CLAUDE'S COUNTEREXAMPLE TO THE JACOBIAN CONJECTURE

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This is an expository note to explain the recent counterexample to the Jacobian conjecture, discovered by Claude Fable during the World Cup final, July 19, 2026. The counterexample is given by explicit polynomials, so checking it is an easy (if tedious) calculation. The purpose of this note is to show how to produce the example using only classical geometry of conics and lines, without any calculations in coordinates. The only non-classical fact, which is used as a black box, is that every $\mathbb{A}^1$-bundle over $\mathbb{A}^n$ is trivial.

The Jacobian conjecture is the following. Consider a map $F : \mathbb{A}^n \to \mathbb{A}^n$ given by n polynomials $f_1, \ldots, f_n$ in n variables $x_1, \ldots, x_n$. Let

$$J(f) = \left[\frac{\partial f_i}{\partial x_j}\right]_{1 \le i,j \le n}$$

be its Jacobian matrix of partial derivatives.

Conjecture 0.1 (Jacobian conjecture)**.** *The map F has a polynomial inverse map G : $\mathbb{A}^n \to \mathbb{A}^n$ if and only if* $\det J(f)$ *is a nonzero constant.*

The forward direction is obvious since $F \circ G$ is then the identity function, so $\det(F \circ G) = 1$ and equals the product $\det(F)\det(G)$.

Theorem 0.2 (Claude Fable, 2026)**.** *The Jacobian conjecture is false. In particular, there exists a map $F : \mathbb{A}^3 \to \mathbb{A}^3$ with* $\det J(f)$ *a nonzero constant, but F is not injective.*

In fact, the example found by Claude turns out to be entirely understandable using classical geometry of points, lines and conics in the projective plane. The purpose of this note is to explain how to derive this example 'easily', without any calculations in coordinates, and without using any deep or opaque tools of algebraic geometry.

Acknowledgments. The author claims no originality for anything in this note. The details of the construction below were worked out by first having my own Claude Fable subscription analyze the posted counterexample; Claude quickly noticed that there was an apparent connection to discriminants of cubics. I discussed the example off and on with Claude and with my friend Nic Ford for a couple of days, until I could render the details to my own satisfaction in the coordinate-free way below. I came across a similar discussion on the Secret Blogging Seminar here, also looking to express the locus X as an iterated $\mathbb{A}^1$-bundle, including the citation to [3] for the fact on triviality of such bundles.

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1. Setup for the example: Cubics with a choice of root

Notation. We work on $\mathbb{P}^1$, fixing two points $0, \infty \in \mathbb{P}^1$. Let $\operatorname{Sym}^d \mathbb{P}^1 \cong \mathbb{P}^d$ be the space of multisets of d points on $\mathbb{P}^1$, identified with the projective space $\mathbb{P}^d$ of degree-d polynomials via

$$\{p_1, \ldots, p_d\} \Leftrightarrow \left\langle \prod_{i=1}^{d} (t - p_i) \right\rangle,$$

where the p_i correspond to the roots of the polynomial. A point $p_i = \infty$ corresponds to a lower-degree polynomial, i.e. one with a root at infinity. I will use the notation $\{\cdots\}$ for both sets and multisets when no confusion should arise.

Multiplication of polynomials (or multiset union) induces a map

$$\operatorname{Sym}^d \mathbb{P}^1 \times \operatorname{Sym}^e \mathbb{P}^1 \to \operatorname{Sym}^{d+e} \mathbb{P}^1.$$

As a special case, we consider

$$\begin{aligned} \mu : \mathbb{P}^1 \times \operatorname{Sym}^2 \mathbb{P}^1 &\to \operatorname{Sym}^3 \mathbb{P}^1, \\ (p, \{q, r\}) &\mapsto \{p, q, r\}. \end{aligned} \tag{1.1}$$

We may think of the source as the space of "cubics with a specified choice of root", mapping to the space of cubics. The map is therefore generically 3-to-1, but is branched over the locus of cubics with a double or triple root. The ramification locus in $\mathbb{P}^1 \times \operatorname{Sym}^2 \mathbb{P}^1$ is where the chosen root is the repeated root, i.e. it is

$$L = \{(p, D) : p \in D\} \subset \mathbb{P}^1 \times \operatorname{Sym}^2 \mathbb{P}^1. \tag{1.2}$$

Thus μ is étale, of degree 3, away from L. We note that for each $p \in \mathbb{P}^1$, the fiber

$$L_p = \{D : p \in D\} \subset \operatorname{Sym}^2 \mathbb{P}^1 \tag{1.3}$$

is a line (this justifies the notation L).

We also consider the locus of *depressed cubics*,

$$W = \Big\{\{p, q, r\} : p + q + r = 0\Big\} \subset \operatorname{Sym}^3 \mathbb{P}^1, \tag{1.4}$$

or equivalently, cubics for which the t^2 term of $(t-p)(t-q)(t-r)$ vanishes. This gives a plane in $\operatorname{Sym}^3 \mathbb{P}^1$, since the condition that the t^2 term vanishes is linear in the coefficients of the cubic. (We note that the definition of W depends on our choice of points 0 and ∞.) If $\infty \in \{p, q, r\}$, the condition means ∞ occurs at least twice. We consider its preimage

$$H = \{(p, D) : p + D = 0\} \subset \mathbb{P}^1 \times \operatorname{Sym}^2 \mathbb{P}^1. \tag{1.5}$$

Its fiber is again a line, namely

$$H_p = \{D : p + D = 0\} \subset \operatorname{Sym}^2 \mathbb{P}^1. \tag{1.6}$$

Thus L and H yield, for each p, lines L_p, H_p in $\operatorname{Sym}^2 \mathbb{P}^1$, which are distinct if $p \neq \infty$, while if $p = \infty$, we have $L_\infty = H_\infty$. We let

$$X = (\mathbb{P}^1 \times \operatorname{Sym}^2 \mathbb{P}^1) \setminus (L \cup H). \tag{1.7}$$

Since W is a plane, $\operatorname{Sym}^3 \mathbb{P}^1 \setminus W \cong \mathbb{A}^3$. By construction,

$$\mu|_X : X \to \mathbb{A}^3$$

is étale of degree 3.

Proposition 1.1 (Claude Fable / Anthropic, 2026)**.** *There is an isomorphism $X \cong \mathbb{A}^3$.*

In particular, $\mu|_X$ gives an étale map $\mathbb{A}^3 \to \mathbb{A}^3$ of degree 3, disproving the Jacobian conjecture. (Explicitly, the Jacobian of $\mu|_X$ is a polynomial with no solutions, since the ramification locus is empty. Hence it is a nonzero constant. On the other hand, $\mu|_X$ is not injective since it is generically 3-to-1.)

Remark 1.2. The map μ and the loci above are not new, and it appears that loci similar to X have previously been studied as candidate counterexamples to the Jacobian conjecture, where it can be shown that X has the same invariants as affine space, but might not actually be isomorphic to it (an 'exotic affine space'); see e.g. [1, 2]. This is not my area of specialty in algebraic geometry, so in this writeup I have focused on explaining this example in isolation (which ends up being mostly elementary).

2. Showing $X \cong \mathbb{A}^3$

We will present X as an iterated $\mathbb{A}^1$-bundle, $X \to X^* \to \mathbb{A}^1$. We use as a black box:

Proposition 2.1 (Triviality of $\mathbb{A}^1$-bundles on affine space)**.** *Every $\mathbb{A}^1$-bundle over affine space $\mathbb{A}^n$ is isomorphic to $\mathbb{A}^{n+1}$.*

Proof sketch. This follows from the vanishing of $H^1(\mathbb{A}^n, \mathcal{O}_{\mathbb{A}^n})$ and $\operatorname{Pic}(\mathbb{A}^n)$. See e.g. [3]. □

Remark 2.2. In practice, this cohomological vanishing means that, if we actually did write out explicit equations for these maps, we would be guaranteed to be able to find explicit polynomials exhibiting an isomorphism $X \cong \mathbb{A}^3$, perhaps with some effort. However, the purpose of this note is to avoid calculating anything.

2.1. **Conics and lines in** $\operatorname{Sym}^2 \mathbb{P}^1$**.**

We set up the geometry of lines and conics in $\operatorname{Sym}^2 \mathbb{P}^1$.

First, we have the rational normal conic C, the image of the map

$$\gamma : \mathbb{P}^1 \to \operatorname{Sym}^2 \mathbb{P}^1, \quad \gamma(p) = \{p, p\}.$$

We note that the tangent line to C at $\{p, p\}$ is the line

$$L_p = \{D \in \operatorname{Sym}^2 \mathbb{P}^1 : p \in D\}$$

introduced earlier.

Next, let $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ be the automorphism $\varphi(p) = -2p$. Using φ, we define an auxiliary conic C_φ by

$$s : \mathbb{P}^1 \to \operatorname{Sym}^2 \mathbb{P}^1, \quad s(p) = \{p, \varphi(p)\}.$$

The conics C and C_φ intersect at exactly the two points

$$s(0) = \gamma(0) = \{0, 0\} \text{ and } s(\infty) = \gamma(\infty) = \{\infty, \infty\},$$

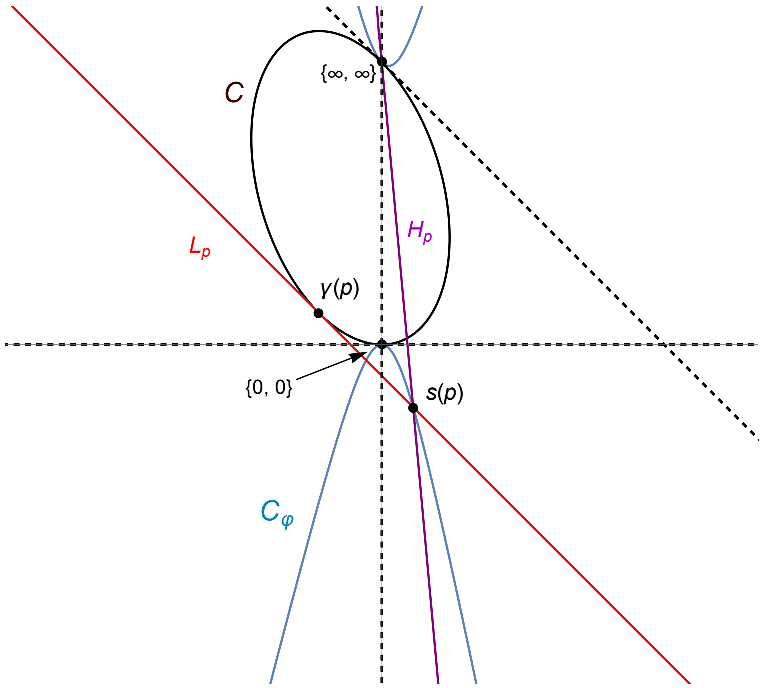

FIGURE 2.1. The two conics C and C_φ and the two lines L_p and H_p in the projective plane of conics $\mathbb{P}^2 \cong \mathrm{Sym}^2 \mathbb{P}^1$. The discriminant curve C (in black) is the set of conics with a double root $\{p, p\}$. The auxiliary curve C_φ (in blue) is the locus of conics whose roots form a pair of the form $s(p) = \{p, -2p\}$. The line L_p (in red) is the tangent line to C at the point $\gamma(p) = \{p, p\}$. The line H_p (in purple) is the line through infinity and $s(p)$; equivalently, the set of conics that become depressed cubics after multiplying by $t - p$. (The equivalence is because $p + p + (-2p) = 0$.) The point $s(p)$ lies on both L_p and H_p; note that L_p intersects C_φ at a second point, namely $s(\varphi^{-1}(p))$.

and they are tangent at those two points. We also note the easy observation

$$s(p) \in L_p \text{ for all } p.$$

We also note for later that the second point of intersection of $L_p \cap C_\varphi$ is $s(\varphi^{-1}(p))$.

In fact, so far these observations are all true for any choice of automorphism $p \mapsto \lambda p$ with $\lambda \neq -1$ (for $\lambda = -1$, the map s doesn't give a conic, but rather a double cover of the line through $\{0, 0\}$ and $\{\infty, \infty\}$). So, why the scalar -2? Because then the relation

$$p + p + \varphi(p) = 0$$

holds for all p, which says that $s(p) \in H_p$. We also always have $\{\infty,\infty\} \in H_p$, so H_p is the line joining $\{\infty,\infty\}$ to $s(p)$. (If $p=\infty$, we just have instead $H_\infty = L_\infty$, which still contains $s(\infty)$).

Thus, geometrically, H is the pencil of lines through $\{\infty,\infty\}$, and H_p is the member of the pencil that passes through $s(p)$ (or, for $p=\infty$, is the common tangent line to C and C_φ at $\{\infty,\infty\}$).

This description is depicted in Figure 2.1 above.

2.2. **First $\mathbb{A}^1$-bundle.** We first consider "linear projection from $s(p)$". We recall the locus

$$X = (\mathbb{P}^1 \times \operatorname{Sym}^2 \mathbb{P}^1) \setminus (L \cup H).$$

Thus, for each p, the fiber X_p is $\operatorname{Sym}^2 \mathbb{P}^1 \setminus (L_p \cup H_p)$.

Let $(\operatorname{Sym}^2 \mathbb{P}^1)^*$ denote the space of lines in $\operatorname{Sym}^2 \mathbb{P}^1$. Let

$$\begin{aligned} \pi : X \subset \mathbb{P}^1 \times \operatorname{Sym}^2 \mathbb{P}^1 &\dashrightarrow \mathbb{P}^1 \times (\operatorname{Sym}^2 \mathbb{P}^1)^*, \\ (p, D) &\mapsto (p, \text{the line } \ell \text{ through } s(p) \text{ and } D). \end{aligned} \tag{2.1}$$

This map is defined as long as $s(p)$ and D are distinct points of $\operatorname{Sym}^2 \mathbb{P}^1$. This always holds on X because then, by definition,

$$D \in \operatorname{Sym}^2 \mathbb{P}^1 \setminus (L_p \cup H_p),$$

whereas $s(p)$ is in fact contained in (both) L_p and H_p. Let $X^* = \pi(X)$. By construction, D never lies on L_p or H_p, so we may describe the image as

$$X^* = \{(p,\ell) : s(p) \in \ell \text{ and } \ell \neq L_p, H_p\}.$$

The condition $s(p) \in \ell$ is closed and makes X^* of dimension 2.

Lemma 2.3. *$\pi : X \to X^*$ is an $\mathbb{A}^1$-bundle.*

Proof. Let $(p,\ell) \in X^*$. Then ℓ is a line through $s(p)$. Then D can be any point along ℓ other than $s(p)$ itself (since $s(p)$ is part of the deleted locus $L_p \cup H_p$). Thus

$$\pi^{-1}(p,\ell) \cong \ell \setminus \{s(p)\} \cong \mathbb{P}^1 \setminus \{\text{a point}\} = \mathbb{A}^1. \qquad \square$$

2.3. **Second $\mathbb{A}^1$-bundle.** The next observation we need is: if $\ell \subset \operatorname{Sym}^2 \mathbb{P}^1$ is a line through $s(p)$, then ℓ intersects C_φ at a unique second point. (If ℓ is the tangent line to C_φ, then the second point is just $s(p)$ again.) Call this point $\rho(p,\ell)$. This gives

$$\begin{aligned} \rho : \{(p,\ell) : s(p) \in \ell\} &\to C_\varphi, \\ (p,\ell) &\mapsto \rho(p,\ell), \end{aligned} \tag{2.2}$$

and for each fixed p, varying ℓ gives an isomorphism from the set of lines ℓ through $s(p)$ to the points of C_φ. To see how deleting the choices $\ell = L_p$ and $\ell = H_p$ affects this map: as mentioned earlier, L_p intersects C_φ at the points $s(p)$ and $s(\varphi^{-1}(p))$, while H_p intersects C_φ at the points $s(p)$ and $\{\infty,\infty\}$. Thus

$$\rho(p, L_p) = s(\varphi^{-1}(p)) \text{ and } \rho(p, H_p) = \{\infty,\infty\}.$$

Equivalently, we see that $\rho^{-1}(\{\infty,\infty\})$ is precisely the set of points of the form (p, H_p), and $\rho^{-1}(s(q))$ always includes the point (p, L_p) where $p = \varphi(q)$.

Lemma 2.4. *ρ maps X^* onto $C_\varphi \setminus \{\infty,\infty\} \cong \mathbb{A}^1$.*

Proof. The condition $\ell \neq H_p$ deletes the points where $\rho(p,\ell) = \{\infty,\infty\}$, making that fiber empty. On the other hand, taking $p = \infty$, we do not delete any other choices of ℓ through $s(\infty)$ (since $L_\infty = H_\infty$), so every other point of $\mathbb{A}^1$ remains in the image. $\square$

Lemma 2.5. *$X^* \to C_\varphi \setminus \{\infty,\infty\}$ is an $\mathbb{A}^1$-bundle.*

Proof. As discussed above, the fiber $\rho^{-1}(\{\infty,\infty\})$ is the set of points (p,ℓ) where $\ell = H_p$. The condition $\ell \neq H_p$ deletes those points and has no effect on the other fibers. The condition $\ell \neq L_p$ instead deletes one point from each other fiber. Fixing $s(q) \in C_\varphi \setminus \{\infty,\infty\}$, the deleted point is (p, L_p) where $p = \varphi(q)$. For every other p, the line $\ell = \ell(s(p), s(q))$ joining $s(p)$ and $s(q)$ is not L_p. Thus

$$\rho^{-1}(s(q)) = \big\{(p, \ell(s(p), s(q))) : p \in \mathbb{P}^1 \setminus \{\varphi(q)\}\big\} \cong \mathbb{P}^1 \setminus \{\varphi(q)\} \cong \mathbb{A}^1. \qquad \square$$

References

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