2 + 2\cdot 0 - 3(2+2+2)$$

$$= 8 + 12 - 2 - 18 \quad \checkmark$$

㊟ 3264 & All that: §13.6.3. Variant using $Bl_C(\mathbb{P}^3)$.
(exterior)
instead of $\mathbb{P}(N \oplus \mathbb{1})$.

Conics.

$\mathbb{P}^5$ = conics in $\mathbb{P}^2$.

$\uparrow$ veronese.

$\mathbb{P}^2_{abc}$ = double lines. $(aX+bY+cZ)^2$

$v_2^* \mathcal{O}(1) = \mathcal{O}(2)$.
(hyperplane $\subseteq \mathbb{P}^5$ cuts out conic on $\mathbb{P}^2$.)

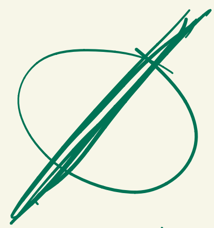
For C a conic, have

$H_C = \{$ conics tangent to $C \}$.

hypersurface of deg 6.

$\supseteq \underbrace{v_2(\mathbb{P}^2)}$ always!

This is the base locus of the H_C's.



$\Rightarrow$ A double line is always "tangent".

Conics tangent to 5 conics: $\bigcap H_{C_i} = v_2(\mathbb{P}^2) \cup \underbrace{X}_{\text{finite}}$.

Step 1: $N_{v_2(\mathbb{P}^2)/\mathbb{P}^5}$ rank 3.

On $\mathbb{P}^n$ have: (Euler sequence)

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}(1)^{\oplus n+1} \to T\mathbb{P}^n \to 0$$

So, $$c(T\mathbb{P}^n) = \frac{c(\mathcal{O}(1)^{\oplus n+1})}{\cancel{c(\mathcal{O})}} = \frac{c(\mathcal{O}(1))^{n+1}}{1}$$

$$= (1+H)^{n+1}$$

Since $\overset{(\mathbb{P}^5)}{H}$ pulls back to $\overset{(\mathbb{P}^2)}{2L}$,

$$0 \to \underbrace{T\mathbb{P}^2}_{c(\,) = (1+L)^3} \longrightarrow \underbrace{T\mathbb{P}^5\big|_{v^2(\mathbb{P}^2)}}_{(1+2L)^6} \to N_{v_2(\mathbb{P}^2)/\mathbb{P}^5} \to 0.$$

Since $v_2^*\mathcal{O}(1) = \mathcal{O}(2)$.

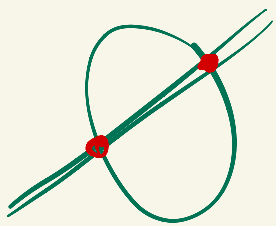
So, $$c(N_{v_2(\mathbb{P}^2)/\mathbb{P}^5}) = \frac{(1+2L)^6}{(1+L)^3} = 1 + 9L + 30\underbrace{L^2}_{\substack{[pt] \\ \text{on } \mathbb{P}^2}}$$

So, the Chow ring is

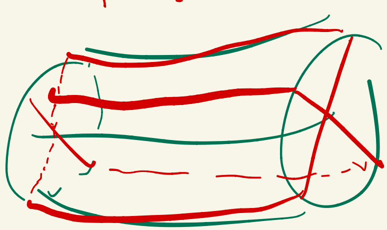
$$A(\mathbb{P}(\underbrace{N \oplus \mathbb{1}}_{\text{rank } 4})) = \frac{A(\mathbb{P}^2)[\zeta]}{(\zeta^4 + \zeta^3 \cdot 9L + \zeta^2 \cdot 30L^2)}$$

Step 2: Examine cones $C_{v_2(\mathbb{P}^2)}(H_c)$.

H_c is singular along $v_2(\mathbb{P}^2)$ because a double line is tangent to a conic twice:



H_c has multiplicity 2 along $v_2(\mathbb{P}^2)$.



Our cone is

$$\left[\mathbb{P}(C_{v_2(\mathbb{P}^2)}(H_c) \oplus \mathbb{1})\right] = a\zeta + b\,\underbrace{\pi^* L}_{\text{from } \mathbb{P}^2}$$

(divisor)

- $a = 2$ = multiplicity.

(J is the relative hyperplane class, H_c looks like 2 hyperplanes locally.)

Relatedly: in $Bl_{v_2(\mathbb{P}^2)}(\mathbb{P}^5)$, $\pi^* H_c$

$$= 2[\text{exc. div}] + [\text{strict transform of } H_c].$$

- For b, apply the Gysin map $s^* = (\pi^*)^{-1}$

$$s^*[\text{cone}] = s^*\left(\cancel{a J} + \underbrace{b\, \pi^* L}_{b \cdot L}\right).$$

$s^*[\text{cone}]$

This is the <u>definition</u> of $v_2(\mathbb{P}^2) \underset{\mathbb{P}^5}{\cdot} H_c$!

$$= [v_2(\mathbb{P}^2)] \cdot \underbrace{[H_c]}_{\deg 6} \text{ in } \mathbb{P}^5$$

$= 12L$ on $\mathbb{P}^2$.

So $\underline{\underline{b = 12}}$.

__Step 3:__ __Multiply:__ $(2\zeta + 12\pi^* L)^5$

$=$ ⋮ (algebra)

(Use defining relations of $A(\mathbb{P}(N \oplus \mathbb{1}))$.)

$= 4512 \underbrace{\zeta^3 \pi^* L^2}_{[pt] \text{ on } \mathbb{P}(N \oplus \mathbb{1})}$.

__Step 4:__ In $\mathbb{P}^5$:

$$\underbrace{(H_{c_1} \cdot \dots \cdot H_{c_5})}_{6^5} = (H_{c_1} \cdot \dots \cdot H_{c_5})^{v_2(\mathbb{P}^2)} + \deg(X).$$

$$7776 = 4512 + \deg(X).$$

$$7776 - 4512 = \underline{\underline{3264 [pt]}} = \deg(X).$$