The distribution of the maximum of character sums

Dimitris Koukoulopoulos
(joint work with J. Bober, L. Goldmakher and A. Granville)

Université de Montréal

Analytic Number Theory, a conference to honor É. Fouvry
Centre International de Rencontres Mathématiques
June 18, 2013
Background & motivation

Let $\chi$ be a Dirichlet character modulo $q$ and define

$$M(\chi) = \max_{1 \leq z \leq q} \left| \sum_{n \leq z} \chi(n) \right|.$$

If $\chi$ is non-principal, then Pólya and Vinogradov showed in 1918 that

$$M(\chi) \ll \sqrt{q} \log q.$$

Assuming GRH, Montgomery and Vaughan improved this in 1977 to

$$M(\chi) \ll \sqrt{q} \log \log q.$$

This is best possible: Paley had already shown in 1932 that

there is a sequence $q_n \to \infty$ such that $M \left( \left( \frac{q_n}{.} \right) \right) \gg \sqrt{q_n} \log \log q_n$.

However, such extremal examples should be rather rare. Our goal is to study how rare they are.
The distribution of $M(\chi)$: random models

We shall study

$$P_q(\tau) := \frac{\# \{ \chi \pmod{q} : M(\chi) > \frac{e^\gamma}{\pi} \tau \sqrt{q} \}}{\phi(q)} = \text{Prob} \left( M(\chi) > \frac{e^\gamma}{\pi} \tau \sqrt{q} \right).$$

We always assume for simplicity that $q$ is prime.

Two questions:

1. Is there a random model that describes $P_q(\tau)$ accurately?
2. How big is $P_q(\tau)$?

Let $(X_p)_{p\mid q}$ be a sequence of independent random variables, uniformly distributed on $\{ z \in \mathbb{C} : |z| = 1 \}$, and $X_p = 0$ if $p \mid q$.
(They should model $\chi(p)$ as $\chi$ runs through characters modulo $q$.)

Then we define $X_n = \prod_{p^r \mid n} X_p^r$, which serves a model for $\chi(n)$.

**First attempt:** model $\sum_{n \leq z} \chi(n)$ by $\sum_{n \leq z} X_n$.

For $z$ large compared to $q$, this will fail: periodicity is not taken into account.
The distribution of $M(\chi)$: random models, continued

$$P_q(\tau) = \text{Prob} \left( M(\chi) > \frac{e^{\gamma}}{\pi} \tau \sqrt{q} \right).$$

$(X_p)_{p|q}$ sequence of independent random variables, uniformly distributed on $\{z \in \mathbb{C} : |z| = 1\}$, $X_p = 0$ if $p|q$, $X_n = \prod_{p|r|n} X_p$.

**Second attempt:** use Pólya’s expansion ($\chi$ primitive, $e(x) = e^{2\pi i x}$):

$$\sum_{n \leq z} \chi(n) = \frac{\tau(\chi)}{2\pi i} \sum_{1 \leq |n| \leq w} \frac{\overline{\chi}(n)(1 - e(-nz/q))}{n} + O \left( \frac{q \log q}{w} \right) \quad (1 \leq w \leq q).$$

Our model for $\sum_{n \leq z} \chi(n)$ then becomes

$$S(z) := \frac{\tau(\chi)}{2\pi i} \sum_{1 \leq |n| \leq q} \frac{\overline{X}_n \cdot (1 - e(-nz/q))}{n}.$$

This model captures the periodicity of $\chi$.

**Remark.** The standard deviation of $S(z)/\sqrt{q}$ is $\ll 1$. (Compare this to 1st model: the SD of $T(z) = \sum_{n \leq z} X_n$ is $\sqrt{z}$, and one might expect $T(z)/\sqrt{z}$ to get large relatively often.) As a result, $P_q(\tau)$ will be rather small.
Known results on $P_q(\tau)$

In 1979, Montgomery and Vaughan showed that

$$\frac{1}{\phi(q)} \sum_{\chi \pmod{q}} M(\chi)^{2k} \ll_k q^k.$$  

An immediate corollary is that

$$P_q(\tau) = \text{Prob} \left( M(\chi) > \frac{e^\gamma}{\pi} \tau \sqrt{q} \right) \ll A \frac{1}{\tau^A}.$$  

In 2011, Bober-Goldmakher proved that, for fixed $\tau$ and $q \to \infty$ over primes,

$$\exp \left\{ -\frac{Ce^\tau}{\tau} (1 + o_{\tau \to \infty}(1)) \right\} \leq P_q(\tau) \leq \exp \left\{ -e^{B\sqrt{\tau}/(\log \tau)^{1/4}} \right\},$$

where $C = 1.09258 \ldots$ This supports the claim that $P_q(\tau)$ is very small.

Question: why do the tails of the distribution of $M(\chi)$ have this double exponential decay?
The distribution of $M(\chi)$ vs the distribution of $L(1, \chi)$

$$\sum_{n \leq \alpha q} \chi(n) = \frac{\tau(\chi)}{2\pi i} \sum_{1 \leq |n| \leq q} \overline{\chi}(n) \frac{(1 - e(-n\alpha))}{n} + O(\log q).$$

In view of Pólya’s expansion, one might conjecture that

$$P_q(\tau) := \text{Prob} \left( M(\chi) > \frac{e^\gamma}{\pi} \tau \sqrt{q} \right) \approx \text{Prob} \left( |L(1, \chi)| > c\tau \right).$$

Granville-Soundararjan: for $q$ prime and $e^\tau = o(\log q)$,

$$\text{Prob} \left( |L(1, \chi)| > e^\gamma \tau \right) = \exp \left\{ -\frac{Ce^\tau}{\tau} (1 + o_{\tau \to \infty}(1)) \right\}.$$

Compare this to

$$\exp \left\{ -\frac{Ce^\tau}{\tau} (1 + o_{\tau \to \infty}(1)) \right\} \leq P_q(\tau) \leq \exp \left\{ -e^{B\sqrt{\tau}/(\log \tau)^{1/4}} \right\}.$$

The distribution of $L(1, \chi)$: main ideas

1. We shall take moments of $L(1, \chi)$, so we need to ‘shorten’ it. We have that $\log L(1, \chi) = \sum_p \chi(p)/p + C_\chi$, where $C_\chi$ is a constant.

PNT $\Rightarrow \log L(1, \chi) \sim \sum_{p \leq e^{q^\epsilon}} \chi(p)/p + C_\chi$,

GRH $\Rightarrow \log L(1, \chi) \sim \sum_{p \leq (\log q)^{2+\epsilon}} \chi(p)/p + C_\chi$.

But we study $L(1, \chi)$ statistically: for most $\chi \pmod q$,

Zero-density estimates $\Rightarrow \log L(1, \chi) \sim \sum_{p \leq (\log q)^{100}} \chi(p)/p + C_\chi$.

2. Take moments of

$$L(1, \chi; y) := \prod_{p \leq y} \left(1 - \frac{\chi(p)}{p}\right)^{-1} = \sum_{p | n \Rightarrow p \leq y} \frac{\chi(n)}{n} \quad (y = (\log q)^{100}) :$$

$$\frac{1}{\phi(q)} \sum_{\chi \pmod q} |L(1, \chi; y)|^{2k} = \frac{1}{\phi(q)} \sum_{\chi \pmod q} \left| \sum_{p | n \Rightarrow p \leq y} \frac{\tau_k(n)\chi(n)}{n} \right|^2$$

$$= \sum_{m \equiv n \pmod q} \frac{\tau_k(m)\tau_k(n)}{mn}.$$
The distribution of $L(1, \chi)$, continued

Ignoring the off-diagonal terms (assumption that $X_n$ is a good model for $\chi(n)$), and assuming that $q$ is prime,

$$M_{2k} := \frac{1}{\phi(q)} \sum_{\chi \pmod{q}} |L(1, \chi; y)|^{2k} = \sum_{m \equiv n \pmod{q}} \frac{\tau_k(m)\tau_k(n)}{mn} \quad p | mn \Rightarrow p \leq y, p | q$$

$$\approx \sum_{p | n \Rightarrow p \leq y} \frac{\tau_k(n)^2}{n^2} = \prod_{p \leq y} \left(1 + \frac{\tau_k(p)^2}{p^2} + \frac{\tau_k(p^2)}{p^4} + \cdots\right).$$

Then Granville and Soundararajan proceed to show that $\log M_{2k} = 2e^{\gamma}k + C'k/\log k + O(k/\log^2 k)$, which allows them to estimate $\text{Prob} (|L(1, \chi)| > e^{\gamma}\tau)$ quite accurately.

**Remark.** In fact, they observe that

$$\log \left(1 + \frac{\tau_k(p)^2}{p^2} + \frac{\tau_k(p^2)}{p^4} + \cdots\right) = \log l_0(2k/p) + O(k/p^2),$$

where $l_0(t) = \sum_{n \geq 0} \left(\frac{t/2}{n!}\right)^2$ is the modified Bessel function of the 1st kind. In particular, most of the contribution to $M_{2k}$ comes from primes $p \approx k$. 
New results on $P_q(\tau) = \text{Prob} \left( M(\chi) > \frac{e^{\gamma}}{\pi} \tau \sqrt{q} \right)$

Recall Bober-Goldmakher’s result: for $\tau$ fixed and $q \to \infty$ over primes,

$$\exp \left\{ - \frac{Ce^\tau}{\tau} (1 + o_{\tau \to \infty}(1)) \right\} \leq P_q(\tau) \leq \exp \left\{ -e^{B\sqrt{\tau}/(\log \tau)^{1/4}} \right\}.$$

There are two issues to be addressed:

- There is a discrepancy between upper and lower bounds.
- The result is not uniform in $\tau$ and $q$.

**Theorem (Bober, Goldmakher, Granville, K. (2013))**

Let $\theta > 14/15$, $q$ be prime and $2 \leq \tau \leq \log \log q - \log \log \log q - 5$. Then

$$\exp \left\{ - \frac{Ce^\tau}{\tau} (1 + o_{\tau \to \infty}(1)) \right\} \leq P_q(\tau) \leq \exp \left\{ -e^{\tau + O_{\theta}(\tau^\theta)} \right\}.$$

**Remark.** On GRH, the theorem holds when $\tau \leq \log_2 q - \log_4 q + O(1)$. It seems likely that it can be shown unconditionally $\tau = o(\log q)$ can be obtained unconditionally.
A reduction to the distribution of $L(1, \chi)$: lower bounds

For lower bounds on $P_q(\tau)$, we follow Bober-Goldmakher and note that

$$
\sum_{n \leq q/2} \chi(n) \sim \frac{\tau(\chi)}{2\pi i} \sum_{1 \leq |n| \leq q} \frac{\overline{\chi}(n)(1-e(-n/2))}{n}
$$

$$
= \frac{\tau(\chi)}{\pi i} \sum_{1 \leq |n| \leq q} \frac{\overline{\chi}(n)}{n} = \frac{2\tau(\chi)}{\pi i} \left\{ \begin{array}{ll}
\sum_{1 \leq n \leq q} \frac{\overline{\chi}(n)}{n} & \text{if } \chi(-1) = -1, \\
0 & \text{if } \chi(-1) = 1.
\end{array} \right.
$$

When $\chi$ is odd, the right hand side is essentially $L(1, \overline{\chi})$, divided by the Euler factor at $p = 2$.

One can then obtain the claimed lower bound on $P_q(\tau)$ using the methods of Granville-Soundararajan.

The upper bound is significantly harder. The main issue is to understand where $\sum_{n \leq z} \chi(n)$ is maximized (ideas about pretentious characters).
A detour: pretentious characters

Granville-Soundararajan (2006) and Goldmakher (2010) improved the previously known bounds for $M(\chi)$ when $\chi$ has odd order $g$ to

$$M(\chi) \ll \begin{cases} \sqrt{q}(\log q)^{1-\delta_g+o(1)} & \text{unconditionally,} \\ \sqrt{q}(\log \log q)^{1-\delta_g+o(1)} & \text{on GRH.} \end{cases}$$

$\delta_g = 1 - g \frac{\sin \pi}{\pi} g$

Idea of the proof: $g$ odd $\Rightarrow$ $\chi(-1) = 1$. So Pólya’s expansion becomes

$$\sum_{n \leq \alpha q} \chi(n) \sim \frac{\tau(\chi)}{2\pi i} \sum_{1 \leq |n| \leq q} \frac{\overline{\chi}(n)(1 - e(-n\alpha))}{n} = -\frac{\tau(\chi)}{2\pi i} \sum_{1 \leq |n| \leq q} \frac{\overline{\chi}(n)e(-n\alpha)}{n}.$$

Let $|\alpha - a/b| < 1/(bB)$, $b \leq B := e^{\sqrt{\log q}}$. Montgomery-Vaughan showed

$$\sum_{n \leq x} \frac{\chi(n)e(n\alpha)}{n} \ll \log \log x + \log b + \frac{(\log b)^{3/2}}{\sqrt{b}} \log x \quad (x \geq 2).$$

So, we may assume that $b \leq (\log q)^{1/3}$. Also, let $\alpha = a/b$ for simplicity.
Pretentious characters, continued

ord(χ) = g = odd, χ(−1) = 1, α = a/b, b ≤ (log q)1/3. We need to estimate

$$\sum_{n \leq \alpha q} \chi(n) \sim -\frac{\tau(\chi)}{2\pi i} \sum_{1 \leq |n| \leq q} \frac{\bar{\chi}(n)e(-na/b)}{n}.$$ 

Expand $e(-na/b)$ in terms of characters $\psi (\mod d)$, $d | b$, to replace $\sum_{n \leq \alpha q} \chi(n)$ by sums of the form

$$S = \sum_{1 \leq |n| \leq z} \frac{\bar{\chi}(n)\psi(n)}{n} = (1 - \chi(-1)\psi(-1)) \sum_{n \leq z} \frac{\bar{\chi}(n)\psi(n)}{n}.$$ 

For $S$ to be big, $\chi$ must be ‘close’ to $\psi$ ($\chi(p) \approx \psi(p)$). Indeed,

$$\sum_{n \leq z} \frac{\chi(n)\bar{\psi}(n)}{n} \ll \frac{\log z}{\exp\{\mathcal{D}(\chi, \psi; z)/2\}}, \quad \mathcal{D}^2(\chi, \psi; z) = \sum_{p \leq z} \frac{1 - \Re(\chi(p)\bar{\psi}(p))}{p}.$$ 

If $\mathcal{D}(\chi, \psi; z)$ is small, we say that $\chi$ pretends to be $\psi$.

Also, $\psi(-1) = -\chi(-1) = -1 \implies \text{ord}(\psi) = \text{even} \neq g$.

But then $\chi \approx \psi \implies 1 = \chi^g \approx \psi^g \neq 1$, a contradiction.
A reduction to the distribution of $L(1, \chi)$: upper bounds

In bounding $P_q(\tau)$ from above, the key step is the following:

**Theorem (Bober, Goldmakher, Granville, K. (2013))**

Let $\theta > 14/15$, $q$ be prime and $2 \leq \tau \leq \log \log q - \log \log \log q - 5$. With the exception of $\ll q \exp\{-20\tau e^\tau\}$ characters mod $q$, if $M(\chi) > \frac{e\gamma}{\pi} \tau \sqrt{q}$, then $\chi$ is odd, and there is a $b \leq \tau^{10}$ such that

$$\left| \sum_{\substack{n \in \mathbb{N}, (n, b) = 1 \\ p | n \Rightarrow p \leq e^\tau}} \frac{\chi(n)}{n} \right| \geq e^\gamma \tau + O_\theta(\tau^\theta).$$

Then $P_q(\tau) \leq \exp\left\{-e^\tau + O_\theta(\tau^\theta)\right\}$, by Granville-Soundararajan.

Main ideas involved in proving the above theorem:

1. A high moment bound to truncate Pólya’s expansion.
2. Use “pretentious characters” to locate the max of $\left| \sum_{n \leq x} \chi(n) \right|$.
3. Slow variance of $\sum_{n \leq x} \chi(n)$ (Lipschitz bounds).
Truncating Pólya’s expansion

When $\chi$ is primitive, we have that

$$M(\chi) = \max_{\alpha \in [0,1]} \left| \sum_{n \leq \alpha q} \chi(n) \right| = \frac{\sqrt{q}}{2\pi} \max_{\alpha \in [0,1]} \left| \sum_{1 \leq |n| \leq q} \frac{\chi(n)(1 - e(n \alpha))}{n} \right|.$$  

Using a moments argument, we show that, for most $\chi$,

$$\sum_{1 \leq |n| \leq q} \frac{\chi(n)(1 - e(n \alpha))}{n} \sim \sum_{1 \leq |n| \leq q, P^+(n) \leq y} \frac{\chi(n)(1 - e(n \alpha))}{n},$$

with $y \approx e^\tau$ (here $P^+(n) = \max\{p|n\}$ and $P^-(n) = \min\{p|n\}$). This is done by observing that their difference equals

$$\sum_{1 \leq |n| \leq q, P^+(n) > y} \frac{\chi(n)(1 - e(n \alpha))}{n} = \sum_{1 \leq |g| \leq q, P^+(g) \leq y} \frac{\chi(g)}{g} \sum_{y<h \leq \frac{q}{g}, P^-(h) > y} \frac{\chi(h)(1 - e(gh \alpha))}{h},$$

$$\ll \sum_{P^+(g) \leq y} \frac{1}{g} \max_{\alpha \in [0,1]} \left| \sum_{y<h \leq \frac{q}{g}, P^-(h) > y} \frac{\chi(h)e(h \alpha)}{h} \right|.$$
Truncating Pólya’s expansion, continued

\[
\sum_{1 \leq |n| \leq q \atop P^+(n) > y} \frac{\chi(n)(1 - e(n\alpha))}{n} \ll \sum_{P^+(g) \leq y} \frac{1}{g} \max_{\alpha \in [0,1]} \sum_{y < h \leq q/g, P^-(h) > y} \frac{\chi(h)e(h\alpha)}{h}.
\]

We raise both sides to \(2k\). Then \(\max_{\alpha \in [0,1]}\) is removed by noticing that \(|\alpha - r/R|\) for some \(r \in \{1, \ldots, R\}\). It remains to estimate

\[
\sum_{\chi (\mod q)} \left| \sum_{y < h \leq q/g, P^-(h) > y} \frac{\chi(h)e(hr/R)}{h} \right|^{2k}.
\]

Then we find that this is \(\lesssim \sum_{P^-(n) > y, n > y^k} \tau_k(n)^2 / n^2 = o(1)\) if \(k \leq y/(\log y)^{100}\). (If \(y > k\), the primes \(p \approx k\) that give most of the contribution to the sum \(\sum_{n \geq 1} \tau_k(n)^2 / n^2\) are not present.)

\[
y \approx e^\tau \Rightarrow P_q(\tau) \sim \text{Prob} \left( \max_{\alpha \in [0,1]} \left| \sum_{P^+(|n|) \leq e^\tau} \frac{\chi(n)(1 - e(n\alpha))}{n} \right| > 2e^{\gamma \tau} \right).
\]
Locating the maximum

\[ P_q(\tau) \sim \text{Prob} \left( \max_{\alpha \in [0,1]} \left| \sum_{P^+(|n|) \leq e^\tau} \frac{\chi(n)(1 - e(n\alpha))}{n} \right| > 2e^{\gamma \tau} \right). \]

Write \( N(\chi) \) for the above maximum, and let \( \alpha_\chi \) be its location.

Let \( |\alpha_\chi - a/b| < 1/(bB) \), \( b \leq B := e^{\sqrt{\tau}} \). Also, let \( \xi \) be the primitive character of conductor \( \leq \tau \) that lies the ‘closest’ to \( \chi \), i.e.

\[ \mathbb{D}(\chi, \xi; e^\tau) = \min_{\psi \text{ mod } d \leq \tau} \mathbb{D}(\chi, \psi; e^\tau), \quad \mathbb{D}^2(f, g; y) = \sum_{p \leq y} \frac{1 - \Re(f(p)\overline{g}(p))}{p}. \]

Claim: If \( N(\chi) > 2e^{\gamma \tau} \), then \( \xi = 1 \) and \( \chi \) is odd.

Assume not. Then

\[ \sum_{P^+(|n|) \leq e^\tau} \frac{\chi(n)}{n} = (1 - \chi(-1)) \sum_{P^+(n) \leq e^\tau} \frac{\chi(n)}{n} = o(\tau). \]

Also, \( b \leq \tau^{1/10} \); else, \( N(\chi) \sim \sum_{P^+(|n|) \leq e^\tau} \chi(n)e(n\alpha)/n = o(\tau) \), by Montgomery-Vaughan, a contradiction to “\( N(\chi) > 2e^{\gamma \tau} \)”.
Locating the maximum, continued

If $\xi \pmod{D}$ is the ‘closest’ character to $\chi$, and either $\xi \neq 1$ or $\chi$ is even:

$$2e^{\gamma \tau} < N(\chi) \sim \sum_{P^+(|n|) \leq e^\tau} \frac{\chi(n)e(n\alpha)}{n}, \quad |\alpha \chi-a/b| \leq \frac{1}{be^{\sqrt{\tau}}}, \quad b \leq \tau^{1/10}.$$ 

Assume that $\alpha \chi = a/b$, and expand $e(na/b)$ using characters, to get sums

$$\sum_{P^+(|n|) \leq e^\tau} \frac{\chi(n)\overline{\psi}(n)}{n} = (1 - \chi(-1)\psi(-1)) \sum_{P^+(n) \leq e^\tau} \frac{\chi(n)\overline{\psi}(n)}{n}.$$

Small unless $\chi \overline{\psi}$ odd and $\chi \approx \psi$. So $\psi$ induced by $\xi$ and $\chi \overline{\xi}$ odd. Then

$$\sum_{P^+(|n|) \leq e^\tau} \frac{\chi(n)e(na/b)}{n} \sim \frac{2D^{1/2}}{b} \left| \sum_{D|d|b} \frac{\chi(b/d)\mu(d/D)\overline{\xi}(d/D)}{\phi(d)/d} \sum_{P^+(n) \leq e^\tau} \frac{\chi(n)\overline{\xi}(n)}{n} \right|.$$

$$\Rightarrow 2e^{\gamma \tau} \lesssim \frac{2\sqrt{D}}{b} \sum_{D|d|b} \frac{d}{\phi(d)} \cdot \frac{\phi(d)}{d} e^{\gamma \tau} = \frac{2e^{\gamma \tau}(b/D)}{\sqrt{Db}/D} \leq \frac{2e^{\gamma \tau}}{\sqrt{D}}.$$

If $D > 1$, this is a contradiction. So $\xi = 1$ and $\chi$ is odd.
Locating the maximum, continued

To summarize,

\[
N(\chi) := \max_{\alpha \in [0,1]} \left| \sum_{P^+(|n|) \leq e^\tau} \frac{\chi(n)(1 - e(n\alpha))}{n} \right| > 2e^\gamma \Rightarrow \chi \approx 1 \text{ and } \chi \text{ odd.}
\]

Also, recall that \(\alpha_\chi\) location of max, \(|\alpha_\chi - a/b| < 1/(be^{\sqrt{\tau}})\).

**Claim.** \(\exists c \leq \tau\) with \(|\sum_{P^+(n) \leq e^\tau} \chi(n)/n| \gtrsim e^\gamma \tau \phi(b)/b\).

If \(b > \tau\), we take \(c = 1\) (by Mont-Vaughan: \(N(\chi) \sim 2|\sum_{P^+(n) \leq e^\tau} \frac{\chi(n)}{n}|\)).

If \(b \leq \tau\), then we use a result of Fouvry-Tenenbaum on smooth numbers in APs to get asymptotics for the sum \(\sum_{n \leq N, P^+(n) \leq e^\tau} \chi(n)/n\).

We then find that \(\exists N \in (e^{\sqrt{\tau}}, e^{\tau \log \tau}]\) (related to \(|\alpha_\chi - a/b|\)) such that

\[
\left| \sum_{n \leq N, (n,b)=1} \frac{\chi(n)}{n} \right| \left| \sum_{n > N, (n,b)=1} \frac{\chi(n)}{n} \right| \gtrsim \frac{\phi(b)}{b} e^\gamma \tau.
\]
Lipschitz bounds for averages of $\chi$

\[ S_1 = \sum_{n \leq N, (n,b) \leq e^\tau} \frac{\chi(n)}{n}, \quad S_2 = \sum_{n > N, (n,b) \leq e^\tau} \frac{\chi(n)}{n}. \]

We have $|S_1| + |S_2| \gtrsim e^{\gamma \tau} \frac{\phi(b)}{b}$, we want to show that $|S_1 + S_2| \gtrsim e^{\gamma \tau} \frac{\phi(b)}{b}$.

Note that $|S_1| + |S_2| \lesssim e^{\gamma \tau} \frac{\phi(b)}{b}$.

So, if $S_j = \lambda_j |S_j|$ with $|\lambda_j| = 1$, $j \in \{1, 2\}$, then

\[
0 \leq \sum_{n \leq N, (n,b) = 1} \frac{\chi(n)\overline{\lambda_1}}{n} + \sum_{n > N, (n,b) = 1} \frac{\chi(n)\overline{\lambda_2}}{n} = o(\tau \phi(b)/b).
\]

So $\chi(n) \sim \lambda_1$ for most $n \leq N$ and $\chi(n) \sim \lambda_2$ for most $n > N$.

Averages of mult. fncts vary slowly. Ideas from Halász’s theorem + $\chi \approx 1$:

\[
\frac{\sum_{n \leq x^{1+\delta}} \chi(n)}{x^{1+\delta}} - \frac{\sum_{n \leq x} \chi(n)}{x} \lesssim \delta \log(1/\delta) \quad (\delta \geq 1/\log x).
\]

So $\lambda_1 \sim \lambda_2$, which implies that $|S_1| + |S_2| \sim |S_1 + S_2|$. 

Thank you!