When the sieve works

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The general sieve problem

Given \( A \subset \mathbb{N} \) and a set of primes \( P \), what is the size of

\[
S(A, P) := \#\{a \in A : p \mid a \Rightarrow p \notin P\}
\]

Examples:

- Taking \( A = \mathbb{N} \cap [1, x] \), \( P = \{p \leq \sqrt{x}\} \) we count primes.
- Taking \( A = \{n(n+2) : n \leq x\} \), \( P = \{p \leq \sqrt{x}\} \) we count twin primes.
- Taking \( A = \{n \leq x : n \equiv 1 \pmod{4}\} \), \( P = \{p \leq \sqrt{x} : p \equiv 3 \pmod{4}\} \) we count (a dense subset of) numbers that can be written as the sum of two squares (Iwaniec).
- Taking \( A = \mathbb{N} \cap [1, x] \), \( P = \{p > y\} \) we count \( y \)-smooth/friable numbers.

Goal of classical sieve methods: Given \( A \), estimate \( S(A, P) \) for \( P \subset \{p \leq y\} \) with \( y \) as large as possible (ideally, with \( y^2 \approx \max\{p \mid \prod_{a \in A} a\} \)).
A heuristic argument

We focus on the case when $A = \mathbb{N} \cap [1, x]$, $P \subset \{p \leq x\}$. We let

$$S(x, P) = \#\{n \leq x : p|n \Rightarrow p \notin P\}.$$

Heuristically, for a prime $p$

$$\text{Prob} (n \leq x : p|n) = \frac{\lfloor x/p \rfloor}{\lfloor x \rfloor} \approx \frac{1}{p}.$$

In general, for primes $p_1 < p_2 < \cdots < p_r$

$$\text{Prob} (n \leq x : p_1 \cdots p_r|n) = \frac{\lfloor x/(p_1 \cdots p_r) \rfloor}{\lfloor x \rfloor} \approx \frac{1}{p_1} \cdots \frac{1}{p_r}.$$

So, we expect that

$$\frac{S(x, P)}{\lfloor x \rfloor} = \text{Prob} (n \leq x : p \nmid n \forall p \in P) \approx \prod_{p \in P} \left(1 - \frac{1}{p}\right).$$
Expectations and reality

We know that \( \#\{p \leq x\} \sim x/\log x \). However, the heuristic predicts that

\[
\frac{\#\{p \leq x\}}{x} \sim \frac{S(x, \{p \leq \sqrt{x}\})}{x} \sim \prod_{p \leq \sqrt{x}} \left(1 - \frac{1}{p}\right) \sim \frac{2e^{-\gamma}}{\log x}; \quad 2e^{-\gamma} > 1.
\]

In general,

\[
S(x; P) \ll x \prod_{p \in P} \left(1 - \frac{1}{p}\right).
\]

Also, if \( \max P \leq x^{1/2-\epsilon} \), then

\[
S(x; P) \asymp_{\epsilon} x \prod_{p \in P} \left(1 - \frac{1}{p}\right).
\]

But if \( P = \{x^{1/u} < p \leq x\} \), then \( S(A, P) = x/u^{(1+o(1))u} \), whereas the prediction is that \( S(A, P) \approx x/u \).
When does the sieve work?

Question

When does the sieve work or, more precisely, when is it true that

\[ S(x, P) \asymp x \prod_{p \in P} \left(1 - \frac{1}{p}\right) \quad ? \]

Hildebrand showed that the smooth primes are the extreme example:

Let \( u \geq 1 \) and \( P \subset \{p \leq x\} \).

\[ \sum_{p \in P} \frac{1}{p} \lesssim \log u \quad \Rightarrow \quad S(x, P) \gtrsim S(x, \{x^{1/u} < p \leq x\}) = \frac{x}{u(1+o(1))}. \]

It is generally expected that if \( P^c \) contains enough many big primes, then (*) should hold.

For this reason, we use the complementary notation

\[ Q = \{p \leq x\} \setminus P, \quad \psi(x; Q) = S(x, P) = \#\{n \leq x : p | n \Rightarrow p \in Q\}. \]
The effect of the big primes

**Proposition**

If \( Q \subset \{ p \leq x^{1-\epsilon} \} \), \( u \in [1, \log x] \) and \( \kappa = \sum_{q \in Q \cap [x^{1/u}, x]} 1/q \), then

\[
\frac{\psi(x; Q)}{x} \ll \epsilon \left( \kappa + u^{-\epsilon u/2} + x^{-1/10} \right) \cdot \prod_{p \leq x, p \notin Q} \left( 1 - \frac{1}{p} \right).
\]

**Proposition**

If \( \epsilon > 0 \), \( Q \subset \{ p \leq x \} \) and \( u \in [1, \log x] \) are such that

\[
\sum_{q \in Q, x^{1/u} < q \leq x} \frac{1}{q} > \epsilon,
\]

then \( \exists t \in [x^{1/u}, x] : \frac{\psi(t; Q)}{t} \gg \frac{\epsilon \min \{ 1, \epsilon u \}}{\log u} \prod_{p \leq t, p \notin Q} \left( 1 - \frac{1}{p} \right). \)
A different extremal example

The key is how big $\sum_{q \in Q \cap [x^{1/u}, x]} 1/q$ is. Consider

$$Q = \bigcup_{m=1}^{N-1} \left\{ x^{\frac{m}{N+1}} < p < x^{\frac{m}{N}} \right\},$$

If $n \leq x$ has all its prime factors in $Q$, then $n \in \bigcup_{m=1}^{N} \left( x^{\frac{m}{N+1}}, x^{\frac{m}{N}} \right)$. 

$$\psi(x; Q) = O(x^{1-1/N}) + \sum_{x^{\frac{N}{N+1}} < n \leq x} 1 \ll_N \frac{x}{\log^2 x}.$$ 

Note that

$$\sum_{q \in Q} \frac{1}{q} = (N - 1) \log \frac{N + 1}{N} \sim 1 - \frac{3/2 + o(1)}{N} < 1.$$
A problem in additive combinatorics

Estimating $\psi(x; Q)$ is essentially equivalent to finding solutions to

$$\log p_1 + \cdots + \log p_r = \log x + O(1) \quad (r \in \mathbb{N}, \ p_1, \ldots, p_r \in Q).$$

**Theorem (Bleichenbacher)**

Let $T \subset (0, 1)$ be open. If $\int_T dt \cdot t > 1$, then there are $t_1, \cdots, t_k \in T$ such that $t_1 + \cdots + t_k = 1$. This is optimal, as the example $T = \bigcup_{m=1}^N \left( \frac{m}{N+1}, \frac{m}{N} \right)$ shows.

**Corollary (Lenstra-Pomerance)**

Let $Q \subset \{ p \leq x \}$, $u \geq 1$.

$$\sum_{q \in Q, \ x^{1/u} < q < x} \frac{1}{q} > 1 + \epsilon \quad \Rightarrow \quad \frac{\psi(x; Q)}{x} \gg \epsilon \frac{e^{-O(u)}}{(\log x)^{u-1}} \prod_{p \leq x, \ p \notin P} \left(1 - \frac{1}{p}\right).$$
The main defect of Bleichenbacher’s theorem is that it does not say anything about how many solutions there are to \( e_1 + \cdots + e_k = 1 \) other than that there is at least 1.

It is easier to look at the discrete analogue of this problem: Given \( A \subset [1, N] \cap \mathbb{N} \) with \( \sum_{a \in A} 1/a > 1 \), how many solutions are there to \( a_1 + \cdots + a_k = N + O(1) \) with \( k \in \mathbb{N}, a_1, \ldots, a_k \in A \)?

**Theorem (Granville-K-Matomäki)**

\[ \exists \lambda > 1, c > 0 \text{ such that if } 1 \leq u \leq c\sqrt{N}, \ A \subset [N/u, N] \cap \mathbb{N} \text{ satisfy } \sum_{a \in A} 1/a \geq \lambda, \text{ then } \exists k \in \mathbb{N}, n \in [N - k, N] \text{ such that } \]

\[ \sum_{(a_1, \ldots, a_k) \in A^k \atop a_1 + \cdots + a_k = n} \frac{1}{a_1 \cdots a_k} \gg \frac{u^{-O(u)}}{N} \left( \sum_{a \in A} \frac{1}{a} \right)^k. \]
Motivated by Bleichenbacher’s theorem, we conjecture that this result holds for any $\lambda' > 1$. 

$\exists \lambda' > 1, c' > 0$ such that if $Q \subset \{p \leq x\}$, $1 \leq u \leq c' \sqrt{\log x}$, then

$$\sum_{q \in Q, x^{1/u} < q \leq x} \frac{1}{q} \geq \lambda' \Rightarrow \frac{\psi(x; Q)}{x} \gg \frac{1}{u^{O(u)}} \prod_{p \leq x, p \notin Q} \left(1 - \frac{1}{p}\right).$$
Sketch of the proof

For sets of integers $C, D$, let $C + D = \{c + d : c \in C, d \in D\}$. 

$\exists v \in [1, u]$ such that the set $B = A \cap [1, N/v]$ has $\geq \frac{\lambda N}{2v^2}$ elements. We will show that

$$\exists k : \#\{(b_1, \ldots, b_k) \in B^k : b_1 + \cdots + b_k \in [N - k, N]\} \geq \frac{|B|^k}{u^{O(u)}N}.$$ 

Varying Ruzsa-Chang: if $|B + B| \leq 4|B|$, then $B + B + B$ contains a GAP $P = \{a_0 + a_1k_1 + \cdots + a_dk_d : |k_j| \leq K_j\}$ of size $|P| \gg |B|$ and rank $d \ll 1$. Also, $r_{B+B+B}(n) \gg |B|^2 \forall n \in P$. So (*) follows.

If $|B + B| > 4|B|$, replace $B$ with $2B = B + B$ and repeat.

$$2B \subset [1, 2N/v] = [1, N/(v/2)] \quad \text{and} \quad |2B| > 4 \cdot \frac{\lambda N}{2v^2} = \frac{\lambda N}{2(v/2)^2}.$$ 

Apply induction; this process terminates at some $k$ with $2^k \leq 2v/\lambda$.

**Problem:** We need to keep track of the representations!

Use instead restricted sumsets $\{n \in B + B : r_{B+B}(n) \geq \eta|B|\}$ (ideas from Balog-Szemeredi-Gowers theorem).
An application

Let $f$ be a Hecke eigencuspsform for $SZ_2(\mathbb{Z})$ of weight $k$. It has $k/12 + O(1)$ zeroes on the upper half plane $\mathbb{H}$, which are equidistributed by QUE (Rudnick).

Ghosh and Sarnak initiated the study of "real" zeroes of $f$, i.e. zeroes on the geodesics

$$
\delta_1 = \{ z \in \mathbb{H} : \Re(z) = 0 \}, \quad \delta_2 = \{ z \in \mathbb{H} : \Re(z) = 1/2 \}
$$

$$
\delta_3 = \{ z \in \mathbb{H} : |z| = 1, 0 \leq \Re(z) \leq 1/2 \}.
$$

They showed that

$$
N(f) := \# \{ z \in \delta_1 \cup \delta_2 : f(z) = 0 \} \gg_{\epsilon} k^{1/4 - 1/80 - \epsilon}.
$$

Matokäki, using methods described before, showed that $N(f) \gg_{\epsilon} k^{1/4 - \epsilon}$. 
How sieve methods enter the picture

\[ f(z) = \sum_{n=1}^{\infty} \lambda(n)n^{(k-1)/2} e^{2\pi i n z}. \]

Ghosh-Sarnak: If \( C \leq m \leq \epsilon \sqrt{k/\log k} \), \( \alpha \in \mathbb{R} \), and \( y_m = \frac{k-1}{4\pi m} \), then

\[ \left( \frac{e}{m} \right)^{(k-1)/2} f(\alpha + iy_m) = \lambda(m)e^{2m\pi i \alpha} + O(k^{-\delta}). \]

So if \( m_1 \) is even, \( m_2 \) is odd, and \( |\lambda(m_1)|, |\lambda(m_2)| \geq k^{-\delta/2} \), then \( f \) has a zero in the line segment connecting \( \alpha + iy_{m_1} \) and \( \alpha + iy_{m_2} \) for \( \alpha = 0 \) or \( \alpha = 1/2 \), i.e in \( \delta_1 \cup \delta_2 \).

Since \( \lambda(p)^2 = \lambda(p^2) + 1 \), we have that \( \max\{|\lambda(p)|, |\lambda(p^2)|\} \geq 1/2 \).

So we need to show that \( N_1 \cup N_2 \) contains many integers, where

\( N_j = \{ n \in \mathbb{N} : n \text{ square-free and odd, } p|n \Rightarrow |\lambda(p^j)| \geq 1/2 \} \quad (j = 1, 2). \)

Even though we don’t have much control over the location of the primes in \( P_j = \{ p > 2 : |\lambda(p^j)| \geq 1/2 \} \) for \( j = 1, 2 \), the methods described before are general enough that can handle this problem.
Thank you for your attention!