that we cannot conclude, from our premises, that no parallelogram is a triangle, for there is nothing that forces us to keep the region representing the class of all parallelograms from cutting into the region representing the class of all triangles.

Euler’s diagrammatic device can be used in a great variety of situations, and it is recommended to the person unfamiliar with logical procedure.

We shall not, for the present, go beyond the above superficial study of inductive and deductive reasoning. As already indicated, deductive reasoning has the advantage that its conclusions are unquestionable if the premises are accepted, and it has the additional advantage of considerable economy: before a bridge is built and put into use, deductive reasoning can determine the outcome. But in spite of these particular advantages, deductive reasoning does not supplant the inductive approach; actually, each way of obtaining knowledge has its advantages and disadvantages. The significant thing, from the point of view of our present study, is that the ancient Greeks found in deductive reasoning the vital element of the modern mathematical method.

1.3 Early Greek Mathematics and the Introduction of Deductive Procedures

The origin of early Greek mathematics is clouded by the greatness of Euclid’s *Elements*, written about 300 B.C., because this work so clearly excelled many preceding Greek writings on mathematics that the earlier works were thenceforth discarded. As the great mathematician David Hilbert (1862–1943) once remarked, one can measure the importance of a scientific work by the number of earlier publications rendered superfluous by it.

The debt of Greek mathematics to ancient oriental mathematics is difficult to evaluate, nor has the path of transmission from the one to the other yet been satisfactorily
uncovered. That the debt is considerably greater than formerly believed became evident with twentieth-century researches of Babylonian and Egyptian records. Greek writers themselves expressed respect for the wisdom of the East, and this wisdom was available to anyone who could travel to Egypt and Babylonia. There are also internal evidences of a connection with the East. Early Greek mysticism in mathematics smacks strongly of oriental influence, and some Greek writings, like those of Heron and Diophantus, exhibit a Hellenic perpetuation of the more arithmetic tradition of the orient. Also, there are strong links connecting Greek and Mesopotamian astronomy.

But whatever the strength of the historical connection between Greek and ancient oriental mathematics, the Greeks transformed the subject into something vastly different from the set of empirical conclusions worked out by their predecessors. The Greeks insisted that mathematical facts must be established, not by empirical procedures, but by deductive reasoning; mathematical conclusions must be assured by logical demonstration rather than by laboratory experimentation.

This is not to say that the Greeks shunned preliminary empirical and experimental methods in mathematics, for it is probably quite true that few, if any, significant mathematical facts have ever been found without some preliminary empirical work of one form or another. Before a mathematical statement can be proved or disproved by deduction, it must first be thought of, or conjectured, and a conjecture is nothing but a guess made more or less plausible by intuition, observation, analogy, experimentation, or some other form of empirical procedure. Deduction is a convincing formal mode of exposition, but it is hardly a means of discovery. It is a set of complicated machinery that needs material to work upon, and the material is frequently furnished by empirical considerations. Even the steps of a deductive proof or disproof are not dictated to us by the deductive apparatus itself but must be arrived at by trial and error, experience, and shrewd guessing. Indeed, skill in the art of good guessing is one of the prime ingredients in the make-up of a worthy mathematician. What is important here is that the Greeks insisted that a conjectured or laboratory-obtained mathematical statement must be followed up with a rigorous proof or disproof by deduction and that no amount of verification by experiment is sufficient to establish the statement.

It is difficult to give a wholly adequate explanation of just why the Greeks of 600 to 400 B.C. decided to abandon empirical methods of establishing mathematical knowledge and to insist that all mathematical conclusions be established only by deductive reasoning. This completely new viewpoint on mathematical method is usually explained by the peculiar mental bias of the Greeks of classical times toward philosophical inquiries. In philosophical speculations, reasoning centers about abstract concepts and broad generalizations and is concerned with inevitable conclusions following from assumed premises. Now the empirical method affords no way of discriminating between a valid and an invalid argument and so is hardly applicable to philosophic considerations. It is deductive reasoning that philosophers find to be their indispensable tool, and so the Greeks naturally gave preference to this method when they began to consider mathematics.

Another explanation of the Greek preference for deduction stems from the Hellenic love for beauty. Appreciation of beauty is an intellectual as well as an emotional experience,
and from this point of view the orderliness, consistency, completeness, and conviction found in deductive argument are very satisfying.

A still further explanation for the Greek preference for deductive procedures has been found in the nature of Greek society in classical times. Philosophers, mathematicians, and artists belonged to a social class that in general disdained manual work and practical pursuits, which were carried on by a large slave class. In Greek society the slave class ran the businesses and managed the industries, took care of households, and did both the technical and the unskilled work of the time. This slave basis naturally fostered a separation of theory from practice and led the members of the privileged class to a preference, for deduction and abstraction and a disdain for experimentation and practical application.

It is disappointing that, unlike the situation with ancient Egyptian and Babylonian mathematics, there exist virtually no source materials for contemporary study that throw much light on early Greek mathematics. We are forced to rely on manuscripts and accounts that are dated several hundred years after the original treatments were written. In spite of this difficulty, however, scholars of classicism have been able to build up a rather consistent, though somewhat hypothetical, account of the history of early Greek mathematics and have even plausibly restored many of the original Greek texts. This work required amazing ingenuity and patience; it was carried through by painstaking comparisons of derived texts and by the examination of countless literary fragments and scattered remarks made by later authors, philosophers, and commentators.

Our principal source of information concerning very early Greek mathematics is the so-called Eudemian Summary of Proclus. This summary constitutes a few pages of Proclus’s Commentary on Euclid, Book I, and is a very brief outline of the development of Greek geometry from the earliest times to Euclid. Although Proclus lived in the fifth century A.D., a good thousand years after the inception of Greek mathematics, he still had access to a number of historical and critical works that are now lost to us except for the fragments and allusions preserved by him and others. Among these lost works was apparently a full history of Greek geometry, covering the period before 335 B.C., written by Eudemus, a pupil of Aristotle. The Eudemian Summary is so named because it is based on this earlier work.

According to the Eudemian Summary, Greek mathematics appears to have started in an essential way with the work of Thales of Miletus in the first half of the sixth century B.C. This versatile genius, declared to be one of the “seven wise men” of antiquity, was a worthy founder of systematic mathematics and is the first known individual with whom the use of deductive methods in mathematics is associated. Thales, the summary tells us, sojourned for a time in Egypt and brought back geometry with him to Greece, where he began to apply to the subject the deductive procedures of philosophy. In particular, he is credited with the following elementary geometrical results:

1. A circle is bisected by any diameter.
2. The base angles of an isosceles triangle are equal.
3. Vertical angles formed by two intersecting straight lines are equal.
4. Two triangles are congruent if two angles and a side in one are equal respectively to two angles and the corresponding side of the other. (It is thought that Thales used this
result to determine the distance of a ship from shore.)

5. An angle inscribed in a semicircle is a right angle. (The Babylonians of some 1400 years earlier were acquainted with this geometrical fact.)

We are not to measure the value of these results by their content but rather by the belief that Thales supported them with a certain amount of logical reasoning instead of intuition and experiment. For the first time a student of mathematics was committed to a form of deductive reasoning, crude and incomplete though it may have been. Moreover, the fact that the first deductive thinking was done in the field of geometry instead of algebra, for instance, inaugurated a tradition in mathematics that was maintained, as we shall see, until very recent times.

The next outstanding Greek mathematician mentioned in the *Eudemian Summary* is Pythagoras, who is claimed to have continued the purification of geometry that was begun some fifty years earlier by Thales. Pythagoras was born about 572 B.C. on the island of Samos, one of the Aegean islands near Thales’s home city of Miletus, and it may be that he studied under the older man. It seems that he visited Egypt and perhaps traveled even more extensively about the orient. When, on returning home, he found Samos under the tyranny of Polycrates and Ionia under Persian dominion, he decided to migrate to the Greek seaport of Crotona in southern Italy. Here he founded the celebrated Pythagorean school, a brotherhood knit together with secret and cabalistic rites and observances and committed to the study of philosophy, mathematics, and natural science.

The philosophy of the Pythagorean school was built on the mystical assumption that whole number is the cause of the various qualities of man and matter. This oriental outlook, perhaps acquired by Pythagoras in his eastern travels, led to the exaltation and study of number relations and to a perpetuation of numerological nonsense that has lasted even into modern times. However, in spite of the unscientific nature of much of Pythagorean study, members of the society contributed, during the two hundred or so years following the founding of their organization, a good deal of sound mathematics. They developed the properties of parallel lines and used them to prove that the sum of the angles of any triangle is equal to two right angles. They contributed in a noteworthy manner to Greek geometrical algebra; they effected the geometrical equivalent of addition, subtraction, multiplication, division, extraction of roots, and even the complete solution of the general quadratic equation insofar as it has real roots. They developed a fairly complete theory of proportion, though it was limited only to commensurable magnitudes, and used it to deduce properties of similar figures. They were aware of the existence of at least three of the regular polyhedral solids, and they discovered the incommensurability of a side and a diagonal of a square. Although much of this information was already known to the Babylonians of earlier times, the deductive aspect of mathematics is thought to have been considerably exploited and advanced in this work of the Pythagoreans. Chains of propositions in which successive propositions were derived from earlier ones in the chain began to emerge. As the chains lengthened, and some were tied to others, the bold idea of developing all of geometry in one long chain suggested itself. It is claimed in the *Eudemian Summary* that the Pythagorean, Hippocrates of Chios, was the first to attempt, with at least partial success, a logical presentation of geometry in the form of a single chain of propositions based upon a few initial definitions and assumptions.
The famous Greek philosopher, Plato, was strongly influenced by the Pythagoreans, and Plato, in turn, exerted a considerable influence on the development of mathematics in Greece. Plato’s influence was not due to any mathematical discoveries he made but rather to his enthusiastic conviction that the study of mathematics furnished the finest training field for the mind, and hence was essential in the cultivation of philosophers and those who should govern his ideal state. This belief explains the renowned motto over the door of his Academy, “Let no one unversed in geometry enter here.” Thus, because of its logical element and the pure attitude of mind that he felt its study creates, mathematics seemed of utmost importance to Plato, and for this reason it occupied a valued place in the curriculum of the Academy. Some see in certain of Plato’s dialogues what may perhaps be considered the first serious attempt at a philosophy of mathematics. Certainly mathematics in Greece at the time of Plato had advanced a long way from the empirical mathematics of ancient Egypt and Babylonia.

1.4 Material Axiomatic

Much was accomplished by the Greeks during the three hundred years between Thales in 600 B.C. and Euclid in 300 B.C. Not only did the Pythagoreans and others develop the material that ultimately was organized into the Elements of Euclid, but there were developed notions concerning infinitesimals and summation processes (notions that did not attain final clarification until the rigorization of the calculus in modern times) and also considerable higher geometry (the geometry of curves other than the circle and the straight line and of surfaces other than the sphere and plane). Curiously enough, much of this higher geometry originated in continued attempts to solve the three famous construction problems of antiquity—the duplication of a cube, the trisection of an arbitrary angle, and the quadrature of a circle—illustrating the principle that the growth of mathematics is stimulated by the presence of outstanding unsolved problems.

Also, some time during the first three hundred years of Greek mathematics, there developed the Greek notion of a logical discourse as a sequence of statements obtained by deductive reasoning from an accepted set of initial statements. Certainly, if one is going to present an argument by deductive procedure, any statement of the argument will have to be derived from some previous statement or statements of the argument, and such a previous statement must itself be derived from some still more previous statement or statements. Clearly this cannot be continued backward indefinitely, nor should one resort to illogical circularity by deriving statement q from statement p and then later deriving statement p from statement q. The only way out of the difficulty is to set down, toward the start of the discourse, a collection of fundamental statements whose truths are to be accepted and then to proceed by purely deductive reasoning to derive all the other statements of the discourse. Now both the initial and the derived statements of the discourse are statements about the technical matter of the discourse and hence involve special or technical terms. The meanings of these terms must be made clear to the reader, and so, the Greeks felt, the discourse should start with a list of explanations and definitions of these technical terms. After these explanations and definitions have been given, the initial statements, called axioms and/or postulates of the discourse, are to be listed. These initial statements, according to the viewpoint held by some of the Greeks, should be so carefully chosen that their truths are quite acceptable to the reader in view of
the explanations and definitions already cited.

A discourse that is conducted according to the above plan is described today as a development by *material axiomatics*. Certainly the most outstanding contribution of the early Greeks to mathematics was the formulation of axiomatic procedure and the insistence that mathematics be systematized by such a procedure. Euclid’s *Elements* is the earliest extensively developed example of axiomatic procedure that has come down to us; it largely follows the pattern of material axiomatics, and we shall certainly want to examine it in some detail. In more recent years, the pattern of material axiomatics has been significantly refined to yield a more abstract form of discourse known as *formal axiomatics* (see Chapter 6). For the time being we will content ourselves by summarizing the pattern of material axiomatics.

**Pattern of Material Axiomatics**

1. Initial explanations of certain basic technical terms of the discourse are given, the intention being to suggest to the reader what is to be meant by these basic terms.
2. Certain primary statements that concern the basic terms and that are felt to be acceptable as true on the basis of the properties suggested by the initial explanations are listed. These primary statements are called the *axioms* or *postulates* of the discourse.
3. All other technical terms of the discourse are defined by means of previously introduced terms.
4. All other statements of the discourse are logically deduced from previously accepted or established statements. These derived statements are called the *theorems* of the discourse.

To gain a feeling for the pattern of material axiomatics, let us consider an example. Suppose one is faced with the task of developing a logical discourse on carpentry. The subject of carpentry contains many special or technical terms, such as nail, spike, brad, screw, wood, hard wood, soft wood, board, strut, beam, hammer, saw, screw driver, plane, or chisel. Some of these technical terms can be defined in terms of others. For example, a spike and a brad can each be defined as a special kind of nail; hard wood and soft wood can be defined as certain special kinds of wood; board, strut, and beam can be defined as pieces of wood of certain shapes used for certain purposes; various kinds of hammers and saws can be defined in terms of the basic hammer and saw. It is certainly logical, then, to commence the discourse with some sort of explanation or description of the basic technical terms—say, nail, wood, hammer, saw, and others—and then to define further technical terms, either at the start or as needed, in terms of the basic ones. After giving these initial explanations and possible definitions, the next thing to do is to list some fundamental statements about the explained and defined terms that will be assumed so that the discourse may get under way. Now these assumed statements, from the point of view of material axiomatics, should be such that the reader is perfectly willing to accept them on account of the initial explanations of the basic terms involved. For example, one may wish to assume that *it is always possible to drive a nail with a hammer into a piece of wood, that it is always possible with a saw to cut a piece of wood in two by a planar cut, etc.* That two boards of desired lengths can be fastened together with nails now follows as
a consequence of these assumptions, and is thus a theorem of the discourse. Probably enough has been said to illustrate the Greek notion of material axiomatics.

The theory of some simple games can be rather easily developed by material axiomatics. Consider, for example, the familiar game of tic-tac-toe. Among the technical terms of this game are the *playing board, nought, cross, a win, a draw*, and so on. These technical terms are to be explained or defined. The rules of the game are then stated as the postulates of the discourse, these rules being perfectly acceptable once one understands the basic terms of the discourse. From these rules one can then proceed to deduce the theory of the game, proving as a theorem, for example, that *with sufficiently good playing, the player who starts a game need not lose the game*.

### 1.5 The Origin of the Axiomatic Method

We do not know with whom the axiomatic method originated. By the account given in the *Eudemian Summary*, the method seems to have evolved with the Pythagoreans as a natural outgrowth and refinement of the early application of deductive procedures to mathematics. This is the traditional and customary account and is based principally on Proclus’s summary, which, in turn, is based on the lost history of geometry written by Eudemus about 335 B.C. The account may be the true one, and, if so, we must concede to the Pythagoreans a very high place in the history of the development of mathematics.

There are some historians of ancient mathematics who find the account of the early history of Greek mathematics, as reconstructed from the *Eudemian Summary*, somewhat difficult to believe and who feel that the traditional stories about Thales and Pythagoras must be discredited as purely legendary and unhistorical in content. For example, the *Eudemian Summary* says that Thales proved that a circle is bisected by any one of its diameters. The realization that so obvious a matter as this should need demonstration seems to reflect a mathematical sophistication of a much more advanced period, when the importance and delicacy of initial assumptions had become much clearer. Eudemus may have hypothetically restored the sequence of events so that they accorded with the state of the theory of his time, as many historians do when source material is not available. Actually, we can have very little idea of the roles played in the history of mathematics by Thales and Pythagoras, and it may be much closer to reality to assume that early Greek mathematics cannot have differed greatly from the oriental type. An essential turn in the development of a subject is usually brought about by some crucial circumstance, and in mathematics such a circumstance arose some time in the fifth century B.C. with the devastating discovery of the irrationality of $\sqrt{2}$.

Let us pause a moment to consider the significance of the last statement. Since the rational numbers consist of all numbers of the form $p/q$, where $p$ and $q$ are integers with $q \neq 0$, the discovery alluded to states that there are no integers $p$ and $q$ such that $p/q = \sqrt{2}$; that is, $\sqrt{2}$ is not a rational number and hence, by definition, is an *irrational* (nonrational) number. The traditional proof of this fact, apparently known to Aristotle (384–322 B.C.), is simple and runs as follows: Suppose, on the contrary, that there are two integers $p$ and $q$ such that $p/q = \sqrt{2}$, where, without any loss of generality, we may assume that $p$ and $q$ have no common positive integral factor other than unity. Then $p^2 = 2q^2$. Since $p^2$ is twice an integer, we see that $p^2$, and hence $p$, must be even. So we may put $p = 2r$. Then we find
4r^2 = 2q^2, or 2r^2 = q^2, from which we conclude that q^2, and hence q, must be even. But this is impossible, since we assumed that p and q have no common integral factor different from unity. The supposition that \(\sqrt{2}\) is rational has led to a contradictory situation, whence it follows that \(\sqrt{2}\) must be irrational. This result was surprising and disturbing on several grounds. First of all, it seemed to deal a mortal blow to the Pythagorean philosophy that all depends on the integers. Next, it seemed contrary to common sense, for it was felt intuitively that any magnitude could be expressed by some rational number. The geometrical counterpart was equally startling, for who could doubt that for any two given line segments one is able to find some third line segment, perhaps very very small, that can be marked off a whole number of times into each of the given segments. But take as the two given segments a side s and a diagonal d of a square. Now if there exists a third segment t which can be marked off a whole number of times into s and d we would have s = qt and d = pt, where p and q are integers. But d = s\sqrt{2}, whence pt = qt\sqrt{2}, or \(\sqrt{2} = p/q\), a rational number. Contrary to intuition, then, there exist line segments having no common unit of measure. But the whole Pythagorean theory of proportion was built on the seemingly obvious assumption that any two line segments are commensurable, that is, do have some common unit of measure.

No wonder the discovery of the irrationality of \(\sqrt{2}\) led to some consternation in the Pythagorean ranks. The situation must have caused a profound reaction in mathematical thinking, and must have very considerably emphasized the extreme importance of careful agreement on what can be taken for basic assumptions. A crisis, like this one of the discovery of irrational numbers, could well account for the origin of the axiomatic method, and, if so, the credit for the invention might largely go to Eudoxus, the genius of the time who finally resolved the crisis that had arisen. 6

This second explanation of the possible origin of the axiomatic method has other points in its favor. For example, it places less stress on any peculiar mentality possessed by the Greeks of very early times, and it accounts for the relatively large number of Greek papyrus fragments containing texts after the pattern of oriental mathematics. These texts, like the similar ones from Babylonian times, probably formed the backbone of instruction in elementary mathematics. At this elementary level the highly sophisticated axiomatic method had as little influence as it has today in much of our elementary teaching. Writings of this sort, then, do not reflect any degeneration of the so-called Greek spirit in mathematics but simply exhibit the continuance, on an elementary level, of older traditions. Heron’s geometry, for example, can be properly considered a Hellenic form of oriental tradition; it should not be regarded as a sign of decline in Greek mathematics just because it does not employ the refined procedures of the axiomatic method.

Perhaps it is needless to hypothesize about the origin of the axiomatic method. Certainly, by the middle of the fourth century B.C., the method had been fairly well developed, for in Aristotle’s Analytica posteriora, we find a good deal of light thrown on some of its features. Aristotle was not a mathematician, but as the systematizer of classical logic, he found in elementary mathematics excellent models of logical reasoning, and his mathematical illustrations tell us a great deal about the principles of the axiomatic method as accepted in his time. By the turn of the century the stage was set for Euclid’s magnificent and epoch-making application of the axiomatic method.