Group structures of elliptic curves over finite fields

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Warm-up

\[ E : y^2 = x^3 + ax + b, \quad (*) \]

where \( a \) and \( b \) are some fixed integers with \( 4a^3 + 27b^2 \neq 0 \).

The set

\[ E(\mathbb{Q}) = \{\text{set of solutions to } (*) \text{ over } \mathbb{Q}\} \cup \{\infty\} \]

is a finitely generated abelian group.

Consider the reduction of \( E \) over \( \mathbb{F}_p \), \( p \nmid 4a^3 + 27b^2 \):

\[ E_p : y^2 = x^3 + a_p x + b_p \quad (a_p = a (\text{mod } p), \ b_p = b (\text{mod } p)), \quad (*)_p \]

The set

\[ E_p(\mathbb{F}_p) = \{\text{set of solutions to } (*)_p \text{ over } \mathbb{F}_p\} \cup \{\infty\} \]

is a finite abelian group of rank at most 2, i.e. \( \exists \) unique \( m, k \in \mathbb{N} \) such that

\[ E_p(\mathbb{F}_p) \simeq \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/mk\mathbb{Z}. \]
**Possible group structures**

**Question**

What are the possibilities for \( E_p(\mathbb{F}_p) \) as a group, as \( p \) runs over all primes and \( E \) over all elliptic curves?

\[ \mathcal{I} = \{ (m, k) : \exists \ p \text{ and } E/\mathbb{F}_p \text{ such that } E(\mathbb{F}_p) \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/mk\mathbb{Z} \} \]

- If \((m, k) \in \mathcal{I}\) and \( N = m^2 k \), Hasse’s bound implies that
  \[ |p + 1 - N| \leq 2 \sqrt{p} \iff N - 2\sqrt{N} + 1 < p < N + 2\sqrt{N} + 1. \]

- If \((m, k) \in \mathcal{I}\), then \( \exists p, E/\mathbb{F}_p \) with \( E(\mathbb{F}_p) \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/mk\mathbb{Z} \). So
  \[ E(\mathbb{F}_p)[m] \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \subset E(\mathbb{F}_p) \xrightarrow{\text{Weil pairing}} p \equiv 1 \ (\text{mod } m). \]

**Lemma 1**

\((m, k) \in \mathcal{I}\) if and only if there is some \( p \equiv 1 \ (\text{mod } m) \) in
\((N - 2\sqrt{N} + 1, N + 2\sqrt{N} + 1) = (m^2 k - 2m\sqrt{k} + 1, m^2 k + 2m\sqrt{k} + 1).\)
Characterization of admissible groups points

\[ \mathcal{L} = \{ (m, k) : \exists p \text{ and } E/\mathbb{F}_p \text{ such that } E(\mathbb{F}_p) \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/mk\mathbb{Z} \} \].

Lemma 1

\((m, k) \in \mathcal{L}\) if and only if there is some \(p \equiv 1 \pmod{m}\) in

\((N - 2\sqrt{N} + 1, N + 2\sqrt{N} + 1) = (m^2 k - 2m\sqrt{k} + 1, m^2 k + 2m\sqrt{k} + 1)\).

Lemma 2 (Rück)

\(N = m^2 k = \prod_\ell \ell^{h_\ell}, \ |p - N - 1| < 2\sqrt{N}\). Then \((m, k) \in \mathcal{L}\) if-f

\[ \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/mk\mathbb{Z} \cong \mathbb{Z}/p^{h_p}\mathbb{Z} \times \prod_{\ell \neq p} \left( \mathbb{Z}/\ell^{b_\ell}\mathbb{Z} \times \mathbb{Z}/\ell^{h_\ell - b_\ell}\mathbb{Z} \right) \]

with \(0 \leq b_\ell \leq \min\{v_\ell(p - 1), h_\ell/2\}\).

Let \(p \equiv 1 \pmod{m}\) in \((m^2 k - 2m\sqrt{k} + 1, m^2 k + 2m\sqrt{k} + 1), \ N = km^2\).

\(v_\ell(m) \leq \lfloor h_\ell/2 \rfloor\) and \(v_\ell(p - 1) \geq v_\ell(m), \ \forall \ell \mid m\).

Take \(b_\ell = v_\ell(m)\) in Lemma 2 to deduce Lemma 1.
An average question

Lemma 1

\((m, k) \in \mathcal{I}\) if and only if there is some \(p \equiv 1 \pmod{m}\) in

\((N - 2\sqrt{N} + 1, N + 2\sqrt{N} + 1) = (m^2k - 2m\sqrt{k} + 1, m^2k + 2m\sqrt{k} + 1)\).

- If \(k\) is small, the group \(\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/mk\mathbb{Z}\) might not occur. e.g. if \(m = 11, k = 1\), there are no primes \(p \equiv 1 \pmod{11}\) in \((100, 144)\), so the group \(\mathbb{Z}/11\mathbb{Z} \times \mathbb{Z}/11\mathbb{Z}\) is not realized as \(E(\mathbb{F}_p)\).

- When \(m = 1\), we ask for primes in \((k - 2\sqrt{k} + 1, k + 2\sqrt{k} + 1)\), which should always contain a prime. So all finite cyclic groups should be realized as \(E(\mathbb{F}_p)\). Proving this is beyond RH.

Banks-Pappalardi-Shparlinski study

\[ S(M, K) = \#\{m \leq M, k \leq K : (m, k) \in \mathcal{I}\} \]

\[ = \#\{m \leq M, k \leq K : \exists p \equiv 1 \pmod{m} \text{ such that } m^2k - 2m\sqrt{k} + 1 < p < m^2k + 2m\sqrt{k} + 1\} \]

\[ = \#\{m \leq M, k \leq K : \exists j, |j| < 2\sqrt{k}, \text{ with } m^2k + jm + 1 \text{ prime}\}. \]
Conjectures and heuristics

\[ S(M, K) = \# \{ m \leq M, k \leq K : \exists j, \ |j| < 2\sqrt{k}, \text{ with } m^2 k + jm + 1 \text{ prime} \}. \]

**Conjecture (Banks-Shparlinski-Pappalardi)**

\[
S(M, K) = \begin{cases} 
  o(MK) & \text{if } K \leq (\log M)^{2-\epsilon}, \ K \to \infty \\
  (1 + o(1))MK & \text{if } K \geq (\log M)^{2+\epsilon}, \ M \to \infty.
\end{cases}
\]

\# \{ p \leq x \} \sim \int_2^x \frac{dt}{\log t}. \text{ So } n \text{ is prime with probability } 1/\log n.

\[
\text{Prob} \left( \{ m^2 k + jm + 1 \text{ is not prime } \forall j \in [-2\sqrt{k}, 2\sqrt{k}] \} \right) 
\approx \prod_{|j| \leq 2\sqrt{k}} \left( 1 - \frac{1}{\log(m^2 k + jm + 1)} \right) 
\approx \left( 1 - \frac{1}{m^2 k} \right)^{4\sqrt{k}} \to \begin{cases} 0 & \text{if } \frac{\sqrt{k}}{\log m} \to \infty, \\
 1 & \text{if } \frac{\sqrt{k}}{\log m} \to 0. \end{cases}
\]
### Results on $S(M, K)$

$S(M, K) = \# \{ m \leq M, k \leq K : \exists j, |j| < 2\sqrt{k}, \text{ with } m^2k + jm + 1 \text{ prime} \}$.

#### Theorem (Banks-Pappalardi-Shparlinski)

\[
\begin{align*}
S(M, K) & \ll K M / \log M \quad \text{for all } M, K \geq 2, \\
S(M, K) & \gg \epsilon MK / \log K \quad \text{if } M \leq K^{43/94-\epsilon}, \\
S(M, K) & \gg \epsilon MK / (\log K)^2 \quad \text{if } M \leq K^{1/2-\epsilon}.
\end{align*}
\]

First part implies BPS conjecture when $K \ll 1$.

#### Theorem (Chandee-David-K-Smith)

\[
\begin{align*}
S(M, K) & \ll MK^{3/2} / \log M \quad \text{for all } M, K \geq 2, \\
S(M, K) & = (1 + o_\epsilon(1))MK \quad \text{if } M \leq K^{1/4-\epsilon}, \\
S(M, K) & \gg MK \quad \text{if } M \leq K^{1/2}.
\end{align*}
\]

First part implies the full first part of the BPS conjecture, i.e. $S(M, K) = o(MK)$ when $K \leq (\log M)^{2-\epsilon}$. Second part implies the second part of the BPS conjecture when $M \leq K^{1/4-\epsilon}$. 

Proof of $S(M, K) \ll MK^{3/2}/\log M$

$$S(M, K) = \#\{m \leq M, k \leq K : \exists j, \ |j| < 2\sqrt{k}, \text{ with } m^2k + jm + 1 \text{ prime} \}$$

$$\leq \sum_{m \leq M} \sum_{k \leq K} \sum_{|j| \leq 2\sqrt{k}} 1 = \sum_{k \leq K} \sum_{|j| < 2\sqrt{k}} \sum_{m \leq M} 1 \quad m^2k + jm + 1 \text{ is prime}$$

$$\ll \sum_{k \leq K} \sum_{|j| < 2\sqrt{k}} \frac{M}{\log M} \frac{k}{\phi(k)} \prod_{p \leq M} \left(1 - \frac{\left(\frac{j^2 - 4k}{p}\right)}{p}\right).$$

Elliott: Zero-density estimates imply that . . .

$$S(M, K) \ll \frac{M}{\log M} \sum_{k \leq K} \frac{k}{\phi(k)} \sum_{|j| < 2\sqrt{k}} \prod_{p \leq (\log M)^{100}} \left(1 - \frac{\left(\frac{j^2 - 4k}{p}\right)}{p}\right)$$

$$\ll \frac{M}{\log M} \cdot K \cdot K^{1/2} = \frac{MK^{3/2}}{\log M}.$$
\[ S(M, K) \sim MK \text{ when } M \leq K^{1/4-\epsilon} \]

\[ S(M, K) = \#\{m \leq M, k \leq K : \exists p \equiv 1 \pmod{m} \text{ such that } m^2k - 2m\sqrt{k} + 1 < p < m^2k + 2m\sqrt{k} + 1\} . \]

**Theorem (K)**

Let \(1 \leq h \leq x\) and \(1 \leq Q^2 \leq h/x^{1/6+\epsilon}\). For every \(A > 0\)

\[
\frac{1}{x} \int_x^{2x} \sum_{q \leq Q} \max_{(a,q)=1} \left| \sum_{\substack{y < p \leq y + h \\ p \equiv a \pmod{q}}} \log p - \frac{h}{\phi(q)} \right| \ dy \ll_A \frac{h}{(\log x)^A} .
\]

This is proven using zero-density estimates. Most likely it is possible to improve the range \(M \leq K^{1/4-\epsilon}\) using sieve-theoretic ideas.

To control \(S(M, K)\), we apply the theorem with \(x = M^2K\), \(h = M\sqrt{K}\), \(Q = M\). We need \(M^2 \leq M\sqrt{K}/(M^2K)^{1/6+\epsilon}\).
\[ S(M, K) \gg MK \text{ when } M \leq \sqrt{K} \]

\[ S(M, K) = \# \{ m \leq M, k \leq K : \exists p \equiv 1 \pmod{m} \text{ such that } p \in I_{m^2k} = (m^2k - 2m\sqrt{k} + 1, m^2k + 2m\sqrt{k} + 1) \} . \]

The Brun-Titchmarsh inequality implies that

\[ \# \{ p \in I_{m^2k} : p \equiv 1 \pmod{m} \} \ll \frac{m\sqrt{k}}{\phi(m) \log(2k)} . \]

So

\[ S(M, K) \gg \sum_{m \leq M} \sum_{k \leq K} \frac{\# \{ p \in I_{m^2k} : p \equiv 1 \pmod{m} \}}{\frac{m^2k}{\phi(m) \log(2k)}} \]

\[ \gg \frac{\log K}{\sqrt{K}} \sum_{m \leq M} \frac{\phi(m)}{m} \sum_{k \leq K} 1 \]

\[ = \frac{\log K}{\sqrt{K}} \sum_{m \leq M} \frac{\phi(m)}{m} \sum_{p \leq M^2K} \sum_{k \leq K} 1 \]

\[ \text{such that } |k - p - 1| < 2\sqrt{p}/m^2 \]
Thank you!