DIVISORS OF SHIFTED PRIMES

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Abstract. We bound from below the number of shifted primes $p+s \le x$ that have a divisor in a given interval $(y,z]$. Kevin Ford has obtained upper bounds of the expected order of magnitude on this quantity as well as lower bounds in a special case of the parameters y and z. We supply here the corresponding lower bounds in a broad range of the parameters y and z. As expected, these bounds depend heavily on our knowledge about primes in arithmetic progressions. As an application of these bounds, we determine the number of shifted primes that appear in a multiplication table up to multiplicative constants.

1. Introduction

When one studies the multiplicative structure of the integers a natural and important question that arises is how many integers possess a divisor in a prescribed interval $(y,z]$. More precisely, for $y<z$ and $x\ge 1$ define

$$H(x,y,z) = |\{n \le x : \exists d|n \text{ with } y<d\le z\}|.$$

The study of this function was initiated by Besicovitch [2] and was further developed by Erdős [6], [7], [9] and Tenenbaum [24], [25], who obtained bounds on $H(x,y,z)$ in various cases of the parameters y and z. In his seminal paper [26] Tenenbaum focused on estimating $H(x,y,z)$ for all x,y,z and he obtained reasonably sharp bounds on it. A consequence of Tenenbaum's work was the realization that, for fixed x and y, as z varies in $(y,x]$ the behavior of $H(x,y,z)$ changes when z is around $y+y(\log y)^{-\log 4+1}$, $2y$ and y^2. The problem of establishing the correct order of magnitude of $H(x,y,z)$ was completely resolved by Ford in his profound work [11], where he discovered a striking connection between the distribution of the prime factors of integers with a divisor in $(y,z]$ and random walks with certain constraints. We state here the core of the main theorem in [11]. First, for a given pair (y,z) with $2\le y<z$ define η,u,β and ξ by

$$z = e^{\eta} y = y^{1+u}, \quad \eta = (\log y)^{-\beta}, \quad \beta = \log 4 - 1 + \frac{\xi}{\sqrt{\log\log y}}. \tag{1.1}$$

Furthermore, put

$$z_0(y) = y\exp\{(\log y)^{-\log 4+1}\} \approx y + y(\log y)^{-\log 4+1},$$

$$G(\beta) = \begin{cases} \dfrac{1+\beta}{\log 2}\log\Big(\dfrac{1+\beta}{e\log 2}\Big)+1 & 0\le\beta\le\log 4-1, \\ \beta & \log 4-1\le\beta, \end{cases}$$

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and

$$\delta = 1 - \frac{1 + \log\log 2}{\log 2} = 0.086071\dots$$

Lastly, here and for the rest of this paper the notation $f \asymp g$ means that $f \ll g$ and $g \ll f$. Constants implied by $\ll$, $\gg$ and $\asymp$ are absolute unless otherwise specified, e.g. by a subscript.

Theorem 1.1 (Ford [11]). *Let $x > 100000$ and $100 \le y \le z-1$ with $z \le x$.*

(a) *If $y \le \sqrt{x}$, then*

$$\frac{H(x,y,z)}{x} \asymp \begin{cases} \log(z/y) = \eta & y+1 \le z \le z_0(y), \\ \dfrac{\beta}{\max\{1,-\xi\}(\log y)^{G(\beta)}} & z_0(y) \le z \le 2y, \\ u^{\delta}\left(\log \dfrac{2}{u}\right)^{-3/2} & 2y \le z \le y^2, \\ 1 & z \ge y^2. \end{cases}$$

(b) *If $y > \sqrt{x}$, then*

$$H(x,y,z) \asymp \begin{cases} H\left(x, \dfrac{x}{z}, \dfrac{x}{y}\right) & \frac{x}{y} \ge \frac{x}{z} + 1, \\ \eta x & \textit{else}. \end{cases}$$

When the interval $(y, z]$ is relatively short, Tenenbaum established an asymptotic formula for $H(x, y, z)$.

Theorem 1.2 (Tenenbaum [26]). *If $z \le \sqrt{x}$ and $\xi \to \infty$, then*

$$H(x,y,z) \sim \eta x \quad (y \to \infty,\ z - y \to \infty).$$

A natural generalization of $H(x, y, z)$ arises from restricting the range of n to be some subset of the natural numbers $\mathscr{A}$. To this end we define

$$H(x,y,z;\mathscr{A}) = |\{n \in [0,x] \cap \mathscr{A} : \exists d|n \text{ with } y < d \le z\}|.$$

If $\mathscr{A}$ is reasonably well-distributed in arithmetic progressions, then a simple heuristic shows that we should have

$$H(x,y,z;\mathscr{A}) \approx \frac{|\mathscr{A} \cap [0,x]|}{x} H(x,y,z).$$

In the case that $\mathscr{A}$ is an arithmetic progression Ford, Khan, Shparlinski and Yankov [12] obtained upper bounds on $H(x, y, z; \mathscr{A})$. In the present paper we focus on the special and important case when $\mathscr{A} = P_s := \{p + s : p \text{ prime}\}$ for fixed $s \ne 0$. It is well-known that P_s is well-distributed in arithmetic progressions $a \pmod q$ with $(a - s, q) = 1$. Making this precise using sieving arguments and combining it with the methods developed in [11] can lead to bounds on $H(x, y, z; P_s)$ of the expected order of magnitude. The upper bounds were settled by Ford in [11]. We state below a short interval version of Theorem 6 in [11]; for a proof of it see the proofs of Theorem 6 and Lemma 6.1 in [11].

Theorem 1.3 (Ford [11])**.** *Fix $s \in \mathbb{Z} \setminus \{0\}$. Let $2 \le y \le \sqrt{x}$, $y+1 \le z \le x$ and $x(\log z)^{-10} \le \Delta \le x$. Then*

$$H(x,y,z;P_s) - H(x-\Delta,y,z;P_s) \ll_s \begin{cases} \dfrac{\Delta}{x}\dfrac{H(x,y,z)}{\log x} & z \ge y + (\log y)^{2/3}, \\ \dfrac{\Delta}{\log x} \displaystyle\sum_{\substack{y<d\le z \\ (d,s)=1}} \dfrac{1}{\phi(d)} & z \le y + (\log y)^{2/3}. \end{cases}$$

Remark 1.1. The reason that the upper bound in Theorem 1.3 has this particular shape is due to our incomplete knowledge about the sum $\sum_{y<d\le z} \frac{1}{\phi(d)}$ when the interval $(y, z]$ is very short. The main theorem in [23] implies that

$$\sum_{y<d\le z} \frac{1}{\phi(d)} \asymp \log(z/y) \quad (z \ge y + (\log y)^{2/3}),$$

whereas standard conjectures on Weyl sums would yield that

$$\sum_{y<d\le z} \frac{1}{\phi(d)} \asymp \log(z/y) \quad (z \ge y + \log\log y). \tag{1.2}$$

The range of y and z in (1.2) is the best possible one can hope for, since it is well-known that the order of $n/\phi(n)$ can be as large as $\log\log n$ if n has many small prime factors.

In general, lower bounds on $H(x,y,z;P_s)$ are more difficult because they rely on more precise knowledge about the distribution of primes in arithmetic progressions, which is a notoriously difficult problem. A special case was worked out by Ford.

Theorem 1.4 (Ford [11])**.** *For fixed s, a, b with $s \in \mathbb{Z} \setminus \{0\}$ and $0 \le a < b \le 1$ we have*

$$H(x, x^a, x^b; P_s) \gg_{s,a,b} \frac{x}{\log x}.$$

The purpose of this paper is to provide lower bounds on $H(x,y,z;P_s)$ in a broader range of the parameters y and z. We split our results according to the range of the parameter $\eta = \log(z/y)$. For small values of η lower bounds on $H(x,y,z;P_s)$ depend heavily on inequalities of the form

$$\pi(x;q,a) \ge \frac{cx}{\phi(q)\log x} \quad \text{for} \quad (a,q)=1 \tag{1.3}$$

for some $c > 0$, uniformly in some range of q with a possible 'small' exceptional set, namely reverse Brun-Titchmarsh inequalities. Such results have been proven by Alford, Granville and Pomerance [1] and Harman [16]. Also, Bombieri, Friedlander and Iwaniec proved in [3] an asymptotic formula for

$$\sum_{\substack{q\le Q \\ (q,a)=1}} \pi(x;q,a),$$

when $Q \le x^{1-\epsilon}$ and a is fixed. Combining these results with the arguments leading to Theorem 1.2 we show the following theorem. Here and for the rest of this paper $x_0(\cdot)$

denotes a sufficiently large positive constant which depends only on the parameters given, e.g. $x_0(s)$, and its meaning might change from statement to statement.

Theorem 1.5 (Small values of η). *Fix $s \in \mathbb{Z} \setminus \{0\}$. Let $3 \le y+1 \le z \le x$ with $y \le \sqrt{x}$ and $\{y < d \le z : (d,s) = 1\} \ne \emptyset$.*

(a) *Let $\epsilon > 0$. If $x \ge x_0(s,\epsilon)$, $z \le x^{5/12-\epsilon}$ and*

$$y + \log\log y \le z \le y + \frac{y}{(\log y)^2},$$

then

$$H(x,y,z;P_s) \gg \begin{cases} \dfrac{H(x,y,z)}{\log x} & z \ge y + (\log y)^{2/3}, \\ \dfrac{x}{\log x} \displaystyle\sum_{\substack{y<d\le z \\ (d,s)=1}} \frac{1}{\phi(d)} & z \le y + (\log y)^{2/3} \end{cases} \tag{1.4}$$

with the implied constant depending on s and ϵ. If, in addition, $(z-y)/\log\log y \to \infty$ as $y \to \infty$, then

$$H(x,y,z;P_s) \sim_{\epsilon,s} \begin{cases} f(s)\dfrac{315\zeta(3)}{2\pi^4}\dfrac{\eta x}{\log x} & \text{if } \dfrac{z-y}{(\log y)^{2/3}} \to \infty, \\ \dfrac{x}{\log x} \displaystyle\sum_{\substack{y<d\le z \\ (d,s)=1}} \frac{1}{\phi(d)} & \text{otherwise}, \end{cases}$$

as $y \to \infty$, where $f(s) = \prod_{p|s} \frac{(p-1)^2}{p^2-p+1}$.

(b) *If $x \ge x_0(s)$, $z \le x^{0.472}$ and*

$$y + \exp\{4.532(\log y)^{1/4}\} \le z \le y + \frac{y}{(\log y)^2},$$

then (1.4) holds with the implied constant depending on s.

(c) *If (1.3) holds for some $c > 0$, uniformly in $q \le Q$ for some $Q = Q(x) \le \sqrt{x}$, $x \ge x_0(s,c)$ and*

$$z \le y + \frac{y}{(\log y)^2},$$

then (1.4) is valid for $z \le Q$ with the implied constant depending on s and c.

(d) *Let $B \ge 2$ be fixed. If*

$$z \ge y + \frac{y}{(\log y)^B} \quad \text{and} \quad \xi \to \infty,$$

then

$$H(x,y,z;P_s) \sim_{s,B} f(s)\frac{315\zeta(3)}{2\pi^4}\frac{\eta x}{\log x} \quad (y \to \infty).$$

For intermediate and large values of η we need results about primes in arithmetic progressions *on average* in order to control error terms coming from the linear sieve. The most famous such result is the Bombieri-Vinogradov theorem [4, p. 161]. This theorem allows one to get the expected order of $H(x,y,z;P_s)$ for $y \le x^{1/2-\epsilon}$. To go beyond this threshold we make use of Theorem 9 in [3].

Theorem 1.6 (Intermediate and large values of η; short intervals)**.** *Fix $s \in \mathbb{Z} \setminus \{0\}$ and $B \ge 2$. Let $x \ge x_0(s,B)$, $x(\log x)^{-B} \le \Delta \le x$ and $3 \le y+1 \le z \le x$ with $\{y < d \le z : (d,s)=1\} \ne \emptyset$, $y \le \sqrt{x}$ and*

$$z \ge y + \frac{y}{(\log y)^B}.$$

Then

$$H(x,y,z;P_s) - H(x-\Delta,y,z;P_s) \gg_{s,B} \frac{\Delta}{x}\frac{H(x,y,z)}{\log x}.$$

We may combine Theorems 1.3 and 1.6 with an argument given in [11] to obtain the expected order of $H(x,y,z;P_s)$ in the full range of the parameters y and z, when $\eta \ge (\log y)^{-B}$ for some fixed $B \ge 2$.

Theorem 1.7 (Intermediate and large values of η)**.** *Fix $s \in \mathbb{Z} \setminus \{0\}$ and $B \ge 2$. Let $x \ge x_0(s,B)$ and $3 \le y+1 \le z \le x$ with $\{y < d \le z : (d,s)=1\} \ne \emptyset$ and*

$$z \ge y + \frac{y}{(\log y)^B}.$$

Then

$$H(x,y,z;P_s) \asymp_{s,B} \frac{H(x,y,z)}{\log x}.$$

Finally, when η is very large we are able to establish an asymptotic formula for $H(x,y,z;P_s)$, similar to the one given for $H(x,y,z)$ in Theorem 21(iv) of [15].

Theorem 1.8 (Very large values of η)**.** *Let $s \in \mathbb{Z} \setminus \{0\}$. If $2 \le y \le z \le x$, then*

$$H(x,y,z;P_s) = \frac{x}{\log x}\left(1 + O_s\left(\frac{\log y}{\log z}\right)\right).$$

Shifted primes in the multiplication table. A straightforward application of Theorem 1.7 is to the multiplication table problem. This problem, which was first posed by Erdős [8],[9], is to count the number of distinct integers of the form ab with $1 \le a,b \le N$, namely to estimate the quantity

$$A(N) := |\{ab : 1 \le a,b \le N\}|.$$

A related question is to estimate

$$A(N;P_s) := |\{ab \in P_s : 1 \le a,b \le N\}|,$$

that is how many shifted primes appear in the multiplication table. The order of $A(N)$ was determined by Ford in [11], where he proved that

$$A(N) \asymp \frac{N^2}{(\log N)^{\delta}(\log\log N)^{3/2}}.$$

This follows by the elementary inequalities

$$H\Big(\frac{N^2}{2},\frac{N}{2},N\Big) \le A(N) \le \sum_{m\ge0} H\Big(\frac{N^2}{2^m},\frac{N}{2^{m+1}},\frac{N}{2^m}\Big)$$

and Theorem 1.1. Similarly, using Theorem 1.7 we establish the order of magnitude of $A(N;P_s)$.

Corollary 1.1. *If $N \ge N_0(s)$, then*

$$A(N;P_s) \asymp_s \frac{A(N)}{\log N}.$$

2. Background material

Notation. We make use of some standard notation. If $a(n)$, $b(n)$ are two arithmetic functions, then we denote with $a*b$ their Dirichlet convolution. Furthermore, for $n\in\mathbb{N}$ and $1\le y\le z$ we put $\omega(n;y,z)=|\{p \text{ prime}: p|n,\ y<p\le z\}|$ and $\Omega(n;y,z)=\sum\{a:p^a\|n,\ y<p\le z\}$, where $p^a\|n$ means that $p^a|n$ and $p^{a+1}\nmid n$. Also, for brevity let $\omega(n;z)=\omega(n;1,z)$ and $\Omega(n;z)=\Omega(n;1,z)$. For $n\in\mathbb{N}$ we use $P^+(n)$ and $P^-(n)$ to denote the largest and smallest prime factor of n, respectively, with the notational conventions that $P^+(1)=0$ and $P^-(1)=+\infty$. Given $1\le y<z$, $\mathscr{P}(y,z)$ denotes the set of all integers n such that $P^+(n)\le z$ and $P^-(n)>y$. In addition, $\pi(x;q,a)$ stands for the number of primes up to x in the arithmetic progression $a \pmod q$. Lastly, for a Dirichlet character χ, $N(\sigma,V,\chi)$ denotes the number of zeros $\rho=\beta+i\gamma$ of its associated L-function with $|\gamma|\le V$ and $\beta\ge\sigma$.

In this section we state various preliminary results that are needed in order to prove Theorems 1.5, 1.6, 1.7 and 1.8. First, we list a series of results on primes in arithmetic progressions. We start with a lemma which is a direct corollary of Theorem 2.1 in [1].

Lemma 2.1. *Let $\epsilon\in(0,1/12)$. There exists $x_\epsilon\ge1$ such that for every $x\ge x_\epsilon$, there is a set $\mathcal{D}_\epsilon(x)\subset\mathbb{N}\cap[\log x,x]$ with $|\mathcal{D}_\epsilon(x)|\ll_\epsilon 1$ such that for every $(a,q)=1$ with $q\le x^{5/12-\epsilon}$,*

$$\Big|\pi(x;q,a)-\frac{\mathrm{li}(x)}{\phi(q)}\Big| \le \epsilon\frac{\mathrm{li}(x)}{\phi(q)},$$

with the possible exception of $q\in\mathcal{M}\mathcal{D}_\epsilon(x)=\{md: m\in\mathbb{N}, d\in\mathcal{D}_\epsilon(x)\}$.

Harman [16], allowing a larger set of exceptional moduli, gave a variation of Lemma 2.1. His starting point is the following result.

Lemma 2.2. *Given $\epsilon>0$, there are constants $K(\epsilon)\ge2$ and $c(\epsilon)>0$ such that if $K(\epsilon)<q<x^{0.472}$ and for every $d|q$ with χ a primitive character $\pmod d$ we have*

$$L(\sigma+it,\chi)\ne0 \quad \text{for } \sigma>1-\frac{1}{(\log q)^{3/4}},\ |t|\le\exp\{\epsilon(\log q)^{3/4}\},$$

then for any a with $(a,q)=1$ we have

$$\pi(x;q,a)\ge\frac{c(\epsilon)x}{\phi(q)\log x}.$$

Using Lemma 2.2 along with estimates on averages of $N(\sigma, V, \chi)$ Harman showed a variation of Lemma 2.1. The main part of the argument is given in [16], but the result is not stated explicitly; we state it and prove it here for the sake of completeness.

Lemma 2.3. *There exist absolute positive constants c_1, c_2 and x_0 so that for all $x \geq x_0$ there is a set $\mathcal{E}(x) \subset \mathbb{N} \cap [\log x, x]$ satisfying the following:*

(1) $|\mathcal{E}(x)| \leq \exp\{3.641(\log x)^{1/4}\}$;
(2) $|\mathcal{E}(x) \cap [1, \exp\{c_1(\log x)^{3/4}\}]| \ll 1$;
(3) *For every $(a,q) = 1$ with $q \leq x^{0.472}$ we have*

$$\pi(x; q, a) \geq \frac{c_2 x}{\phi(q) \log x},$$

with the possible exception of $q \in \mathcal{ME}(x) = \{me : m \in \mathbb{N}, e \in \mathcal{E}(x)\}$.

Proof. Set $W = (0.4166 \log x)^{3/4}$. From [4, p. 93, 95] there is an absolute constant c_1 such that there is at most one primitive character χ_1 to a modulus $q_1 \leq V = \exp\{c_1(\log x)^{3/4}\}$ whose L-function has a zero ρ with $|\mathrm{Im}(\rho)| \leq V$ and $\mathrm{Re}(\rho) > 1 - 1/W$. By [4, p. 96], this exceptional modulus q_1 satisfies $q_1 \geq \log x$. In addition, Montgomery showed in [20] that

$$\sum_{q \leq Q} \sideset{}{^*}\sum_{\chi \ (\mathrm{mod}\ q)} N(\sigma, V, \chi) \ll (Q^2 V)^{2(1-\sigma)/\sigma} (\log QV)^{13} \quad (4/5 \leq \sigma \leq 1), \tag{2.1}$$

where $\sum^*$ means that the sum runs over primitive characters only. Inequality (2.1) with $Q = x^{0.472}$ and $\sigma = 1 - 1/W$ yields that $N(\sigma, V, \chi) = 0$ for all primitive characters to every moduli $q \leq x^{0.472}$ with at most $\exp\{3.64094(\log x)^{1/4}\}$ exceptions. Call this exceptional set $\mathcal{E}_1(x)$. This set contains no elements $\leq \log x$ and at most one element $\leq V$, by the discussion in the beginning of the proof. Next, applying Lemma 2.1 with $\epsilon_0 = 2/3 \times 10^{-4}$ we obtain a set $\mathcal{D}_{\epsilon_0}(x) \subset [\log x, x]$ with boundedly many elements and the property that if $q \leq x^{0.4166}$ and $q \notin \mathcal{MD}_{\epsilon_0}(x)$, then

$$\pi(x; q, a) \geq (1 - \epsilon_0) \frac{x}{\phi(q) \log x} \quad \text{for} \quad (a, q) = 1. \tag{2.2}$$

Set

$$\mathcal{E}(x) = \mathcal{E}_1(x) \cup \mathcal{D}_{\epsilon_0}(x).$$

Clearly, conditions (1) and (2) hold for $\mathcal{E}(x)$. Also, if $q \leq x^{0.4166}$ is such that $q \notin \mathcal{ME}(x)$, then (3) holds by (2.2). Finally, if $q \in [x^{0.4166}, x^{0.472}]$ and $q \notin \mathcal{ME}(x)$, then the hypothesis of Lemma 2.2 is met and we deduce (3). This completes the proof of the lemma. □

Below we state the Brun-Titchmarsh inequality [14, Theorem 3.7].

Lemma 2.4. *Uniformly in $1 \leq q < y \leq x$ and $(a, q) = 1$ we have that*

$$\pi(x; q, a) - \pi(x - y; q, a) \ll \frac{y}{\phi(q) \log(2y/q)}.$$

In addition, we will need a generalization of Lemma 2.4, which is an easy application of the results and methods in [22]. Let $\mathcal{M}$ denote the class of functions $F:\mathbb{N}\to[0,+\infty)$ for which there exist constants A_F and $B_{F,\epsilon}$, $\epsilon>0$, such that

$$F(nm)\le\min\{A_F^{\Omega(m)},B_{F,\epsilon}m^{\epsilon}\}F(n)$$

for all $(m,n)=1$ and all $\epsilon>0$.

Lemma 2.5. *Let $F\in\mathcal{M}$, $a\in\mathbb{Z}\setminus\{0\}$ and $1\le q\le h\le x$ such that $(a,q)=1$ and $x>|a|$. If $q\le x^{1-\epsilon}$ and $\frac{h}{q}\ge(\frac{x-a}{q})^{\epsilon}$ for some $\epsilon>0$, then*

$$\sum_{\substack{x-h<p\le x\\ p\equiv a \pmod q}}F\Big(\frac{p-a}{q}\Big)\ll_{a,\epsilon,F}\frac{h}{\phi(q)(\log x)^2}\sum_{n\le x}\frac{F(n)}{n};$$

the implied constant depends on F only via the constants A_F and $B_{F,\alpha}$, $\alpha>0$.

Proof. Observe that it suffices to show the lemma for the function $\widetilde{F}$ defined for $n=2^r m$ with $(m,2)=1$ by

$$\widetilde{F}(n)=\min\{A_F^r,\min_{\epsilon>0}(B_{F,\epsilon}2^{r\epsilon})\}F(m).$$

We have that $\widetilde{F}\in\mathcal{M}$ with parameters A_F and $B_{F,\alpha}^2$, $\alpha>0$. Without loss of generality we may assume that $\widetilde{F}(1)=1$. Also, suppose that $x\ge x_0(\epsilon,a,F)$, where $x_0(a,\epsilon,F)$ is a sufficiently large constant; otherwise, the result is trivial. Put

$$q_1=\begin{cases}q & \text{if } 2|aq\\ 2q, & \text{if } 2\nmid aq,\end{cases}$$

and let $X=(x-a)/q_1$ and $H=h/q_1$. Note that if $p\equiv a\pmod q$ and $p>2$, then $p\equiv a\pmod{q_1}$. So if we set $p=q_1m+a$ for $p>2$, then

$$\begin{aligned}\sum_{\substack{x-h<p\le x\\ p\equiv a \pmod q}}\widetilde{F}\Big(\frac{p-a}{q}\Big)&\le\sum_{\substack{X-H<m\le X\\ P^-(q_1m+a)>\sqrt{X}}}\widetilde{F}\Big(\frac{q_1}{q}m\Big)+\sum_{\substack{X-H<m\le X\\ 3\le q_1m+a\le\sqrt{X}}}\widetilde{F}\Big(\frac{q_1}{q}m\Big)+O_{a,F}(1)\\ &\ll_{a,F}\sum_{\substack{X-H<m\le X\\ P^-(q_1m+a)>\sqrt{X}}}\widetilde{F}(m)+\sum_{\substack{X-H<m\le X\\ m\le\sqrt{X}-a}}\widetilde{F}(m)+1,\end{aligned}$$

since $q_1/q\in\{1,2\}$ and $\widetilde{F}(2m)\ll_F\widetilde{F}(m)$ for all $m\in\mathbb{N}$. Let $F_1(n)=\widetilde{F}(n)$ and $F_2(n)$ be the characteristic function of integers n such that $P^-(n)>\sqrt{X}$. Let $Q_1(x)=x$, $Q_2(x)=q_1x+a$ and $Q=Q_1Q_2$. Also, if $P(x)\in\mathbb{Z}[x]$, then let $\rho_P(m)$ be the number of solution of the congruence $P(x) \equiv 0 \pmod{m}$. By Corollary 3 in [22], we have that

$$\sum_{\substack{X-H<m\le X\\ P^-(q_1m+a)>\sqrt{X}}} \widetilde{F}(m) = \sum_{X-H<m\le X} F_1(m)F_2(mq_1+a)$$

$$\ll_{a,\epsilon,F} H \prod_{p\le X}\Big(1-\frac{\rho_Q(p)}{p}\Big)\prod_{j=1}^{2}\sum_{n\le X}\frac{F_j(n)\rho_{Q_j}(n)}{n} \tag{2.3}$$

$$\ll_{a,\epsilon} \frac{h}{\phi(q)}\frac{1}{\log^2 x}\sum_{n\le X}\frac{\widetilde{F}(n)}{n},$$

since $q\le x^{1-\epsilon}$ and the discriminant of Q depends only on a. Also, if the sum

$$\sum_{\substack{X-H<m\le X\\ m\le\sqrt{X}-a}} \widetilde{F}(m)$$

is non-zero, then $H\ge X/2$. In this case Corollary 3 in [22] implies that

$$\sum_{\substack{X-H<m\le X\\ m\le\sqrt{X}-a}} \widetilde{F}(m) \ll_{a,\epsilon,F} \frac{\sqrt{X}}{\log X}\sum_{n\le X}\frac{\widetilde{F}(n)}{n} \ll_{a,\epsilon} \frac{h}{q\log^2 x}\sum_{n\le X}\frac{\widetilde{F}(n)}{n},$$

which together with (2.3) completes the proof of the lemma. □

Using Lemma 2.5 we prove the following estimate.

Lemma 2.6. *Let* $1\le v\le v_0<2$, $a\in\mathbb{Z}\setminus\{0\}$, $1\le q\le x$ *and* $3/2\le y\le (x-a)/q$ *with* $(a,q)=1$ *and* $x>|a|$. *If* $q\le x^{1-\epsilon}$ *for some* $\epsilon>0$, *then*

$$\sum_{\substack{p\le x\\ p\equiv a \pmod q}} v^{\Omega(\frac{p-a}{q};y)} \ll_{a,\epsilon,v_0} \frac{x}{\phi(q)\log x}(\log y)^{v-1}.$$

Proof. We may assume that $x\ge x_0(a,\epsilon,v_0)$, where $x_0(a,\epsilon,v_0)$ is a sufficiently large constant. Let $X=(x-a)/q$ and write $v^{\Omega(n;y)-\omega(n;y)}=(1*b)(n)$, where b is the multiplicative function that satisfies

$$b(p^l)=\begin{cases}0 & \text{if } l=1 \text{ or } p>y,\\ v^{l-2}(v-1) & \text{if } l\ge 2 \text{ and } p\le y.\end{cases}$$

Then

$$v^{\Omega(n;y)} = v^{\omega(n;y)}\sum_{kf=n} b(k) \le \sum_{kf=n} b(k)v^{\omega(k;y)}v^{\omega(f;y)}$$

and consequently

$$\sum_{\substack{p\le x\\ p\equiv a \pmod q}} v^{\Omega(\frac{p-a}{q};y)} \le \sum_{k\le X} v^{\omega(k;y)}b(k) \sum_{\substack{p\le x\\ p\equiv a \pmod{qk}}} v^{\omega(\frac{p-a}{qk};y)}. \tag{2.4}$$

If $k \le \sqrt{X}$, then $kq \le x^{1-\epsilon/3}$. So Lemma 2.5 implies that

$$\sum_{\substack{p\le x\\ p\equiv a (\mathrm{mod}\ qk)}} v^{\omega(\frac{p-a}{qk};y)} \ll_{a,\epsilon} \frac{x(\log y)^{v-1}}{\phi(kq)\log x}.$$

If $k > \sqrt{X}$, then

$$\sum_{\substack{p\le x\\ p\equiv a (\mathrm{mod}\ qk)}} v^{\omega(\frac{p-a}{qk};y)} \le \sum_{m\le X/k} v^{\omega(m)} \ll_{a,\epsilon} \frac{x(\log X)^{v-1}}{kq},$$

by Theorem 01 in [15]. Hence the right hand side of (2.4) is

$$\ll_{a,\epsilon} \frac{x(\log y)^{v-1}}{\phi(q)\log x}\sum_{k\le\sqrt{X}}\frac{v^{\omega(k;y)}b(k)}{\phi(k)} + \frac{x(\log X)^{v-1}}{qX^{\alpha/2}}\sum_{\sqrt{X}<k\le X}\frac{v^{\omega(k;y)}b(k)k^{\alpha}}{k} \ll_{a,\epsilon,v_0} \frac{x(\log y)^{v-1}}{\phi(q)\log x},$$

provided that $0<\alpha<1/2$ and $2^{1-\alpha}>v_0$, which completes the proof. □

We complete the results about primes in arithmetic progressions with the following estimate.

Lemma 2.7. *Let $a\in\mathbb{Z}\setminus\{0\}$, $\epsilon>0$ and $A>0$. There exists $B=B(A)$ such that if $R\le x^{1/10-\epsilon}$ and $QR<x(\log x)^{-B}$, then*

$$\sum_{\substack{r\le R\\ (r,a)=1}}\left|\sum_{\substack{q\le Q\\ (q,a)=1}}\left(\pi(x;rq,a)-\frac{\mathrm{li}(x)}{\phi(rq)}\right)\right| \ll_{a,\epsilon,A} \frac{x}{(\log x)^A}.$$

Proof. Use Theorem 9 in [3] plus partial summation. □

We need an estimate on the summatory function of the reciprocals of Euler's ϕ function and other closely related quantities. Such a result was proved by Sitaramachandra [23]. Using the methods of [23] we extend this result according to the needs of this paper.

Lemma 2.8. *Let $a\in\mathbb{N}$, $s\in\mathbb{Z}$ and $x\ge 1$ such that $1\le|s|\le x$. Then*

$$\sum_{\substack{n\le x\\ (n,s)=1}}\frac{\phi(a)}{\phi(an)} = \frac{315\zeta(3)}{2\pi^4}\frac{\phi(s)}{|s|}g(as)\Big(\log x+\gamma-\sum_{p\nmid as}\frac{\log p}{p^2-p+1}+\sum_{p|s}\frac{\log p}{p-1}\Big)$$
$$+O\Big(\tau(s)\frac{a|s|}{\phi(as)}\frac{(\log 2x)^{2/3}}{x}\Big),$$

where $g(as)=\prod_{p|as}\frac{p(p-1)}{p^2-p+1}$.

Proof. Since the proof of this part is along the same lines with the proof of the main result in [23], we simply sketch it. Let $P(x)=\{x\}-1/2$, where $\{x\}$ denotes the fractional part of x. Then using the estimate

$$\sum_{n\le x}\frac{P(x/n)}{n} \ll (\log 2x)^{2/3},$$

which was proved in [27, p. 98], along with an argument similar to the one leading to Lemma 2.2 in [23], we find that

$$\sum_{\substack{n\le x\\(n,m)=1}}\frac{\mu^2(n)}{\phi(n)}P(x/n)\ll\frac{|m|}{\phi(m)}(\log 2x)^{2/3} \tag{2.5}$$

for every $m\in\mathbb{Z}\setminus\{0\}$. Also, by the Euler-McLaurin's summation formula we have

$$\sum_{n\le x}\frac{1}{n}=\log x+\gamma-\frac{P(x)}{x}+O\Big(\frac{1}{x^2}\Big). \tag{2.6}$$

Observe that the arithmetic function $n\to\phi(a)/\phi(an)$ is multiplicative. In particular, we have that

$$\frac{\phi(a)}{\phi(an)}=\sum_{\substack{kf=n\\(k,a)=1}}\frac{\mu^2(k)}{k\phi(k)f}. \tag{2.7}$$

Using relations (2.5), (2.6) and (2.7) and estimating the error terms as in [23] gives us that

$$\begin{aligned}\sum_{\substack{n\le x\\(n,s)=1}}\frac{\phi(a)}{\phi(an)}&=\sum_{\substack{k\le x\\(k,as)=1}}\frac{\mu^2(k)}{k\phi(k)}\sum_{\substack{f\le x/k\\(f,s)=1}}\frac{1}{f}=\sum_{d|s}\frac{\mu(d)}{d}\sum_{\substack{k\le x/d\\(k,as)=1}}\frac{\mu^2(k)}{k\phi(k)}\sum_{l\le x/kd}\frac{1}{l}\\&=\sum_{d|s}\frac{\mu(d)}{d}\sum_{\substack{k\le x/d\\(k,as)=1}}\frac{\mu^2(k)}{k\phi(k)}\Big(\log\frac{x/d}{k}+\gamma-\frac{k}{x/d}P\Big(\frac{x/d}{k}\Big)+O\Big(\frac{k^2}{(x/d)^2}\Big)\Big)\\&=\sum_{d|s}\frac{\mu(d)}{d}\sum_{\substack{k=1\\(k,as)=1}}^{\infty}\frac{\mu^2(k)}{k\phi(k)}\Big(\log\frac{x/d}{k}+\gamma\Big)+O\Big(\frac{\tau(s)a|s|}{\phi(as)}\frac{(\log 2x)^{2/3}}{x}\Big),\end{aligned}$$

since $|s|\le x$. Finally, a simple calculation and the identity

$$\sum_{k=1}^{\infty}\frac{\mu^2(k)}{k\phi(k)}=\frac{315\zeta(3)}{2\pi^4}$$

complete the proof. □

The following result is known as the 'fundamental lemma' of sieve methods. It has appeared in the literature in several different forms (see for example [14, Theorem 2.5, p. 82]). We need a version of it that can be found in [13] and [19].

Lemma 2.9. *Let $D\ge 2$, $D=Z^v$ with $v\ge 3$.*

(a) *Fix $\kappa>0$. There exist two sequences $\{\lambda^+(d)\}_{d\le D}$, and $\{\lambda^-(d)\}_{d\le D}$ such that*

$$|\lambda^{\pm}(d)|\le 1,$$

$$\begin{cases}(\lambda^-*1)(n)=(\lambda^+*1)(n)=1 & \textit{if } P^-(n)>Z,\\(\lambda^-*1)(n)\le 0\le(\lambda^+*1)(n) & \textit{otherwise,}\end{cases}$$

and, for any multiplicative function $\alpha(d)$ *with* $0 \le \alpha(p) \le \min\{\kappa, p-1\}$,

$$\sum_{d \le D} \lambda^{\pm}(d) \frac{\alpha(d)}{d} = \prod_{p \le Z} \Big(1 - \frac{\alpha(p)}{p}\Big)(1 + O_\kappa(e^{-v})).$$

(b) *There exists a sequence* $\{\rho(d)\}_{d \le D}$ *such that*

$$|\rho(d)| \le 1, \tag{2.8}$$

$$\begin{cases} (\rho * 1)(n) = 1 & \text{if } P^-(n) > Z, \\ (\rho * 1)(n) \le 0 & \text{otherwise,} \end{cases} \tag{2.9}$$

and, for any multiplicative function $\alpha(d)$ *satisfying* $0 \le \alpha(p) \le p - 1$ *and*

$$\prod_{y < p \le w} \Big(1 - \frac{\alpha(p)}{p}\Big)^{-1} \le \frac{\log w}{\log y}\Big(1 + \frac{L}{\log y}\Big) \quad (3/2 \le y \le w), \tag{2.10}$$

we have

$$\sum_{d \le D} \rho(d) \frac{\alpha(d)}{d} \gg \prod_{p \le Z} \Big(1 - \frac{\alpha(p)}{p}\Big), \tag{2.11}$$

provided that $D \ge D_0(L)$, *where* $D_0(L)$ *is a constant depending only on* L.

Proof. (a) The result follows by [13, Lemma 5, p. 732].

(b) The construction of the sequence $\{\rho(d)\}_{d \le D}$ and the proof that it satisfies the desired properties is based on [13, Lemma 5] and [19, Lemma 3]. We sketch the proof below. Without loss of generality we may assume that $Z \notin \mathbb{N}$. Set $P(Z) = \prod_{p<Z} p$ and $\rho(d) = \mu(d)\mathbf{1}_A(d)$, where $\mathbf{1}_A$ is the characteristic function of the set

$$A = \{d | P(Z) : d = p_1 \cdots p_r,\ p_r < \cdots < p_1 < Z,\ p_{2l}^3 p_{2l-1} \cdots p_1 < D\ (1 \le l \le r/2)\}.$$

By the proof of Lemma 5 in [13], the sequence $\{\rho(d)\}_{d=1}^\infty$ is supported in $\{d \in \mathbb{N} : d < D\}$ and satisfies (2.8) and (2.9). Finally, by Lemma 3 in [19], there exists a function h, independent of the parameters D, Z and L, such that

$$\sum_{d \le D} \rho(d) \frac{\alpha(d)}{d} \ge \big(h(v) + O(e^{\sqrt{L} - v} (\log D)^{-1/3})\big) \prod_{p < Z} \Big(1 - \frac{\alpha(p)}{p}\Big)$$

for all multiplicative functions $\alpha(d)$ that satisfy $0 \le \alpha(p) \le p - 1$ and (2.10). In addition, h is increasing and $h(3) > 0$, by [18, p. 172-173]. This proves that (2.11) holds too and completes the proof of the lemma. □

We now introduce some notation we will be utilizing later. For a and k in $\mathbb{N}$ and $1 \le y < z$ define

$$\tau(a) = |\{d \in \mathbb{N} : d | n\}|, \quad \tau(a, y, z) = |\{d \in \mathbb{N} : d | n, y < d \le z\}|$$

and

$$\tau_k(a) = |\{(d_1, \dots, d_k) \in \mathbb{N}^k : d_1 \cdots d_k = a\}|.$$

Moreover, for $\sigma > 0$ let

$$\mathscr{L}(a;\sigma) = \{x \in \mathbb{R} : \tau(a, e^x, e^{x+\sigma}) \ge 1\} = \bigcup_{d|a} [\log d - \sigma, \log d)$$

and

$$L(a;\sigma) = \mathrm{meas}(\mathscr{L}(a;\sigma)),$$

where 'meas' denotes the Lebesgue measure on the real line. We note the straightforward inequality

$$L(ab;\sigma) \le \tau(a)L(b;\sigma) \quad \text{for} \quad (a,b) = 1, \tag{2.12}$$

which is item (ii) of Lemma 3.1 in [11].

When η is in the intermediate range of values, the basic result we will use to bound $H(x, y, z; P_s)$ from below is the following estimate.

Lemma 2.10. *Let $\epsilon > 0$, $B > 0$, $x \ge 1$, $3 \le y+1 \le z$ with $z \le x^{2/3}$ and $\eta \in [(\log y)^{-B}, \frac{\log y}{100}]$. Then*

$$H(x, y, z) \asymp_{\epsilon,B} \frac{x}{\log^2 y} \sum_{\substack{a \le y^\epsilon \\ \mu^2(a)=1}} \frac{L(a;\eta)}{a}.$$

The proof of Lemma 2.10 can be found in [11]. Even though this result is not stated explicitly, it is a direct corollary of the methods there: see Theorem 1 and Lemmas 4.1, 4.2, 4.5, 4.8 and 4.9 in [11]. Also, we will need the following result, which is Corollary 1 in [11].

Lemma 2.11. *Suppose $x_1, y_1, z_1, x_2, y_2, z_2$ are real numbers with $2 \le y_i + 1 \le z_i \le x_i$ ($i = 1, 2$), $\log(z_1/y_1) \asymp \log(z_2/y_2)$, $\log y_1 \asymp \log y_2$ and $\log(x_1/z_1) \asymp \log(x_2/z_2)$. Then*

$$\frac{H(x_1, y_1, z_1)}{x_1} \asymp \frac{H(x_2, y_2, z_2)}{x_2}.$$

Finally, we state a covering lemma, which a special case of Lemma 3.15 in [10]. Here for I an interval of the real line we denote by rI the interval that has the same center as I and r times its diameter.

Lemma 2.12. *Let $A = \bigcup_{n=1}^N I_n \subset \mathbb{R}$, where the sets I_n are nonempty intervals of the form $[a, b)$. Then there exists a subcollection $\{I_{i_m}\}_{m=1}^M$ of mutually disjoint intervals such that*

$$A \subset \bigcup_{m=1}^M 3I_{i_m}.$$

3. Small values of η

In this section we give the proof of Theorem 1.5. First, we show an auxiliary result.

Lemma 3.1. *Let $a \in \mathbb{Z} \setminus \{0\}$, $x \ge 2$ and $3 \le Q_1 + 1 \le Q_2 \le 2Q_1$ with $Q_1 \le \sqrt{x}$ and $\{Q_1 < q \le Q_2 : (q, a) = 1\} \ne \emptyset$.*

(a) *Let* $\epsilon \in (0, 1/12)$. *If* $x \ge x_0(a,\epsilon)$ *and* $Q_1 + \log\log Q_1 \le Q_2 \le x^{5/12-\epsilon}$, *then*

$$\sum_{\substack{Q_1<q\le Q_2\\(q,a)=1}} \pi(x;q,a) \gg_{a,\epsilon} \frac{x}{\log x} \sum_{\substack{Q_1<q\le Q_2\\(q,a)=1}} \frac{1}{\phi(q)}. \tag{3.1}$$

If, in addition, $(Q_2-Q_1)/\log\log Q_1 \to \infty$ *as* $Q_1\to\infty$, *then*

$$\sum_{\substack{Q_1<q\le Q_2\\(q,a)=1}} \pi(x;q,a) \sim_{a,\epsilon} \frac{x}{\log x} \sum_{\substack{Q_1<q\le Q_2\\(q,a)=1}} \frac{1}{\phi(q)} \quad (Q_1\to\infty). \tag{3.2}$$

(b) *If* $x \ge x_0(a)$ *and* $Q_1 + \exp\{4.532(\log Q_1)^{1/4}\} \le Q_2 \le x^{0.472}$, *then* (3.1) *holds with the implied constant depending only on* a.

(c) *Let* $B \ge 2$. *If* $Q_2 \ge Q_1 + Q_1(\log Q_1)^{-B}$, *then*

$$\sum_{\substack{Q_1<q\le Q_2\\(q,a)=1}} \pi(x;q,a) \sim_{a,B} f(a)\frac{315\zeta(3)}{2\pi^4}\frac{x\log(Q_2/Q_1)}{\log x}.$$

Proof. (a) For every $\epsilon_1 \in (0,\epsilon]$ and $x \ge x_{\epsilon_1}$ Lemmas 2.1 and 2.4 imply that

$$\begin{aligned}\sum_{\substack{Q_1<q\le Q_2\\(q,a)=1}} \pi(x;q,a) &= (1+\epsilon_1\theta)\mathrm{li}(x) \sum_{\substack{Q_1<q\le Q_2\\ q\notin\mathcal{MD}_{\epsilon_1}(x)\\(q,a)=1}} \frac{1}{\phi(q)} + O\Big(\frac{x}{\log x}\sum_{\substack{Q_1<q\le Q_2\\ q\in\mathcal{MD}_{\epsilon_1}(x)}}\frac{1}{\phi(q)}\Big)\\ &= (1+\epsilon_1\theta)\mathrm{li}(x) \sum_{\substack{Q_1<q\le Q_2\\(q,a)=1}} \frac{1}{\phi(q)} + O\Big(\frac{x}{\log x}\sum_{\substack{Q_1<q\le Q_2\\ q\in\mathcal{MD}_{\epsilon_1}(x)}}\frac{1}{\phi(q)}\Big),\end{aligned} \tag{3.3}$$

for some $|\theta|\le 1$. Fix $d \in \mathcal{D}_{\epsilon_1}(x)$. If $d \ge Q_2 - Q_1$, then the interval $(Q_1/d, Q_2/d]$ contains at most one integer and therefore

$$\sum_{Q_1/d<m\le Q_2/d} \frac{1}{\phi(dm)} \ll \frac{\log\log Q_1}{Q_1}. \tag{3.4}$$

On the other hand, if $d \le Q_2 - Q_1$, then

$$\sum_{Q_1/d<m\le Q_2/d} \frac{1}{\phi(dm)} \ll \frac{\log\log Q_1}{d}\log(Q_2/Q_1). \tag{3.5}$$

Since $d \ge \log x$ and $|\mathcal{D}_{\epsilon_1}(x)| \ll_{\epsilon_1} 1$, relations (3.4) and (3.5) yield that

$$\sum_{\substack{Q_1<q\le Q_2\\ q\in\mathcal{MD}_{\epsilon_1}(x)}} \frac{1}{\phi(q)} \ll_{\epsilon_1} \frac{\log\log Q_1}{Q_1} + \frac{\log\log Q_1}{\log x}\log(Q_2/Q_1). \tag{3.6}$$

Also,

$$\sum_{\substack{Q_1<q\le Q_2\\(q,a)=1}} \frac{1}{\phi(q)} \ge \sum_{\substack{Q_1<q\le Q_2\\(q,a)=1}} \frac{1}{q} \gg_a \log(Q_2/Q_1) \asymp \frac{Q_2-Q_1}{Q_1}, \tag{3.7}$$

uniformly in $Q_1+1\le Q_2\le 2Q_1$ with $\{Q_1<q\le Q_2:(q,a)=1\}\neq\emptyset$. The above inequality, (3.3) and (3.6) imply that

$$\sum_{\substack{Q_1<q\le Q_2\\(q,a)=1}}\pi(x;q,a)=(1+\epsilon_1\theta)\mathrm{li}(x)\sum_{\substack{Q_1<q\le Q_2\\(q,a)=1}}\frac{1}{\phi(q)}\Big(1+O_{a,\epsilon_1}\Big(\frac{\log\log Q_1}{\log x}+\frac{\log\log Q_1}{Q_2-Q_1}\Big)\Big).$$

This proves that (3.2) holds. Next, we show that (3.1) holds. Fix a large positive constant $M=M(\epsilon,a)$ with $M\ge x_\epsilon$. If $Q_1\le M$ and x is large enough, then

$$\sum_{\substack{Q_1<q\le Q_2\\(q,a)=1}}\pi(x;q,a)\ge\max_{\substack{Q_1<q\le Q_2\\(q,a)=1}}\pi(x;q,a)\gg_{a,\epsilon}\frac{x}{\log x}\asymp_{a,\epsilon}\frac{x}{\log x}\sum_{\substack{Q_1<q\le Q_2\\(q,a)=1}}\frac{1}{\phi(q)},$$

by our assumption that $\{Q_1<q\le Q_2:(q,a)=1\}\neq\emptyset$ and the Prime Number Theorem for arithmetic progressions [4, p. 123]. So we may suppose that $Q_1>M$. By (3.3), (3.6) and (3.7) with $\epsilon_1=\epsilon$ we deduce that

$$\sum_{\substack{Q_1<q\le Q_2\\(q,a)=1}}\pi(x;q,a)\ge\frac{x}{2\log x}\Big(\sum_{\substack{Q_1<q\le Q_2\\(q,a)=1}}\frac{1}{\phi(q)}-C_{a,\epsilon}\frac{\log\log Q_1}{Q_1}\Big)\tag{3.8}$$

for some positive constant $C_{a,\epsilon}$. We separate two cases. If

$$\sum_{\substack{Q_1<q\le Q_2\\(q,a)=1}}\frac{1}{\phi(q)}\ge 2C_{a,\epsilon}\frac{\log\log Q_1}{Q_1},\tag{3.9}$$

then (3.1) holds by (3.8). So assume that (3.9) fails. Then, by (3.7) and our assumption that $Q_2\ge Q_1+\log\log Q_1$, we have that

$$\frac{\log\log Q_1}{Q_1}\ll\log\frac{Q_2}{Q_1}\ll_a\sum_{\substack{Q_1<q\le Q_2\\(q,a)=1}}\frac{1}{q}\le\sum_{\substack{Q_1<q\le Q_2\\(q,a)=1}}\frac{1}{\phi(q)}\le 2C_{a,\epsilon}\frac{\log\log Q_1}{Q_1}.\tag{3.10}$$

Also, Lemma 2.1 implies that

$$\sum_{\substack{Q_1<q\le Q_2\\(q,a)=1}}\pi(x;q,a)\ge\frac{x}{2\log x}\sum_{\substack{Q_1<q\le Q_2\\(q,a)=1,\ q\notin\mathcal{MD}_\epsilon(x)}}\frac{1}{\phi(q)}\ge\frac{x}{2\log x}\Big(\sum_{\substack{Q_1<q\le Q_2\\(q,a)=1}}\frac{1}{q}-\sum_{\substack{Q_1<q\le Q_2\\q\in\mathcal{MD}_\epsilon(x)}}\frac{1}{q}\Big).\tag{3.11}$$

By the argument leading to (3.6) we find that

$$\sum_{\substack{Q_1<q\le Q_2\\q\in\mathcal{MD}_\epsilon(x)}}\frac{1}{q}\ll_\epsilon\frac{1}{Q_1}+\frac{\log(Q_2/Q_1)}{\log x}.\tag{3.12}$$

Inserting (3.10) and (3.12) into (3.11) proves (3.1) in the case that (3.9) does not hold too.

(b) When $Q_1\le x^{0.41666}<x^{5/12}$ the result follows from part (a). When $Q_1>x^{0.41666}$ note that

$$Q_2-Q_1\ge\exp\{4.532(\log Q_1)^{1/4}\}\ge\exp\{3.6411(\log x)^{1/4}\}.$$

So following a very similar argument with the one given in part (a) and using Lemma 2.3 in place of Lemma 2.1 we obtain the desired result.

(c) Apply Lemmas 2.7 and 2.8. □

We are now in position to prove Theorem 1.5.

Proof of Theorem 1.5. First, assume that $z \le y + y(\log y)^{-2}$. We treat all four parts of the theorem simultaneously. Let y_0 be a large constant, possibly depending on s, B, ϵ and c, the constant in (1.3), according to the assumptions of each of the parts (a), (b) and (c). If $y \le y_0$, then we trivially have that

$$H(x,y,z;P_s) \ge \max_{\substack{y<d\le z\\(d,s)=1}} \pi(x-s;d,-s) \asymp_{y_0} \frac{x}{\log x},$$

by our assumption that $\{y < d \le z : (d,s) = 1\} \ne \emptyset$ and the Prime Number Theorem for arithmetic progressions [4, p. 123]. So assume that $y > y_0$. By the inclusion-exclusion principle, we have that

$$\sum_{y<d\le z} \pi(x-s;d,-s) - \sum_{y<d_1<d_2\le z} \pi(x-s;[d_1,d_2],-s) \le H(x,y,z;P_s) \le \sum_{y<d\le z} \pi(x-s;d,-s). \tag{3.13}$$

Lemma 2.4 then implies that

$$H(x,y,z;P_s) = \sum_{\substack{y<d\le z\\(d,s)=1}} \pi(x-s;d,-s) + O\Big(\sum_{y<d_1<d_2\le z} \frac{x}{\log(2x/[d_1,d_2])\phi([d_1,d_2])}\Big). \tag{3.14}$$

In the sum over d_1 and d_2 in the right hand side of (3.14) set $m = (d_1, d_2)$ and $d_i = mt_i$, $i = 1, 2$. Since $t_1 + 1 \le t_2$, we get that $m \le d_2 - d_1 \le z - y$. Moreover, notice that

$$\log \frac{2x}{[d_1,d_2]} = \log \frac{2x}{t_1t_2m} \ge \log \frac{2xm}{z^2} \gg \frac{\log 2m \log x}{\log y},$$

uniformly in $y \le \sqrt{x}$. Therefore

$$\begin{aligned}\sum_{y<d_1<d_2\le z} \frac{1}{\log(2x/[d_1,d_2])\phi([d_1,d_2])} &\ll \frac{(\log y)(\log\log y)}{\log x} \sum_{m\le z-y} \frac{1}{m\log 2m} \sum_{y/m<t_1<t_2\le z/m} \frac{1}{t_1t_2}\\ &\le \frac{(\log y)(\log\log y)}{\log x} \sum_{m\le z-y} \frac{1}{m\log 2m} \Big(\sum_{y/m<t\le z/m} \frac{1}{t}\Big)^2\\ &\ll \frac{\eta^2(\log y)(\log\log y)^2}{\log x} \ll \frac{\eta}{\log x}\frac{(\log\log y)^2}{\log y},\end{aligned}$$

which combined with (3.14) yields that

$$H(x,y,z;P_s) = \sum_{\substack{y<d\le z\\(d,s)=1}} \pi(x-s;d,-s) + O_s\Big(\frac{\eta x}{\log x}\frac{(\log\log y)^2}{\log y}\Big).$$

The above estimate together with Lemma 3.1 and the inequality

$$\sum_{\substack{y<d\leq z\\(d,s)=1}}\frac{1}{\phi(d)}\geq\sum_{\substack{y<d\leq z\\(d,s)=1}}\frac{1}{d}\gg_s\eta,$$

which holds uniformly in $y+1\leq z$ with $\{y<d\leq z:(d,s)=1\}\neq\emptyset$, completes the proof of parts (a), (b) and (c) as well as of part (d) when $z\leq y+y(\log y)^{-2}$. It remains to show part (d) when $z>y+y(\log y)^{-2}$, in which case $(\log y)^{-2}\ll\eta\ll(\log y)^{-\log 4+1}$. First, by (3.13) and Lemma 3.1(c), we have that

$$H(x,y,z;P_s)\leq\sum_{\substack{y<d\leq z\\(d,s)=1}}\pi(x-s;d,-s)+O_s(1)\sim_s f(s)\frac{315\zeta(3)}{2\pi^4}\frac{\eta x}{\log x},$$

which proves the desired upper bound. For the lower bound, let χ be the characteristic function of integers n satisfying

$$\Omega(n;y)\leq L(y):=2\log\log y+\psi(y)(\log\log y)^{1/2},$$

where $\psi(y)\to\infty$ as $y\to\infty$ and $\psi(y)\ll(\log\log y)^{1/6}$. Then the inclusion-exclusion principle and Lemma 3.1(c) imply that

$$\begin{aligned}H(x,y,z;P_s)&\geq\sum_{\substack{p+s\leq x\\\tau(p+s,y,z)\geq1}}\chi(p+s)\geq\sum_{p+s\leq x}\chi(p+s)\Big(\sum_{\substack{d|p+s\\y<d\leq z}}1-\sum_{\substack{[d_1,d_2]|p+s\\y<d_1<d_2\leq z}}1\Big)\\&\geq f(s)\frac{315\zeta(3)}{2\pi^4}\frac{\eta x}{\log x}(1-o_s(1))-S-S',\end{aligned}\tag{3.15}$$

where

$$S:=\sum_{\substack{p+s\leq x\\p\nmid s}}(1-\chi(p+s))\sum_{\substack{d|p+s\\y<d\leq z}}1\quad\text{and}\quad S':=\sum_{\substack{p+s\leq x\\p\nmid s}}\chi(p+s)\sum_{\substack{[d_1,d_2]|p+s\\y<d_1<d_2\leq z}}1.$$

To bound S observe that for every $1\leq v\leq 3/2$ we have that

$$\begin{aligned}S&\leq v^{-L(y)}\sum_{\substack{p+s\leq x\\p\nmid s}}v^{\Omega(p+s;y)}\sum_{\substack{d|p+s\\y<d\leq z}}1\leq v^{-L(y)}\sum_{\substack{y<d\leq z\\(d,s)=1}}v^{\Omega(d;y)}\sum_{\substack{p+s\leq x\\p\equiv-s\ (\mathrm{mod}\ d)}}v^{\Omega(\frac{p+s}{d};y)}\\&\ll_s\frac{xv^{-L(y)}(\log y)^{v-1}}{\log x}\sum_{y<d\leq z}\frac{v^{\Omega(d)}}{\phi(d)},\end{aligned}\tag{3.16}$$

by Lemma 2.6. Writing

$$\frac{d}{\phi(d)}=\sum_{k|d}\frac{\mu^2(k)}{\phi(k)}$$

and using Theorem 04 in [15] we find that

$$
\begin{aligned}
\sum_{y<d\le z}\frac{v^{\Omega(d)}}{\phi(d)} &= \sum_{k\le z}\frac{\mu^2(k)v^{\Omega(k)}}{k\phi(k)}\sum_{y/k<f\le z/k}\frac{v^{\Omega(f)}}{f}\\
&\ll \sum_{k\le\sqrt{y}}\frac{\mu^2(k)v^{\Omega(k)}}{k\phi(k)}\big(\eta(\log(y/k))^{v-1}+(\log(y/k))^{v-3}\big)\\
&\quad+\sum_{\sqrt{y}<k\le z}\frac{\mu^2(k)v^{\Omega(k)}}{k\phi(k)}(\log y)^{v-1}\\
&\ll \eta(\log y)^{v-1}+\frac{(\log y)^{v-1}}{y^{1/4}}\sum_{\sqrt{y}<k\le z}\frac{\mu^2(k)v^{\Omega(k)}}{\sqrt{k}\phi(k)}\\
&\ll \eta(\log y)^{v-1},
\end{aligned}
\tag{3.17}
$$

since $\eta\gg(\log y)^{-2}$. Combining inequalities (3.16) and (3.17) we find that

$$S\ll_s\frac{\eta x}{\log x}\frac{(\log y)^{2v-2}}{v^{L(y)}}.$$

Setting $v=L(y)/2\log\log y$ we deduce that

$$S\ll_s\frac{\eta x}{\log x}\exp\Big\{-\frac{\psi(y)^2}{4}+O\Big(\frac{\psi(y)^3}{(\log\log y)^{1/2}}\Big)\Big\}=o\Big(\frac{\eta x}{\log x}\Big)\quad(y\to\infty).\tag{3.18}$$

Next, we turn to the estimation of S'. Note that for every $1/10\le v\le 1$ we have that

$$
\begin{aligned}
S' &\le v^{-L(y)}\sum_{\substack{p+s\le x\\ p\nmid s}}v^{\Omega(p+s;y)}\sum_{\substack{[d_1,d_2]\mid p+s\\ y<d_1<d_2\le z}}1\\
&= v^{-L(y)}\sum_{\substack{y<d_1<d_2\le z\\ (d_1d_2,s)=1}}v^{\Omega([d_1,d_2];y)}\sum_{\substack{p+s\le x,\ p\nmid s\\ p\equiv -s \ (\mathrm{mod}\ [d_1,d_2])}}v^{\Omega(\frac{p+s}{[d_1,d_2]};y)}.
\end{aligned}
\tag{3.19}
$$

Set

$$S_1'=\sum_{\substack{y<d_1<d_2\le z\\ (d_1d_2,s)=1\\ (d_1,d_2)>y^2x^{-3/4}}}v^{\Omega([d_1,d_2];y)}\sum_{\substack{p+s\le x,\ p\nmid s\\ p\equiv -s \ (\mathrm{mod}\ [d_1,d_2])}}v^{\Omega(\frac{p+s}{[d_1,d_2]};y)}$$

and

$$S_2'=\sum_{\substack{y<d_1<d_2\le z\\ (d_1d_2,s)=1\\ (d_1,d_2)\le y^2x^{-3/4}}}v^{\Omega([d_1,d_2];y)}\sum_{\substack{p+s\le x,\ p\nmid s\\ p\equiv -s \ (\mathrm{mod}\ [d_1,d_2])}}v^{\Omega(\frac{p+s}{[d_1,d_2]};y)}.$$

Put $m=(d_1,d_2)$ and $d_i=mt_i$ so that $[d_1,d_2]=mt_1t_2$. Note that $m\le z-y$. First, we deal with S_1'. Since $v^{\Omega(n;y)}\le v^{\omega(n;y)}$ for $v\le 1$ and $[d_1,d_2]\le 2x^{3/4}$ in the range of S_1', Lemma

2.5 gives us that

$$
\begin{aligned}
S_1' &\ll_s \sum_{y^2x^{-3/4}<m\le z-y}\ \sum_{y/m<t_1<t_2\le z/m} \frac{v^{\Omega(mt_1t_2;y)}x(\log y)^{v-1}}{\phi(mt_1t_2)\log x} \\
&\ll \frac{x(\log y)^{v-1}\log\log y}{\log x}\sum_{m\le z-y}\frac{v^{\Omega(m)}}{m}\Big(\sum_{y/m<t\le z/m}\frac{v^{\Omega(t)}}{t}\Big)^2,
\end{aligned} \tag{3.20}
$$

uniformly in $1/10\le v\le 1$, since $\Omega(n;y)\ge\Omega(n)-2$ for $n\le y^3$. By relation (2.39) in [15] we have

$$
\sum_{y/m<t\le z/m}\frac{v^{\Omega(t)}}{t}\ll \eta\Big(\log\frac{1}{\eta}\Big)^{1-v}\Big(\log\frac{y}{m}\Big)^{v-1}\asymp \eta(\log\log y)^{1-v}\Big(\log\frac{y}{m}\Big)^{v-1}, \tag{3.21}
$$

which, combined with (3.20), yields that

$$
S_1'\ll_s \frac{\eta^2x(\log y)^{v-1}(\log\log y)^{3-2v}}{\log x}\sum_{m\le z-y}\frac{v^{\Omega(m)}}{m}\Big(\log\frac{y}{m}\Big)^{2v-2}. \tag{3.22}
$$

We now estimate S_2'. First, for d_1,d_2 in the range of summation of S_2' we have $x(d_1,d_2)/y^2\le x^{1/4}$, by definition. So if S_2' is a non-empty sum, we must have that $y\ge x^{3/8}$ and $m=(d_1,d_2)\le x^{1/4}\le y^{2/3}$. Consequently,

$$
S_2'\le \sum_{\substack{m\le y^{2/3}\\(m,s)=1}}\ \sum_{\substack{y/m<t_1<t_2\le z/m\\(t_1t_2,s)=1}}\ \sum_{\substack{p+s\le x,\ p\nmid s\\ p\equiv -s\ (\mathrm{mod}\ mt_1t_2)}} v^{\Omega(p+s;y)}.
$$

Set $p+s=mt_1t_2k$. Then we have that $k\le x/(yt_1)$, $\frac{z-y}{m}\ge(\frac{z}{m})^{1/2}$ and $mt_1k\le(t_1kz)^{7/8}$. Also, note that $\Omega(n;y)\ge\Omega(n)-2$ for $n\le x$, since $y\ge x^{3/8}$. So

$$
\begin{aligned}
S_2' &\le \frac{1}{v^2}\sum_{\substack{m\le y^{2/3}\\(m,s)=1}}\ \sum_{\substack{y/m<t_1\le z/m\\(t_1,s)=1}}\ \sum_{\substack{k\le x/(yt_1)\\(k,s)=1}} v^{\Omega(mt_1k)}\sum_{\substack{t_1ky<p+s\le t_1kz\\ p\equiv -s\ (\mathrm{mod}\ mt_1k)}} v^{\omega(\frac{p+s}{mt_1k})} \\
&\ll_s \sum_{m\le y^{2/3}}\ \sum_{y/m<t_1\le z/m}\ \sum_{k\le x/(yt_1)}\frac{v^{\Omega(mt_1k)}t_1k(z-y)(\log(t_1kz))^{v-2}}{\phi(mt_1k)} \\
&\ll \frac{\eta y(\log y)^{v-1}\log\log y}{\log x}\sum_{m\le y^{2/3}}\frac{v^{\Omega(m)}}{m}\sum_{y/m<t_1\le z/m}v^{\Omega(t_1)}\sum_{k\le xm/y^2}v^{\Omega(k)} \\
&\ll \frac{\eta x(\log y)^{v-1}\log\log y}{y\log x}\sum_{m\le y^{2/3}}v^{\Omega(m)}(\log 2m)^{v-1}\sum_{y/m<t_1\le z/m}v^{\Omega(t_1)},
\end{aligned}
$$

uniformly in $1/10\le v\le 1$, by Lemma 2.5 and Theorem 01 in [15]. Also,

$$
\begin{aligned}
\sum_{y/m<t_1\le z/m}v^{\Omega(t_1)} &\asymp \frac{y}{m}\sum_{y/m<t_1\le z/m}\frac{v^{\Omega(t_1)}}{t_1}\ll\frac{\eta y(\log\log y)^{1-v}}{m}\Big(\log\frac{y}{m}\Big)^{v-1} \\
&\asymp \frac{\eta y(\log y)^{v-1}(\log\log y)^{1-v}}{m},
\end{aligned}
$$

by (3.21), since $m \le y^{2/3}$. Hence

$$S_2' \ll_s \frac{\eta^2 x(\log y)^{2v-2}(\log\log y)^{2-v}}{\log x} \sum_{m\le y^{2/3}} \frac{v^{\Omega(m)}}{m}(\log 2m)^{v-1}. \tag{3.23}$$

Inequalities (3.19), (3.22) and (3.23) imply that

$$S' \ll_s \frac{\eta^2 x v^{-L(y)}(\log\log y)^{3-2v}}{\log x} \sum_{m\le z-y} \frac{v^{\Omega(m)}}{m}(\log 2m)^{v-1}\Big(\log\frac{y}{m}\Big)^{2v-2}.$$

If we set $v = 1/2$, by partial summation and the estimate $\sum_{n\le x} v^{\Omega(n)} \ll x(\log 2x)^{v-1}$ we find that

$$\sum_{m\le z-y} \frac{v^{\Omega(m)}}{m}(\log m)^{v-1}\Big(\log\frac{y}{m}\Big)^{2v-2} \ll \frac{\log\log y}{\log y}$$

and consequently

$$S' \ll_s \frac{\eta^2 x}{\log x}(\log y)^{\log 4-1} 2^{\psi(y)\sqrt{\log\log y}}(\log\log y)^3.$$

Lastly, putting $\psi(y) = \min\{\xi, (\log\log y)^{1/6}\}$ yields that

$$S' \ll_s \frac{\eta x}{\log x}\frac{(\log\log y)^3}{e^{(1-\log 2)\xi\sqrt{\log\log y}}} = o\Big(\frac{\eta x}{\log x}\Big).$$

Inserting the above estimate and (3.18) into (3.15) gives us that

$$H(x,y,z;P_s) \ge f(s)\frac{315\zeta(3)}{2\pi^4}\frac{\eta x}{\log x}(1-o_s(1)),$$

which completes the proof of part (d) in the case that $z > y + (\log y)^{-2}$ too. □

4. Intermediate and large values of η

To prove Theorem 1.6 we reduce the counting in $H(x,y,z;P_s) - H(x-\Delta,y,z;P_s)$ to the estimation of a sum involving $L(a;\eta)$ as done in [11] for bounding $H(x,y,z)$; then we apply Lemma 2.10. First, we show the following result. Theorem 1.6 will then follow as an easy corollary.

Theorem 4.1. *Fix $s \in \mathbb{Z}\setminus\{0\}$ and $B \ge 2$. Let $x \ge x_0(s,B)$ and $3 \le y+1 \le z$ with $z \le x^{2/3}$, $\eta \in [(\log y)^{-B}, \frac{\log y}{100}]$ and $\{y < d \le z : (d,s) = 1\} \ne \emptyset\}$. Then for*

$$\frac{x}{(\log x)^B} \le \Delta \le \frac{x}{2}$$

we have that

$$H(x,y,z;P_s) - H(x-\Delta,y,z;P_s) \gg_{s,B} \frac{\Delta}{x}\frac{H(x,y,z)}{\log x}.$$

Proof. Fix $\Delta \in (x(\log x)^{-B}, x/2]$ and set $s_1 = 2/(s,2)$. Let $y_0 = y_0(s,B)$ be a large positive constant. If $y \le y_0$, then

$$H(x,y,z;P_s) - H(x-\Delta,y,z;P_s) \ge \max_{\substack{y<d\le z\\(d,s)=1}} \Big(\pi(x-s;d,-s) - \pi(x-\Delta-s;d,-s)\Big)$$
$$\gg_{y_0} \frac{\Delta}{\log x} \asymp_{y_0} \frac{\Delta}{x}\frac{H(x,y,z)}{\log x},$$

by the Prime Number Theorem for arithmetic progressions [4, p. 123] and our assumption that $\{y < d \le z : (d,s) = 1\} \ne \emptyset$. Suppose now that $y > y_0$. Fix an integer $v = v(s) \ge 3$ and set $w = z^{1/20v}$. We will choose v later; till then, all implied constants will be independent of v. Consider integers $n = aqb_1b_2s_1 \in (x-\Delta, x]$ with

(1) $a \le w$, $\mu^2(a) = 1$ and $(a, 2s) = 1$;
(2) $\log(y/q) \in \mathcal{L}(a;\eta)$, $P^-(q) > w$ and $(q, 2s) = 1$;
(3) $b_1 \in \mathscr{P}(w,z)$ and $\tau(b_1) \le v^2$;
(4) $P^-(b_2) > z$;
(5) $n - s$ is prime.

Condition (2) implies that there exists $d|a$ such that $y/d < q \le z/d$; in particular, we have that $\tau(n,y,z) \ge 1$ and thus n is counted by $H(x,y,z;P_s) - H(x-\Delta,y,z;P_s)$. Also, $\Omega(q) \le \frac{\log z}{\log w} = 20v$ and therefore

$$\tau(qb_1) \le 2^{\Omega(q)}\tau(b_1) \le 2^{20v}v^2.$$

Since each n has at most $\tau(qb_1) \le 2^{20v}v^2$ representations of this form, we find that

(4.1)
$$2^{20v}v^2\big(H(x,y,z;P_s) - H(x-\Delta,y,z;P_s)\big) \ge \sum_{\substack{a\le w\\ \mu^2(a)=1\\ (a,2s)=1}} \sum_{\substack{\log(y/q)\in\mathcal{L}(a;\eta)\\ P^-(q)>w\\ (q,2s)=1}} \sum_{\substack{(x-\Delta)/aqs_1<b_1b_2\le x/aqs_1\\ b_1\in\mathscr{P}(w,z),\ P^-(b_2)>z\\ \tau(b_1)\le v^2\\ aqb_1b_2s_1-s \text{ prime}}} 1$$
$$=: \sum_{\substack{a\le w\\ \mu^2(a)=1\\ (a,2s)=1}} \sum_{\substack{\log(y/q)\in\mathcal{L}(a;\eta)\\ P^-(q)>w\\ (q,2s)=1}} B_0(a,q).$$

Given a and q as above, put

$$B(a,q) = \sum_{\substack{(x-\Delta)/aqs_1<b\le x/aqs_1\\ P^-(b)>w\\ aqbs_1-s \text{ prime}}} 1 \quad \text{and} \quad R(a,q) = B(a,q) - B_0(a,q).$$

Given b with $P^-(b) > w$, write $b = b_1b_2$ with $b_1 \in \mathscr{P}(w,z)$ and $P^-(b_2) > z$ and put $F(b) = \tau(b_1)$. Then, for fixed a and q with $(aq, 2s) = 1$, we have that

$$R(a,q) \le \frac{1}{v^2} \sum_{\substack{(x-\Delta)/aqs_1<b\le x/aqs_1\\ P^-(b)>w\\ aqbs_1-s \text{ prime}}} F(b) = \frac{1}{v^2} \sum_{\substack{x-\Delta<p+s\le x\\ p\equiv -s \pmod{aqs_1}\\ P^-(\frac{p+s}{aqs_1})>w}} F\Big(\frac{p+s}{aqs_1}\Big) \ll_s \frac{1}{v}\frac{\Delta}{\phi(aq)\log x\log w},$$

by Lemma 2.5. Inserting the above estimate into (4.1) yields that

$$
\begin{aligned}
2^{20v}v^2\big(H(x,y,z;P_s)-H(x-\Delta,y,z;P_s)\big) \geq & \sum_{\substack{a\le w\\ \mu^2(a)=1\\ (a,2s)=1}} \sum_{\substack{\log(y/q)\in\mathcal{L}(a;\eta)\\ P^-(q)>w\\ (q,2s)=1}} B(a,q) \\
& - O_s\Big(\frac{1}{v}\frac{\Delta}{\log x\log w}\sum_{\substack{a\le w\\ \mu^2(a)=1\\ (a,2s)=1}} \sum_{\substack{\log(y/q)\in\mathcal{L}(a;\eta)\\ P^-(q)>w\\ (q,2s)=1}} \frac{1}{\phi(aq)}\Big).
\end{aligned}
\tag{4.2}
$$

Next, we need to approximate the characteristic function of integers n with $P^-(n)>w$ with a 'smoother' function, the reason being that the error term $\pi(x;rq,a)-\mathrm{li}(x)/\phi(rq)$ in Lemma 2.7 is weighted with the smooth function 1 as q runs through $[1,Q]\cap\mathbb{N}$. To do this we appeal to Lemma 2.9(a) with $Z=w$, $D=z^{1/20}$ and $\kappa=2$. Then

$$
\begin{aligned}
&2^{20v}v^2\big(H(x,y,z;P_s)-H(x-\Delta,y,z;P_s)\big) \\
&\quad\geq \sum_{\substack{a\le w\\ \mu^2(a)=1,(a,2s)=1}} \sum_{\substack{\log(y/q)\in\mathcal{L}(a;\eta)\\ (q,2s)=1}} (\lambda^-*1)(q)B(a,q) - O_s(\mathscr{R}_1) \\
&\quad\geq \sum_{\substack{a\le w\\ \mu^2(a)=1,(a,2s)=1}} \sum_{\substack{\log(y/q)\in\mathcal{L}(a;\eta)\\ (q,2s)=1}} (\lambda^+*1)(q)B(a,q) - O_s(\mathscr{R}_1+\mathscr{R}_2),
\end{aligned}
\tag{4.3}
$$

where

$$
\mathscr{R}_1 := \frac{1}{v}\frac{\Delta}{\log x\log w}\sum_{\substack{a\le w\\ \mu^2(a)=1,(a,2s)=1}} \sum_{\substack{\log(y/q)\in\mathcal{L}(a;\eta)\\ (q,2s)=1}} \frac{(\lambda^+*1)(q)}{\phi(aq)}
$$

and

$$
\mathscr{R}_2 := \sum_{\substack{a\le w\\ \mu^2(a)=1,(a,2s)=1}} \sum_{\substack{\log(y/q)\in\mathcal{L}(a;\eta)\\ (q,2s)=1}} \big((\lambda^+*1)(q)-(\lambda^-*1)(q)\big)B(a,q).
$$

We now bound $\mathscr{R}_2$ from above. For fixed a and q with $(aq,2s)=1$ we have

$$
B(a,q) \ll_s \frac{\Delta}{\phi(aq)\log x\log w},
$$

by the arithmetic form of the large sieve [21] or Lemma 2.5. Since $\lambda^+*1-\lambda^-*1$ is always non-negative, we get that

$$
\mathscr{R}_2 \ll \frac{\Delta}{\log x\log w}\sum_{\substack{a\le w\\ \mu^2(a)=1,(a,2s)=1}} \sum_{\substack{\log(y/q)\in\mathcal{L}(a;\eta)\\ (q,2s)=1}} \frac{(\lambda^+*1)(q)-(\lambda^-*1)(q)}{\phi(aq)}.
\tag{4.4}
$$

Fix $a\le w$ with $(a,2s)=1$ and let $\{I_r\}_{r=1}^R$ be the collection of the intervals $[\log d-\eta,\log d)$ with $d|a$. Then for $I=[\log d-\eta,\log d)$ in this collection Lemmas 2.8 and 2.9(a) imply that

$$\begin{aligned}
&\sum_{\substack{\log(y/q)\in 3I\\ (q,2s)=1}}\frac{(\lambda^+*1)(q)-(\lambda^-*1)(q)}{\phi(aq)}\\
&\qquad=\sum_{\substack{c\le z^{1/20}\\ (c,2s)=1}}(\lambda^+(c)-\lambda^-(c))\sum_{\substack{e^{-\eta}y/cd<m\le e^{2\eta}y/cd\\ (m,2s)=1}}\frac{1}{\phi(acm)}\\
&\qquad=\frac{315\zeta(3)}{2\pi^4}\frac{g(2as)\phi(2s)}{2|s|\phi(a)}\sum_{\substack{c\le z^{1/20}\\ (c,2s)=1}}\frac{\lambda^+(c)-\lambda^-(c)}{c}\frac{g(ac)}{g(a)}\frac{c\phi(a)}{\phi(ac)}\Big(3\eta+O_s(y^{-2/3})\Big)\\
&\qquad\ll_s\frac{\eta}{e^v\phi(a)}\prod_{\substack{p\le w\\ p\nmid 2s,p|a}}\Big(1-\frac{1}{p}\Big)\prod_{\substack{p\le w\\ p\nmid 2sa}}\Big(1-\frac{g(p)}{p-1}\Big)+\frac{1}{\phi(a)\sqrt{y}}\asymp_s\frac{1}{e^v}\frac{\eta}{\phi(a)\log w},
\end{aligned}\tag{4.5}$$

provided that y_0 is large enough, since $g(p)p/(p-1)\le\min\{p-1,2\}$ for $p\ge3$, $g(p)=1+O(p^{-2})$ and $g(a)\asymp1$. By Lemma 2.12, there exists a sub-collection $\{I_{r_t}\}_{t=1}^T$ of mutually disjoint intervals so that

$$\bigcup_{t=1}^T 3I_{r_t}\supset\mathcal{L}(a;\eta).$$

Consequently

$$\begin{aligned}
\sum_{\substack{\log(y/q)\in\mathcal{L}(a;\eta)\\ (q,2s)=1}}\frac{(\lambda^+*1)(q)-(\lambda^-*1)(q)}{\phi(aq)}&\le\sum_{t=1}^T\sum_{\substack{\log(y/q)\in 3I_{r_t}\\ (q,2s)=1}}\frac{(\lambda^+*1)(q)-(\lambda^-*1)(q)}{\phi(aq)}\\
&\ll_s\frac{1}{e^v}\frac{T\eta}{\phi(a)\log w}\\
&=\frac{1}{e^v}\frac{1}{\phi(a)\log w}\mathrm{meas}\Big(\bigcup_{t=1}^T I_{r_t}\Big)\\
&\le\frac{1}{e^v}\frac{L(a;\eta)}{\phi(a)\log w},
\end{aligned}$$

since $\lambda^+*1-\lambda^-*1$ is always non-negative. By the above inequality and (4.4) we get that

$$\mathscr{R}_2\ll_s\frac{1}{e^v}\frac{\Delta}{\log x\log^2 w}\sum_{\substack{a\le w\\ \mu^2(a)=1,(a,2s)=1}}\frac{L(a;\eta)}{\phi(a)}.\tag{4.6}$$

We now bound from the below the sum

$$\mathscr{S}:=\sum_{\substack{a\le w\\ \mu^2(a)=1,(a,2s)=1}}\sum_{\substack{\log(y/q)\in\mathcal{L}(a;\eta)\\ (q,2s)=1}}(\lambda^+*1)(q)B(a,q).$$

We fix a and q with $(aq,2s)=1$ and seek a lower bound on $B(a,q)$. By Lemma 2.9(b) applied with $Z=w$ and $D=w^3$, there exists a sequence $\{\rho(d)\}_{d\le w^3}$ such that $\rho*1$ is bounded above by the characteristic function of integers b with $P^-(b)>w$. So, if we put

$$E(x;k,a)=\pi(x-s;k,a)-\pi(x-\Delta-s;k,a)-\frac{\mathrm{li}(x-s)-\mathrm{li}(x-\Delta-s)}{\phi(k)},$$

then Lemma 2.9(b) and the fact that $2|s_1s$ imply that

$$\begin{aligned}B(a,q)&=\sum_{\substack{x-\Delta<p+s\le x\\ p\equiv -s \pmod{aqs_1}\\ P^-((p+s)/aqs_1)>w}}1\ge\sum_{\substack{x-\Delta<p+s\le x\\ p\equiv -s \pmod{aqs_1}\\ p\nmid s}}(\rho*1)\Big(\frac{p+s}{aqs_1}\Big)\\ &=\sum_{\substack{m\le w^3\\ (m,s)=1}}\rho(m)\big(\pi(x-s;aqs_1m,-s)-\pi(x-s-\Delta;aqs_1m,-s)\big)+O_s(1)\\ &=\big(\mathrm{li}(x-s)-\mathrm{li}(x-s-\Delta)\big)\sum_{\substack{m\le w^3\\ (m,s)=1}}\frac{\rho(m)}{\phi(aqs_1m)}+O_s(1)+\mathscr{R}'_{aqs_1}\\ &\ge C_s\frac{\Delta}{\phi(aq)\log x\log w}+\mathscr{R}'_{aqs_1}\end{aligned}$$

for some positive constant C_s, where

$$\mathscr{R}'_{aqs_1}=\sum_{\substack{m\le w^3\\ (m,s)=1}}\rho(m)E(x;aqs_1m,-s).$$

Since λ^+*1 is always non-negative, we deduce that

$$\mathscr{S}\ge C_s\frac{\Delta}{\log x\log w}\sum_{\substack{a\le w\\ \mu^2(a)=1,(a,2s)=1}}\sum_{\substack{\log(y/q)\in\mathcal{L}(a;\eta)\\ (q,2s)=1}}\frac{(\lambda^+*1)(q)}{\phi(aq)}+\mathscr{R}',\tag{4.7}$$

where

$$\mathscr{R}'=\sum_{\substack{a\le w\\ \mu^2(a)=1,(a,2s)=1}}\sum_{\substack{\log(y/q)\in\mathcal{L}(a;\eta)\\ (q,2s)=1}}(\lambda^+*1)(q)\mathscr{R}'_{aqs_1}.$$

Combining (4.3), (4.6) and (4.7) we get that

$$\begin{aligned}&2^{20v}v^2\big(H(x,y,z;P_s)-H(x-\Delta,y,z;P_s)\big)\\ &\qquad\ge\frac{C_s}{2}\frac{\Delta}{\log x\log w}\sum_{\substack{a\le w\\ \mu^2(a)=1,(a,2s)=1}}\sum_{\substack{\log(y/q)\in\mathcal{L}(a;\eta)\\ (q,2s)=1}}\frac{(\lambda^+*1)(q)}{\phi(aq)}\\ &\qquad\quad-O_s\Big(|\mathscr{R}'|+\frac{1}{e^v}\frac{\Delta}{\log x\log^2 w}\sum_{\substack{a\le w\\ \mu^2(a)=1,(a,2s)=1}}\frac{L(a;\eta)}{\phi(a)}\Big),\end{aligned}\tag{4.8}$$

provided that v is large enough. Fix now $a\le w$ with $(a,2s)=1$ and look at the sum over q on the right hand side of (4.8). Let $\{I_r\}_{r=1}^R$ be the collection of the intervals $[\log d-\eta,\log d)$ with $d|a$. Then, using a similar argument with the one leading to (4.5), we find that for I in this collection

$$\sum_{\substack{\log(y/q)\in I\\ (q,2s)=1}}\frac{(\lambda^+*1)(q)}{\phi(aq)}\gg_s\frac{\eta}{\phi(a)\log w},$$

provided that y_0 and v are large enough. Moreover, by Lemma 2.12, there exists a subcollection $\{I_{r_t}\}_{t=1}^T$ of mutually disjoint intervals so that

$$\eta T=\mathrm{Vol}\Big(\bigcup_{t=1}^T I_{r_t}\Big)\ge\frac{1}{3}\mathrm{Vol}\Big(\bigcup_{r=1}^R I_r\Big)=\frac{L(a;\eta)}{3}.$$

Hence

$$\sum_{\substack{\log(y/q)\in\mathcal{L}(a;\eta)\\ (q,2s)=1}}\frac{(\lambda^+*1)(q)}{\phi(aq)}\ge\sum_{t=1}^T\sum_{\substack{\log(y/q)\in I_{r_t}\\ (q,2s)=1}}\frac{(\lambda^+*1)(q)}{\phi(aq)}\gg_s\frac{\eta T}{\phi(a)\log w}\gg\frac{L(a;\eta)}{\phi(a)\log w},$$

where we used the fact that λ^+*1 is non-negative. Inserting this inequality into (4.8) and choosing a large enough v we conclude that

$$H(x,y,z;P_s)-H(x-\Delta,y,z;P_s)\ge M_s\frac{\Delta}{\log x\log^2 y}\sum_{\substack{a\le w\\ \mu^2(a)=1,(a,2s)=1}}\frac{L(a;\eta)}{\phi(a)}-O_s(|\mathscr{R}'|)\tag{4.9}$$

for some positive constant M_s. Furthermore, note that if a is squarefree, we may uniquely write $a=db$, where $d|2s$, $\mu^2(d)=\mu^2(b)=1$ and $(b,2s)=1$, in which case $L(a;\eta)\le\tau(d)L(b;\eta)$, by inequality (2.12). Thus

$$\sum_{\substack{a\le w\\ \mu^2(a)=1}}\frac{L(a;\eta)}{\phi(a)}\le\sum_{d|2s,\mu^2(d)=1}\frac{\tau(d)}{\phi(d)}\sum_{\substack{b\le w/d\\ \mu^2(b)=1\\ (b,2s)=1}}\frac{L(b;\eta)}{\phi(b)}\le\Big(\sum_{d|s}\frac{\tau(d)}{\phi(d)}\Big)\sum_{\substack{b\le w\\ \mu^2(b)=1\\ (b,2s)=1}}\frac{L(b;\eta)}{\phi(b)},$$

which, combined with (4.9), Lemma 2.10 and the trivial inequality $\phi(a)\le a$, implies that

$$H(x,y,z;P_s)-H(x-\Delta,y,z;P_s)\ge M_s'\frac{\Delta}{x}\frac{H(x,y,z)}{\log x}-O_s(|\mathscr{R}'|)$$

for some positive constant M_s'. In addition, observe that

$$H(x,y,z)\gg\frac{x}{(\log y)^B},$$

by Theorem 1.1 and our assumption that $\eta\ge(\log y)^{-B}$. Hence

$$H(x,y,z;P_s)-H(x-\Delta,y,z;P_s)\gg_s\frac{\Delta}{x}\frac{H(x,y,z)}{\log x}\Big(1-O_s\Big(\frac{(\log x)(\log y)^B|\mathscr{R}'|}{\Delta}\Big)\Big).$$

So it suffices to show that

$$|\mathscr{R}'| \ll_s \frac{\Delta}{(\log x)(\log y)^{B+1}}.$$

For every $a \in \mathbb{N}$ there is a unique set D_a of pairs (d, d') with $d \le d'$, $d|a$ and $d'|a$ such that

$$\mathcal{L}(a;\eta) = \bigcup_{(d,d')\in D_a} [\log d - \eta, \log d')$$

and the intervals $[\log d - \eta, \log d')$ for $(d, d') \in D_a$ are mutually disjoint. With this notation we have that

$$\begin{aligned}
|\mathscr{R}'| &= \Big|\sum_a \sum_m \rho(m) \sum_{(d,d')\in D_a} \sum_{\substack{y/d'<q\le z/d \\ (q,2s)=1}} (\lambda^+ * 1)(q) E(x; ams_1 q, -s)\Big| \\
&= \Big|\sum_a \sum_m \rho(m) \sum_{(d,d')\in D_a} \sum_c \lambda^+(c) \sum_{\substack{y/cd'<g\le z/cd \\ (g,2s)=1}} E(x; ams_1 cg, -s)\Big| \\
&\le \sum_{\substack{a\le w \\ (a,2s)=1}} \sum_{\substack{m\le w^3 \\ (m,s)=1}} \sum_{\substack{c\le z^{1/20} \\ (c,2s)=1}} \sum_{(d,d')\in D_a} \Big| \sum_{\substack{y/cd'<g\le z/cd \\ (g,2s)=1}} E(x; ams_1 cg, -s)\Big|.
\end{aligned}$$

So writing the inner sum as a difference of two sums we obtain that

$$\begin{aligned}
|\mathscr{R}'| &\le 2 \sup_{y\le t\le z} \Big\{ \sum_{\substack{a\le w \\ (a,2s)=1}} \sum_{\substack{m\le w^3 \\ (m,s)=1}} \sum_{\substack{c\le z^{1/20} \\ (c,2s)=1}} \sum_{f|ams_1c} \Big| \sum_{\substack{g\le t/f \\ (g,2s)=1}} E(x; ams_1 cg, -s)\Big|\Big\} \\
&\le 2 \sup_{y\le t\le z} \Big\{ \sum_{\substack{r\le 2z^{7/60} \\ (r,s)=1}} \tau_3(r) \sum_{f|r} \Big| \sum_{\substack{g\le t/f \\ (g,2s)=1}} E(x; rg, -s)\Big|\Big\} \qquad (4.10) \\
&\le 4 \sup_{y\le t\le z} \Big\{ \sum_{\substack{r\le z^{1/8} \\ (r,s)=1}} \tau_3(r) \sum_{f|r} \Big| \sum_{\substack{g\le t/f \\ (g,s)=1}} E(x; rg, -s)\Big|\Big\},
\end{aligned}$$

since $w^4 z^{1/20} \le z^{7/60} \le z^{1/8}/4$ for all $v \ge 3$. Put $\mu = 1 + (\log y)^{-B-7}$ and cover the interval $[1, z^{1/8}]$ by intervals of the form $[\mu^n, \mu^{n+1})$ for $n = 0, 1, \dots, N$. We may take $N \ll (\log y)^{B+8}$. Since $|E(x; k, -s)| \ll \frac{\Delta}{\phi(k)\log x}$ for $k \le z^{9/8} \le x^{3/4}$ with $(k, s) = 1$ by Lemma 2.4, we have that

$$\sum_{\substack{r\le z^{1/8}\\(r,s)=1}}\tau_3(r)\sum_{n=0}^{N}\sum_{\substack{f|r\\ \mu^n\le f<\mu^{n+1}}}\Big|\sum_{\substack{g\le t/f\\(g,s)=1}}E(x;rg,-s)-\sum_{\substack{g\le t/\mu^n\\(g,s)=1}}E(x;rg,-s)\Big|$$

$$\ll\sum_{\substack{r\le z^{1/8}\\(r,s)=1}}\tau_3(r)\sum_{n=0}^{N}\sum_{\substack{f|r\\ \mu^n\le f<\mu^{n+1}}}\sum_{t/\mu^{n+1}<g\le t/\mu^n}\frac{\Delta}{\phi(rg)\log x}$$

$$\ll\frac{\Delta\log\mu}{\log x}\sum_{r\le z^{1/8}}\frac{\tau_3(r)}{\phi(r)}\sum_{f|r}1\ll\frac{\Delta}{(\log x)(\log y)^{B+1}}$$

for all $t\in[y,z]$, by Lemma 2.8, which is admissible. Combining the above estimate with (4.10) we find that

$$|\mathscr{R}'|\ll\sup_{y\le t\le z}\Big\{\sum_{n=0}^{N}\sum_{\substack{r\le z^{1/8}\\(r,s)=1}}\tau_3(r)\tau(r)\Big|\sum_{\substack{g\le t/\mu^n\\(g,s)=1}}E(x;rg,-s)\Big|\Big\}+\frac{\Delta}{(\log x)(\log y)^{B+1}}.\tag{4.11}$$

Finally, since

$$\frac{x}{2}\le x-\Delta\le x\quad\text{and}\quad\Delta\ge\frac{x}{(\log x)^B},$$

Lemma 2.7 applied with $A=5B+56$ in combination with the Cauchy-Schwarz inequality yields that

$$\sum_{\substack{r\le z^{1/8}\\(r,s)=1}}\tau_3(r)\tau(r)\Big|\sum_{\substack{g\le t/\mu^n\\(g,s)=1}}E(x;rg,-s)\Big|$$

$$\ll\Big(\frac{\Delta}{\log x}\sum_{r\le z^{1/8}}\sum_{g\le t/\mu^n}\frac{(\tau_3(r)\tau(r))^2}{\phi(rg)}\Big)^{1/2}\Big(\sum_{\substack{r\le z^{1/8}\\(r,s)=1}}\Big|\sum_{\substack{g\le t/\mu^n\\(g,s)=1}}E(x;rg,-s)\Big|\Big)^{1/2}$$

$$\ll_s\sqrt{\Delta}(\log x)^{18}\frac{\sqrt{x}}{(\log x)^{5B/2+28}}\le\frac{\Delta}{(\log x)^{2B+10}}$$

for all $t\in[y,z]$ and all $n\in\{0,1,\dots,N\}$, since $z^{1/8}\le x^{1/12}$ and $z^{9/8}\le x^{3/4}$. Plugging this estimate into (4.11) gives us that

$$|\mathscr{R}'|\ll_s N\frac{\Delta}{(\log x)^{2B+10}}+\frac{\Delta}{(\log x)(\log y)^{B+1}}\ll\frac{\Delta}{(\log x)(\log y)^{B+1}},$$

which is admissible. □

We are now in position to complete the proof of Theorem 1.6.

Proof of Theorem 1.6. Fix $\Delta\in(x(\log x)^{-B},x]$ and set $\Delta_1=\min\{\Delta,x/2\}$. If $\eta\le\frac{\log y}{100}$, then the theorem follows immediately by Theorem 4.1 and the trivial inequality

$$H(x,y,z;P_s)-H(x-\Delta,y,z;P_s)\ge H(x,y,z;P_s)-H(x-\Delta_1,y,z;P_s).$$

On the other hand, if $\eta \ge \frac{\log y}{100}$, then

$$\begin{aligned} H(x,y,z;P_s) - H(x-\Delta,y,z;P_s) &\ge H(x,y,y^{101/100};P_s) - H(x-\Delta,y,y^{101/100};P_s) \\ &\gg_s \frac{\Delta}{x}\frac{H(x,y,y^{101/100})}{\log x} \asymp \frac{\Delta}{x}\frac{H(x,y,z)}{\log x}, \end{aligned}$$

by Theorem 1.1. In any case, Theorem 1.6 holds. □

Using Theorems 1.3 and 1.6 together with the fact that if $d|n$, then $(n/d)|d$ as well, we show Theorem 1.7.

Proof of Theorem 1.7. We may assume that $y > \sqrt{x}$; else the result follows from Theorems 1.3 and 1.6 with $\Delta = x$. For future reference, note the trivial inequality

$$H(x,y,z;P_s) \ge \pi(z-s) - \pi(y-s) \asymp_{s,B} \frac{z-y}{\log z} \ge \frac{\eta y}{\log x}, \tag{4.12}$$

by the Prime Number Theorem. First, suppose that $\eta \le \log^{-2}(5x/z)$. For $q \in \mathbb{N}$ set

$$A_q = \{p+s \in (qy,qz] : p \equiv -s \pmod q\}.$$

If the shifted prime $p+s \le x$ has a divisor $d \in (y,z]$, then writing $p+s = dq$ we have that $q \le x/y$ and $p+s \in A_q$. So, by Lemma 2.4, we find that

$$\begin{aligned} H(x,y,z;P_s) \le \sum_{\substack{q\le x/y\\(q,s)=1}} |A_q| + O_s(1) &\ll_s \sum_{\substack{q\le x/y\\(q,s)=1}} \frac{q(z-y)}{\phi(q)\log(z-y)} \asymp_B \frac{\eta y}{\log x}\sum_{\substack{q\le x/y\\(q,s)=1}}\frac{q}{\phi(q)} \\ &\ll \frac{\eta x}{\log x} \asymp \frac{H(x,y,z)}{\log x}, \end{aligned} \tag{4.13}$$

by Theorem 1.1. This proves the upper bound in Theorem 1.7 when $\eta \le \log^{-2}(5x/z)$. In order to show the lower bound when $\eta \le \log^{-2}(5x/z)$ it suffices to consider the case $z > x^{2/3}$, since if $z \le x^{2/3}$, then we immediately obtain the result by Theorem 4.1. If $x/z \le 2|s|+2$, then $y \asymp_s x$ and thus

$$H(x,y,z;P_s) \gg_{s,B} \frac{\eta x}{\log x},$$

by (4.12). Combining this with Theorem 1.1 we complete the proof in this case. So assume that $x/z \ge 2|s|+2$, in which case

$$\{x/2z < q \le x/z : (q,s)=1\} \ne \emptyset.$$

It is easy to see that

$$H(x,y,z;P_s) \ge \Big|\bigcup_{\substack{x/2z<q\le x/z\\(q,s)=1}} A_q\Big|. \tag{4.14}$$

If we set

$$T(p) = |\{x/2z < q \le x/z : (q,s)=1,\ p+s \in A_q\}|,$$

then the Cauchy-Schwarz inequality and (4.14) yield that

$$\Big(\sum_{p+s\le x} T(p)\Big)^2 \le H(x,y,z;P_s) \sum_{p+s\le x} T^2(p). \tag{4.15}$$

First, we estimate $\sum_{p+s\le x} T(p)$. Let $C=C(B)>0$ be a constant so that

$$\sum_{\substack{q\le Q\\(q,s)=1}} \pi(X;q,-s) = \mathrm{li}(X) \sum_{\substack{q\le Q\\(q,s)=1}} \frac{1}{\phi(q)} + O_{s,B}\Big(\frac{X}{(\log X)^{B+2}}\Big) \tag{4.16}$$

for all $X\ge 2$ and all $Q\le X(\log X)^{-C}$. Such a constant exists by Lemma 2.7. If $x/z\le(\log x)^{C+1}$, then the Siegel-Walfisz theorem [4, p. 133] and Lemma 2.8 imply that

$$\begin{aligned}\sum_{p+s\le x} T(p) &= \sum_{\substack{x/2z<q\le x/z\\(q,s)=1}} \big(\pi(qz-s;q,-s)-\pi(qy-s;q,-s)\big)\\ &\gg_{s,B} \sum_{\substack{x/2z<q\le x/z\\(q,s)=1}} \frac{q(z-y)}{\phi(q)\log x} \asymp \frac{\eta x}{\log x} \sum_{\substack{x/2z<q\le x/z\\(q,s)=1}} \frac{1}{\phi(q)}\\ &\asymp_s \frac{\eta x}{\log x}.\end{aligned} \tag{4.17}$$

On the other hand, if $x/z\ge(\log x)^{C+1}$, then (4.16) and Lemma 2.8 yield that

$$\begin{aligned}\sum_{p+s\le x} T(p) &\ge \sum_{x/2<p+s\le 2x/3} \sum_{\substack{\frac{p+s}{z}\le q<\frac{p+s}{y}\\(q,s)=1,q\mid(p+s)}} 1\\ &= \sum_{\substack{y<d\le z\\(d,s)=1}} \Big(\pi(2x/3-s;d,-s)-\pi(x/2-s;d,-s)\Big) + O_s(1)\\ &= \Big(\mathrm{li}(2x/3-s)-\mathrm{li}(x/2-s)\Big) \sum_{\substack{y<d\le z\\(d,s)=1}} \frac{1}{\phi(d)} + O_{s,B}\Big(\frac{x}{(\log x)^{B+2}}\Big)\\ &\asymp_{s,B} \frac{\eta x}{\log x},\end{aligned} \tag{4.18}$$

since $\eta\le\log^{-2}5\le\log(3/2)$. Also,

$$\sum_{p+s\le x} T(p) = \sum_{\substack{x/2z<q\le x/z\\(q,s)=1}} |A_q| \ll_{s,B} \frac{\eta x}{\log x}, \tag{4.19}$$

by (4.13). Combining inequalities (4.17), (4.18) and (4.19) we deduce that

$$\sum_{p+s\le x} T(p) \asymp_{s,B} \frac{\eta x}{\log x}, \tag{4.20}$$

uniformly in $\eta \le \log^{-2}(5x/z)$ and $x/z \ge 2|s|+2$. We now bound from above the sum

$$\sum_{p+s\le x} T^2(p) = \sum_{p+s\le x} T(p) + \sum_{p+s\le x} T(p)(T(p)-1). \tag{4.21}$$

We have that

$$\begin{aligned}\sum_{p+s\le x} T(p)(T(p)-1) &= \sum_{p} \sum_{\substack{x/2z<q_1\le x/z\\ \frac{p+s}{z}\le q_1<\frac{p+s}{y}\\ q_1|(p+s),(q_1,s)=1}} \sum_{\substack{x/2z<q_2\le x/z\\ \frac{p+s}{z}\le q_2<\frac{p+s}{y}\\ q_2|(p+s),\ (q_2,s)=1\\ q_2\neq q_1}} 1\\ &= 2\sum_{\substack{\frac{x}{2z}<q_1<q_2\le\frac{x}{z}\\ (q_1q_2,s)=1}} \sum_{\substack{p\equiv -s \pmod{[q_1,q_2]}\\ q_2y<p+s\le q_1z}} 1.\end{aligned} \tag{4.22}$$

Note that we must have $q_2<e^{\eta}q_1$; otherwise, the corresponding summand on the right hand side of (4.22) is trivially zero. Lemma 2.4 and the trivial estimate $\pi(x+h;q,a)-\pi(x;q,a)\le h/q+1$ imply

$$\sum_{\substack{p\equiv -s \pmod{[q_1,q_2]}\\ q_2y<p+s\le q_1z}} 1 \ll_s \frac{q_1z-q_2y}{\phi([q_1,q_2])\log\big(3+(q_1z-q_2y)/[q_1,q_2]\big)}+1. \tag{4.23}$$

Set $m=(q_1,q_2)$ and $q_i=mt_i$, $i=1,2$, in the right hand side of (4.22). Then we will have $m\le x/2z$ and $t_1<t_2<e^{\eta}t_1$. With this notation (4.22) and (4.23) yield that

$$\begin{aligned}\sum_{p+s\le x} T(p)(T(p)-1) \ll_s{}& \log\log(x/y)\sum_{m\le\frac{x}{2z}}\ \sum_{\frac{x}{2mz}<t_1\le\frac{x}{mz}}\ \sum_{t_1<t_2<e^{\eta}t_1} \frac{z/t_2-y/t_1}{\log(3+z/t_2-y/t_1)}\\ &+\frac{x}{z}\log(x/z)+\eta\Big(\frac{x}{z}\Big)^2\end{aligned} \tag{4.24}$$

Fix m and t_1. Recall that we have assumed that $z>x^{2/3}$ and $(\log x)^{-B}\ll\eta\le(\log(5x/z))^{-2}$. So $\log\frac{z-y}{t_1}\gg_B \log x$ and consequently

$$\begin{aligned}\sum_{t_1<t_2<e^{\eta}t_1}\frac{z/t_2-y/t_1}{\log(3+z/t_2-y/t_1)} &\le \int_{t_1}^{e^{\eta}t_1}\frac{z/u-y/t_1}{\log(3+z/u-y/t_1)}du\\ &=\int_0^{(z-y)/t_1}\frac{w}{\log(w+3)}\frac{z}{(w+y/t_1)^2}dw\\ &\asymp_B \frac{\eta^2y}{\log x},\end{aligned}$$

which, combined with (4.20), (4.21) and (4.24), yields that

$$\sum_{p+s\le x}T^2(p)\ll_{s,B}\frac{\eta x}{\log x}+\frac{\eta^2x}{\log x}\log(x/y)\log\log(x/y)\ll\frac{\eta x}{\log x}.$$

Plugging the above estimate and (4.20) into (4.15) gives us that

$$H(x,y,z;P_s) \gg_{s,B} \frac{\eta x}{\log x} \asymp \frac{H(x,y,z)}{\log x},$$

by Theorem 1.1. This completes the proof of the theorem in the case when $\eta \le \log^{-2}(5x/z)$. Assume now that $\eta \ge \log^{-2}(5x/z)$. Fix a large positive constant $y_0 = y_0(s,B)$. If $x/z \le y_0$, then $\eta \ge \log^{-2}(5y_0)$. Hence (4.12) implies that

$$H(x,y,z;P_s) \gg_{s,B} \frac{z-y}{\log y} \gg_{y_0} \frac{z}{\log y} \gg_{y_0} \frac{x}{\log x}.$$

Combining the above inequality with the trivial estimate $H(x,y,z;P_s) \le \pi(x-s)$ and Theorem 1.1 we deduce that

$$H(x,y,z;P_s) \asymp_{y_0} \frac{x}{\log x} \asymp_{y_0} \frac{H(x,y,z)}{\log x},$$

which shows the desired result in this case. So suppose that $x/z > y_0$. We proceed as in the proof of Theorem 1 (iv) in [11]. Partition $(\frac{x}{\log^2(x/z)}, x]$ into intervals $(x_1,x_2]$ with

$$\frac{x_2}{\log^3(x/z)} \le x_2 - x_1 \le \frac{2x_2}{\log^3(x/z)}.$$

Observe that if $p+s \in (x_1,x_2]$, then

$$\tau\Big(p+s, \frac{x_2}{z}, \frac{x_1}{y}\Big) \ge 1 \Rightarrow \tau(p+s,y,z) \ge 1 \Rightarrow \tau\Big(p+s, \frac{x_1}{z}, \frac{x_2}{y}\Big) \ge 1.$$

So

(4.25)

$$H(x,y,z;P_s) \le \sum_{x_1,x_2} \Big\{ H\Big(x_2, \frac{x_1}{z}, \frac{x_2}{y}; P_s\Big) - H\Big(x_1, \frac{x_1}{z}, \frac{x_2}{y}; P_s\Big) \Big\} + O_s\Big(\frac{x}{\log x \log^2(x/z)}\Big).$$

Fix such an interval $(x_1,x_2]$. Then

$$\log\Big(\frac{x_1}{z}\Big) \asymp \log\Big(\frac{x}{z}\Big), \quad x_2 - x_1 \ge \frac{x_2}{\log^4(x_2/y)}, \quad \log\Big(\frac{x_2/y}{x_1/z}\Big) \asymp \eta, \quad \frac{x_1}{z} \le \sqrt{x_2},$$

provided that y_0 is large enough. Therefore Theorems 1.1 and 1.3 and Lemma 2.11 imply that

$$\begin{aligned} H\Big(x_2, \frac{x_1}{z}, \frac{x_2}{y}; P_s\Big) - H\Big(x_1, \frac{x_1}{z}, \frac{x_2}{y}; P_s\Big) &\ll_{s,B} \frac{x_2 - x_1}{x_2 \log x_2} H\Big(x_2, \frac{x_1}{z}, \frac{x_2}{y}\Big) \asymp \frac{x_2 - x_1}{x \log x} H\Big(x, \frac{x}{z}, \frac{x}{y}\Big) \\ &\asymp \frac{x_2 - x_1}{x \log x} H(x,y,z). \end{aligned}$$

Inserting the above inqequality into (4.25) and summing over x_1, x_2 completes the proof of the desired upper bound. The corresponding lower bound is obtained in a similar fashion starting from

$$H(x,y,z;P_s) \ge \sum_{x_1,x_2} \Big\{ H\Big(x_2, \frac{x_2}{z}, \frac{x_1}{y}; P_s\Big) - H\Big(x_1, \frac{x_2}{z}, \frac{x_1}{y}; P_s\Big) \Big\}$$

and using Theorem 1.6 in place of Theorem 1.3. □

We conclude this section with the proof of Theorem 1.8.

Proof of Theorem 1.8. Let $2 \le y \le z \le x$. Let $P = \prod_{y<p\le z} p$ be the product of all prime numbers in $(y, z]$. Then

$$0 \le \pi(x-s) - H(x,y,z;P_s) \le |\{p \le x-s : (p+s,P)=1\}|. \tag{4.26}$$

Lemma 2.5 implies that the right hand side of (4.26) is

$$\ll_s \frac{x}{\log x}\frac{\log y}{\log z},$$

which combined with the Prime Number Theorem completes the proof. □

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