ON THE NUMBER OF INTEGERS IN A GENERALIZED MULTIPLICATION TABLE

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ABSTRACT. Motivated by the Erdős multiplication table problem we study the following question: Given numbers $N_1, \dots, N_{k+1}$, how many distinct products of the form $n_1 \cdots n_{k+1}$ with $1 \le n_i \le N_i$ for $i \in \{1, \dots, k+1\}$ are there? Call $A_{k+1}(N_1, \dots, N_{k+1})$ the quantity in question. Ford established the order of magnitude of $A_2(N_1, N_2)$ and the author the one of $A_{k+1}(N, \dots, N)$ for all $k \ge 2$. In the present paper we generalize these results by establishing the order of magnitude of $A_{k+1}(N_1, \dots, N_{k+1})$ for arbitrary choices of $N_1, \dots, N_{k+1}$ when $2 \le k \le 5$. Moreover, we obtain a partial answer to our question when $k \ge 6$. Lastly, we develop a heuristic argument which explains why the limitation of our method is $k = 5$ in general and we suggest ways of improving the results of this paper.

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Date: May 17, 2013.

The author wishes to acknowledge support in the form of a Research Assistantship from the National Science Foundation grants DMS 05-55367, DMS 08-38434 "EMSW21-MCTP: Research Experience for Graduate Students" and DMS 09-01339.

1. Introduction

1.1. The Erdős multiplication table problem and its generalizations. In 1955 Erdős posed the so-called *multiplication table problem* [E55]: Given a large number N, how many integers can be written as a product ab with $a \le N$ and $b \le N$? Erdős gave the first estimates of this quantity [E55, E60], which were subsequently sharpened by Tenenbaum [T84]. The problem of establishing the order of magnitude of the size of the $N \times N$ multiplication table was completely solved by Ford in [Fo08a, Fo08b], where he showed that

$$A_2(N) := |\{ab : a \le N \text{ and } b \le N\}| \asymp \frac{N^2}{(\log N)^{Q(1/\log 2)}(\log\log N)^{3/2}} \quad (N \ge 3),$$

where

$$Q(u) := \int_1^u \log t \, dt = u \log u - u + 1 \quad (u > 0).$$

More generally, we may ask the same question about higher dimensional analogues of the multiplication table problem, that is to say, we may ask for estimates of

$$A_{k+1}(N) := |\{n_1 \cdots n_{k+1} : n_i \le N \ (1 \le i \le k+1)\}|.$$

In [K10a] the author determined the order of $A_{k+1}(N)$ for every fixed $k \ge 2$: we have that

$$A_{k+1}(N) \asymp_k \frac{N^{k+1}}{(\log N)^{Q(k/\log(k+1))}(\log\log N)^{3/2}} \quad (N \ge 3).$$

In the present paper we broaden our scope and study the number of integers that appear in a $(k+1)$-dimensional multiplication table when the side lengths of the table are different. More precisely, given numbers $N_1, \ldots, N_{k+1}$, we seek uniform bounds on

$$A_{k+1}(N_1, \ldots, N_{k+1}) := |\{n_1 \cdots n_{k+1} : n_i \le N_i \ (1 \le i \le k+1)\}|.$$

Instead of studying $A_{k+1}(N_1, \ldots, N_{k+1})$ directly, we focus on a closely related function: given $x \ge 1$, $\boldsymbol{y} = (y_1, \ldots, y_k) \in \mathbb{R}^k$ and $\boldsymbol{z} = (z_1, \ldots, z_k) \in \mathbb{R}^k$, define

$$H^{(k+1)}(x, \boldsymbol{y}, \boldsymbol{z}) = |\{n \le x : \exists d_1 \cdots d_k | n \text{ such that } y_i < d_i \le z_i \ (1 \le i \le k)\}|.$$

We then have the following theorem, which establishes the expected quantitative relation between $A_{k+1}(N_1, \ldots, N_{k+1})$ and $H^{(k+1)}(x, \boldsymbol{y}, \boldsymbol{z})$; its proof will be given in Subsection 3.4.

Theorem 1.1. *Let $k \ge 1$ and $3 \le N_1 \le N_2 \le \cdots \le N_{k+1}$. Then*

$$A_{k+1}(N_1, \dots, N_{k+1}) \asymp_k H^{(k+1)}\left(N_1 \cdots N_{k+1}, \left(\frac{N_1}{2}, \dots, \frac{N_k}{2}\right), (N_1, \dots, N_k)\right).$$

In light of the above theorem, it suffices to restrict ourselves to the study of $H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y})$, which is slightly easier technically. What is more, bounds on $H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y})$ have applications beyond the multiplication table problem (for example, see [Fo08b] for several such applications when $k = 1$). Before we state the results of this paper, we summarize some already known estimates on $H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y})$ in the theorem below. Briefly, this theorem gives the order of magnitude of $H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y})$ when the numbers $\log y_1, \cdots, \log y_k$ are roughly of the same size. In particular, it establishes the order of magnitude of $H^{(2)}(x, y, 2y)$ for all $2 \le y \le \sqrt{x}$. For a proof of it we refer the reader to [Fo08a, Fo08b] and [K10a]; the first two papers handle the case $k = 1$ and the latter the case $k \ge 2$.

Theorem 1.2 (Ford [Fo08a, Fo08b], Koukoulopoulos [K10a])**.** *Let $k \ge 1$, $c \ge 1$ and $\delta > 0$. Consider numbers $x \ge 3$ and $3 \le y_1 \le \cdots \le y_k \le y_1^c$ with $2^{k+1} y_1 \cdots y_k \le x/y_1^{\delta}$. Then*

$$H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y}) \asymp_{k,c,\delta} \frac{x}{(\log y_1)^{Q(k/\log(k+1))} (\log\log y_1)^{3/2}}.$$

In this paper we extend Theorem 1.2 to a broader range of the parameters $y_1, \dots, y_k$. In particular, when $2 \le k \le 5$ we establish the order of $H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y})$ for any choice of the parameters $y_1, \dots, y_k$. In order to state our results we introduce some notation. Given numbers $3 = y_0 \le y_1 \le \cdots \le y_k$, set

$$\ell_i = \log \frac{3 \log y_i}{\log y_{i-1}} \quad (1 \le i \le k).$$

Also, let i_1 be the smallest element of $\{1, \dots, k\}$ such that

$$\ell_{i_1} = \max\{\ell_i : 1 \le i \le k\}$$

and define $\beta = \beta(k; \boldsymbol{y})$ by

$$\beta = \min\left\{1, \frac{(1 + \ell_1 + \cdots + \ell_{i_1 - 1})(1 + \ell_{i_1+1} + \cdots + \ell_k)}{\ell_{i_1}}\right\}.$$

Lastly, define $\alpha = \alpha(k; \boldsymbol{y})$ implicitly, via the equation

$$\sum_{i=1}^{k} (k - i + 2)^{\alpha} \log(k - i + 2) \ell_i = \sum_{i=1}^{k} (k - i + 1) \ell_i.$$

Note that

$$\alpha \ge \min_{1 \le i \le k} \frac{1}{\log(k - i + 2)} \log\left(\frac{k - i + 1}{\log(k - i + 2)}\right) = \frac{1}{\log 2} \log\left(\frac{1}{\log 2}\right) = 0.528766373\dots$$

as well as

$$\alpha \le \max_{1 \le i \le k} \frac{1}{\log(k - i + 2)} \log\left(\frac{k - i + 1}{\log(k - i + 2)}\right) = \frac{1}{\log(k+1)} \log\left(\frac{k}{\log(k+1)}\right) < 1$$

(here we used Lemma 2.2, which will be stated and proven in Section 2).

Theorem 1.3. *Let $k \in \{2,3,4,5\}$, $x \ge 3$ and $3 \le y_1 \le \cdots \le y_k$ be such that $2^k y_1 \cdots y_k \le x/y_k$. Then*
$$\frac{H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y})}{x} \asymp \frac{\beta}{\sqrt{\log\log y_k}} \prod_{i=1}^{k} \left(\frac{\log y_i}{\log y_{i-1}}\right)^{-Q((k-i+2)^\alpha)}.$$

As we shall see later, the hypothesis that $k \in \{2,3,4,5\}$ in the above theorem is necessary: when $k \ge 6$ there are choices of the parameters $y_1,\dots,y_k$ for which $H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y})$ has genuinely smaller order than what Theorem 1.3 predicts. However, if $\log y_k$ is not much bigger than $\log y_1$, then the conlcusion of Theorem 1.3 is valid. More precisely, we have the following result, which extends Theorem 1.2.

Theorem 1.4. *Let $k \ge 6$, $x \ge 3$ and $3 \le y_1 \le \cdots \le y_k$ such that $2^k y_1 \cdots y_k \le x/y_k$ and $\log y_k \le (\log y_1)^{1+\delta}$ for a sufficiently small positive $\delta = \delta(k)$. Then*
$$\frac{H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y})}{x} \asymp_k \frac{\log \frac{3\log y_k}{\log y_1}}{(\log\log y_1)^{3/2}} \prod_{i=1}^{k} \left(\frac{\log y_i}{\log y_{i-1}}\right)^{-Q((k-i+2)^\alpha)}.$$
.

1.2. **Main results.** Both Theorems 1.3 and 1.4 are consequences of a more general estimate on $H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y})$, which is the main result of this paper.

Theorem 1.5. *Let $k \ge 2$, $x \ge 3$ and $3 \le y_1 \le \cdots \le y_k$ be such that $2^k y_1 \cdots y_k \le x/y_k$. Then*
$$\frac{H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y})}{x} \ll_k \frac{\beta}{\sqrt{\log\log y_k}} \prod_{i=1}^{k} \left(\frac{\log y_i}{\log y_{i-1}}\right)^{-Q((k-i+2)^\alpha)}.$$
If we also assume that
$$\alpha \ge 1 + \epsilon - \frac{1}{\log(k+1)} \log\left(\frac{(k+1)\log(k+1) - 2\log 2}{k-1}\right) \tag{1.1}$$
for some fixed $\epsilon > 0$, then
$$\frac{H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y})}{x} \asymp_{k,\epsilon} \frac{\beta}{\sqrt{\log\log y_k}} \prod_{i=1}^{k} \left(\frac{\log y_i}{\log y_{i-1}}\right)^{-Q((k-i+2)^\alpha)}.$$

Condition (1.1) is essentially optimal in the sense that for every fixed γ that satisfies
$$\frac{1}{\log 2}\log\left(\frac{1}{\log 2}\right) < \gamma < 1 - \frac{1}{\log(k+1)}\log\left(\frac{(k+1)\log(k+1) - 2\log 2}{k-1}\right) \tag{1.2}$$
there is a choice of $y_1 \le \cdots \le y_k$ such that $\alpha = \alpha(k;\boldsymbol{y}) = \gamma$ and for which the order of $H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y})$ is genuinely smaller than the one stated above. We shall discuss this further in the next section using some heuristic arguments. In relation to our comments following the statement of Theorem 1.3, note that the smallest value of k for which the range (1.2) is non-empty is $k = 6$.

Despite its optimality, condition (1.1) is not very easy to work with due to the implicit definition of α. Below we state a weaker version of Theorem 1.5, whose hypotheses are easier to verify.

Corollary 1.6. *Let* $k \geq 2$, $h \in \{1,\dots,k\}$, $x \geq 3$ *and* $3 \leq y_1 \leq \cdots \leq y_k$ *such that* $2^k y_1 \cdots y_k \leq x/y_k$,

$$\frac{1}{\log(k-h+2)}\log\left(\frac{k-h+1}{\log(k-h+2)}\right) > 1 - \frac{1}{\log(k+1)}\log\left(\frac{(k+1)\log(k+1)-2\log 2}{k-1}\right).$$

and $\log y_k \leq (\log y_h)^{1+\delta}$ *for a sufficiently small positive* $\delta = \delta(k)$. *Then*

$$\frac{H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y})}{x} \asymp_k \frac{\beta}{\sqrt{\log\log y_k}}\prod_{i=1}^{k}\left(\frac{\log y_i}{\log y_{i-1}}\right)^{-Q((k-i+2)^\alpha)}.$$

Proof. We consider for the moment δ to be a free parameter. Since $\log y_k \leq (\log y_h)^{1+\delta}$, we have that

$$\sum_{i=1}^{h}(k-i+2)^\alpha\log(k-i+2)\ell_i = \sum_{i=1}^{h}(1+O_k(\delta))(k-i+1)\ell_i.$$

Therefore

$$\begin{aligned}\alpha &\geq \min_{1\leq i\leq h}\frac{1}{\log(k-i+2)}\log\left(\frac{k-i+1}{\log(k-i+2)}\right) - O_k(\delta)\\ &= \frac{1}{\log(k-h+2)}\log\left(\frac{k-h+1}{\log(k-h+2)}\right) - O_k(\delta),\end{aligned}$$

by Lemma 2.2. So if we choose δ small enough, then (1.1) holds and hence the desired result follows. ☐

Applying the above corollary with $h = k \leq 5$ gives us Theorem 1.3 immediately. Similarly, Theorem 1.4 follows by Corollary 1.6 with $h = 1$; we only need to check that

$$\frac{1}{\log(k+1)}\log\left(\frac{k}{\log(k+1)}\right) > 1 - \frac{1}{\log(k+1)}\log\left(\frac{(k+1)\log(k+1)-2\log 2}{k-1}\right) \tag{1.3}$$

or, equivalently, that

$$(k+1)\log(k+1) > k\log 4$$

for $k \geq 2$, which is indeed true.

The main tool we shall use in order to prove Theorems 1.1 and 1.5 is a result that reduces the counting in $H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y})$, which contains information about the local distribution of factorizations of integers, to the estimation of a certain sum which contains information about the global distribution of factorizations of integers. More precisely, for $\boldsymbol{a} = (a_1,\dots,a_k) \in \mathbb{N}^k$ define

$$\mathcal{L}^{(k+1)}(\boldsymbol{a}) = \bigcup_{\substack{d_1\cdots d_i | a_1\cdots a_i\\ 1\leq i\leq k}} [\log(d_1/2),\log d_1) \times \cdots \times [\log(d_k/2),\log d_k)\,,$$

and

$$L^{(k+1)}(\boldsymbol{a}) = \mathrm{Vol}(\mathcal{L}^{(k+1)}(\boldsymbol{a})),$$

where "Vol" denotes the k-dimensional Lebesgue measure. Also, for $1 \leq y < z$ set

$$\mathcal{P}_*(y,z) = \{n \in \mathbb{N} : \mu^2(n) = 1, p|n \Rightarrow y < p \leq z\}$$

and for $\boldsymbol{t} = (t_1, \ldots, t_k)$ with $t_k \geq t_{k-1} \geq \cdots \geq t_1 \geq 1 =: t_0$ set

$$\mathcal{P}_*^k(\boldsymbol{t}) = \{(a_1, \ldots, a_k) \in \mathbb{N}^k : a_i \in \mathcal{P}_*(t_{i-1}, t_i) \ (1 \leq i \leq k)\}.$$

Then we have the following estimate.

Theorem 1.7. *Let $k \geq 1$, $x \geq 1$ and $3 \leq y_1 \leq \cdots \leq y_k$ with $2^k y_1 \cdots y_k \leq x/y_k$. Then*

$$\frac{H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y})}{x} \asymp_k \prod_{i=1}^{k} \left(\frac{\log y_i}{\log y_{i-1}}\right)^{-(k-i+2)} \sum_{\boldsymbol{a} \in \mathcal{P}_*^k(\boldsymbol{y})} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1 \cdots a_k}.$$

When $k = 1$, the above theorem is an immediate consequence of the results and the methods in [Fo08a]: see Lemmas 2.1 and 3.2 there. As an immediate consequence of Theorem 1.7, we have the following result.

Corollary 1.8. *Let $k \geq 1$ be an integer and for $i \in \{1, 2\}$ consider $x_i \geq 1$ and $\boldsymbol{y}_i = (y_{i,1}, \ldots, y_{i,k}) \in [1, +\infty)^k$. Assume that $2^k y_{i,1} \cdots y_{i,k} \leq x/y_{i,k}$ for $i \in \{1, 2\}$ and that there exist constants c and C such that $y_{1,j}^c \leq y_{2,j} \leq y_{1,j}^C$ for all $j \in \{1, \ldots, k\}$. Then*

$$\frac{H^{(k+1)}(x_1, \boldsymbol{y}_1, 2\boldsymbol{y}_1)}{x_1} \asymp_{k,c,C} \frac{H^{(k+1)}(x_2, \boldsymbol{y}_2, 2\boldsymbol{y}_2)}{x_2}.$$

Proof. The result follows by Theorem 1.7, Lemma 2.1(a) and the standard estimate

$$\sum_{a \in \mathcal{P}_*(t, t^B)} \frac{\tau_m(a)}{a} = \prod_{t < p \leq t^B} \left(1 + \frac{m}{p}\right) \asymp_{m,B} 1 \quad (t \geq 1), \tag{1.4}$$

where

$$\tau_m(a) = \sum_{d_1 \cdots d_{m-1} | a} 1 = \sum_{d_1 \cdots d_m = a} 1 \quad (m \in \mathbb{N}, \ a \in \mathbb{N}).$$

□

When $k = 1$, a stronger version of the above corollary is known to be true: see Corollary 1 in [Fo08b].

1.3. **Outline of the paper.** The paper is organized in the following way: In Section 2 we demonstrate a heuristic argument in support of Theorem 1.5 and the optimality of condition (1.1). The first three subsections of Section 3 are devoted to establishing Theorem 1.7, whereas in the last one we prove Theorem 1.1. In Section 4 we develop some estimates related to the probability that a multidimensional Poisson random variable lies close to a hyperplane. Such estimates play a crucial role in the proof of Theorem 1.5. Also, in combination with the heuristic arguments of Section 2, they explain how the parameter α makes its appearance in the statements of our results. In Section 5 we give the proof of the upper bound in Theorem 1.5. The main steps of the proof are described in Subsection 5.1 and proven in Subsection 5.2. The proof of the lower bound in Theorem 1.5 is divided in three sections: in Section 6 we describe the main steps of our argument. The first major such step is then carried out in Section 7. Finally, Section 8 contains the last piece of our argument and completes the proof of Theorem 1.5.

1.4. **Notation.** We make use of some standard notation. For $n \in \mathbb{N}$ we use $P^+(n)$ and $P^-(n)$ to denote the largest and smallest prime factor of n, respectively, with the notational conventions that $P^+(1) = 1$ and $P^-(1) = +\infty$. Also, $\omega(n)$ denotes the number of distinct prime factors of n. Constants implied by $\ll$, $\gg$ and $\asymp$ are absolute unless otherwise specified, e.g. by a subscript. Also, we use the letters c and C to denote constants, not necessarily the same ones in every place, possibly depending on certain parameters that will be specified by subscripts and other means. Also, bold letters always denote vectors whose coordinates are indexed by the same letter with subscripts, e.g. $\boldsymbol{a} = (a_1, \dots, a_k)$ and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_r)$. The dimension of the vectors will not be explicitly specified if it is clear by the context. Finally, we give a table of some basic non-standard notation that we will be using with references to page numbers for its definition.

Symbol	Page	Symbol	Page	Symbol	Page
$Q(u)$	2	α	3	α_i	7
β	3	i_1	3	i_0	7
ℓ_i	3	v_i	33	Δ_r	34
ρ_m	33	$\boldsymbol{e}_k,\ e_{k,i}$	18	$\tau_m(a)$	6
$\tau_{k+1}(\boldsymbol{a})$	7	$\tau_{k+1}(a, \boldsymbol{y}, \boldsymbol{z})$	7	$\mathcal{P}_*(y, z)$	5
$\mathcal{P}_*^k(\boldsymbol{t})$	6	$H^{(k+1)}(x, \boldsymbol{y}, \boldsymbol{z})$	2	$A_{k+1}(N_1, \dots, N_{k+1})$	2
$\mathcal{L}^{(k+1)}(\boldsymbol{a})$	5	$L^{(k+1)}(\boldsymbol{a})$	5	$S^{(k+1)}(\boldsymbol{t})$	15

Acknowledgement. I would like to thank Kevin Ford for many valuable suggestions as well as for discussions that led to an earlier version of Lemma 2.3.

2. Heuristic arguments

In this section we develop a heuristic argument in support of Theorem 1.5. Its main part is given in Subsection 2.1 and it is a generalization of an argument developed by Ford in [Fo08a] for the case $k = 1$ and subsequently by the author in [K10a] for the case $k \geq 2$. In Subsection 2.2 we introduce some new ideas in order to explain the appearance of condition (1.1) in the statement of Theorem 1.5.

Before we delve into the details of this argument, we introduce some additional notation and state two elementary but basic results we will be using throughout the entire paper. For $\boldsymbol{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$ and $\boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^k$ let

$$\tau_{k+1}(\boldsymbol{a}) = |\{(d_1, \dots, d_k) \in \mathbb{N}^k : d_1 \cdots d_i | a_1 \cdots a_i \ (1 \leq i \leq k)\}|$$

and

$$\tau_{k+1}(\boldsymbol{a}, \boldsymbol{y}, \boldsymbol{z}) = |\{(d_1, \dots, d_k) \in \mathbb{N}^k : d_1 \cdots d_i | a_1 \cdots a_i,\ y_i < d_i \leq z_i \ (1 \leq i \leq k)\}|.$$

Finally, set

$$\alpha_i = \frac{1}{\log(i+1)} \log\left(\frac{i}{\log(i+1)}\right) \quad (i \in \mathbb{N})$$

and let i_0 be the minimum element of $\{1, \dots, k\}$ such that

$$|\alpha - \alpha_{k-i_0+1}| = \min\{|\alpha - \alpha_{k-i+1}| : 1 \leq i \leq k\}.$$

Lemma 2.1. *The following assertions hold:*

(a) *For $\boldsymbol{a} \in \mathbb{N}^k$ we have*

$$L^{(k+1)}(\boldsymbol{a}) \le \min\left\{\tau_{k+1}(\boldsymbol{a})(\log 2)^k, \prod_{i=1}^{k}(\log a_1 + \cdots + \log a_i + \log 2)\right\}.$$

(b) *If $(a_1 \cdots a_k, b_1 \cdots b_k) = 1$, then*

$$L^{(k+1)}(a_1 b_1, \dots, a_k b_k) \le \tau_{k+1}(\boldsymbol{a}) L^{(k+1)}(\boldsymbol{b}).$$

Proof. The proof is similar to the proof of Lemma 3.1 in [Fo08a]. □

Lemma 2.2. *The sequence $\{\alpha_i\}_{i\in\mathbb{N}}$ is strictly increasing.*

Proof. The function

$$\frac{1}{\log(x+1)} \log\left(\frac{x}{\log(x+1)}\right)$$

is easily seen to be strictly increasing for $x \ge 15$. Finally, we check numerically that $\alpha_1 < \alpha_2 < \cdots < \alpha_{15}$. □

2.1. **Basic set-up and development of the main argument.** Our goal is to understand when an integer $n \le x$ is counted by $H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y})$. We write $n = a_1 \cdots a_k b$, where

$$a_i = \prod_{\substack{p^\nu \| n \\ 2y_{i-1} < p \le 2y_i}} p^\nu \quad (1 \le i \le k).$$

For simplicity, we assume that the numbers $a_1, \dots, a_k$ are square-free and satisfy $\log a_i \asymp \log y_i$ for all $i \in \{1, \dots, k\}$. Observe that if $\boldsymbol{d} = (d_1, \dots, d_k) \in \mathbb{N}^k \cap \prod_{i=1}^k (y_i, 2y_i]$, then all prime factors of d_i are at most $2y_i$ for all $i \in \{1, \dots, k\}$. Hence $\boldsymbol{d}$ satisfies the relation $d_1 \cdots d_k | n$ if, and only if, $d_1 \cdots d_i | a_1 \cdots a_i$ for all $i \in \{1, \dots, k\}$. Therefore the integer n is counted by $H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y})$ precisely when $\tau_{k+1}(\boldsymbol{a}, \boldsymbol{y}, 2\boldsymbol{y}) \ge 1$. Consider now the set

$$D_{k+1}(\boldsymbol{a}) = \{(\log d_1, \dots, \log d_k) : d_1 \cdots d_i | a_1 \cdots a_i \ (1 \le i \le k)\}.$$

Assume for the moment that $D_{k+1}(\boldsymbol{a})$ is well-distributed in $\prod_{i=1}^k [0, \log(a_1 \cdots a_i)]$. Then we should have that

$$\begin{aligned} \tau_{k+1}(\boldsymbol{a}, \boldsymbol{y}, 2\boldsymbol{y}) = \left| D_{k+1}(\boldsymbol{a}) \cap \prod_{i=1}^{k} (\log y_i, \log y_i + \log 2] \right| &\approx \tau_{k+1}(\boldsymbol{a}) \frac{(\log 2)^k}{\prod_{i=1}^k \log(a_1 \cdots a_i)} \\ &\asymp_k \frac{\prod_{i=1}^k (k-i+2)^{\omega(a_i)}}{\prod_{i=1}^k \log y_i}. \end{aligned} \tag{2.1}$$

The right hand side of (2.1) is at least 1 when

$$\sum_{i=1}^{k} \log(k-i+2)\omega(a_i) \ge \sum_{i=1}^{k} \log\log y_i + O_k(1) = \sum_{i=1}^{k} (k-i+1)\ell_i + O_k(1).$$

Since we expect that

$$|\{n \le x : \omega(a_i) = r_i \ (1 \le i \le k)\}| \approx \frac{x}{\log y_k} \frac{\ell_1^{r_1-1} \cdots \ell_k^{r_k-1}}{(r_1-1)! \cdots (r_k-1)!}$$

(for example, see [T95, Theorem 4, p. 205]), summing the above relation over all vectors $\boldsymbol{r} \in (\mathbb{N} \cup \{0\})^k$ that satisfy

$$\sum_{i=1}^{k} r_i \log(k-i+2) \ge \sum_{i=1}^{k} \ell_i (k-i+1) + O_k(1) \tag{2.2}$$

leads to the estimate

$$H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y}) \approx \frac{x}{\log y_k} \sum_{\substack{\boldsymbol{r} \in (\mathbb{N} \cup \{0\})^k \\ (2.2)}} \frac{\ell_1^{r_1-1} \cdots \ell_k^{r_k-1}}{(r_1-1)! \cdots (r_k-1)!}. \tag{2.3}$$

Using Stirling's formula and Lagrange multipliers, we find that the maximum of $\prod_{i=1}^{k} \ell_i^{r_i-1}/(r_i-1)!$ under condition (2.2) occurs when $r_i \sim (k-i+2)^\alpha \ell_i$ (see Section 4 and, in particular, Remark 4.1 and the proof of Lemma 4.3(a)). In fact, Lemma 4.3(a) implies that

$$\sum_{\substack{\boldsymbol{r} \in (\mathbb{N} \cup \{0\})^k \\ (2.2)}} \frac{\ell_1^{r_1-1} \cdots \ell_k^{r_k-1}}{(r_1-1)! \cdots (r_k-1)!} \asymp_k \frac{\log y_k}{\sqrt{\log\log y_k}} \prod_{i=1}^{k} \left(\frac{\log y_i}{\log y_{i-1}} \right)^{-Q((k-i+2)^\alpha)},$$

so that (2.3) becomes

$$H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y}) \approx \frac{x}{\sqrt{\log\log y_k}} \prod_{i=1}^{k} \left(\frac{\log y_i}{\log y_{i-1}} \right)^{-Q((k-i+2)^\alpha)}. \tag{2.4}$$

If now $\beta \gg 1$, then (2.4) agrees with the conclusion of Theorem 1.5. However, if $\beta = o_{y_1 \to \infty}(1)$, then relation (2.4) overestimates $H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y})$ slightly. The problem lies in our assumption that $D_{k+1}(\boldsymbol{a})$ is well-distributed. Actually, if $\beta = o_{y_1 \to \infty}(1)$, then with high probability the elements of $D_{k+1}(\boldsymbol{a})$ form large clumps. In order to measure the amount of clustering in $D_{k+1}(\boldsymbol{a})$, we use the function $L^{(k+1)}(\boldsymbol{a})$, which we introduced in Section 1. We will show that, unless the prime factors of $a_1, \dots, a_k$ satisfy certain constraints, the measure of $L^{(k+1)}(\boldsymbol{a})$ is small and, as a consequence, the set $D_{k+1}(\boldsymbol{a})$ cannot be well-distributed.

Fix a vector $\boldsymbol{r} \in \mathbb{N}^k$ such that

$$0 \le \sum_{i=1}^{k} r_i \log(k-i+2) - \sum_{i=1}^{k} \ell_i (k-i+1) \le \log(k+1) \tag{2.5}$$

and $r_i \sim (k-i+2)^\alpha \ell_i$ as $\ell_i \to \infty$, for all $i \in \{1, \dots, k\}$, since most of the contribution to the sum in the right hand side of (2.3) comes from such vectors. Consider n with $\omega(a_i) = r_i$ for all $i \in \{1, \dots, k\}$ and write $a_i = p_{i,1} \cdots p_{i,r_i}$ with $2y_{i-1} < p_{i,1} < \cdots < p_{i,r_i} \le 2y_i$. Set

$$U_i = 2\log(k+1) + \sum_{m=1}^{i-1} \ell_m (k-m+1) - \sum_{m=1}^{i-1} r_m \log(k-m+2) \quad (1 \le i \le k).$$

Note that

$$\begin{aligned}&\ell_m(k-m+1)-r_m\log(k-m+2)\\&=(k-m+1-(k-m+2)^{\alpha}\log(k-m+2)+o(1))\ell_m\\&=\log(k-m+2)((k-m+2)^{\alpha_{k-m+1}}-(k-m+2)^{\alpha}+o(1))\ell_m.\end{aligned}$$

So Lemma 2.2 and the definition of i_0 imply that

$$U_i \asymp_k \begin{cases}1+\ell_1+\cdots+\ell_{i-1} & \text{if } 1\le i\le i_0,\\ 1+\ell_i+\cdots+\ell_k & \text{if } i_0+1\le i\le k+1,\end{cases} \tag{2.6}$$

where in the latter case we used (2.5). Assume that there are integers $i\in\{1,\dots,k\}$ and $j\in\{1,\dots,r_i\}$ and a large number C such that

$$0\le\log\log p_{i,j}-\log\log y_{i-1}\le\frac{\log(k-i+2)j-U_i-C}{k-i+1}.$$

We claim that this causes clustering among the elements of $D_{k+1}(\boldsymbol{a})$. Indeed, if we set $b_m=a_m$ for $1\le m<i$, $b_i=p_{i,1}\cdots p_{i,j}$ and $b_m=1$ for $i<m\le k$, then a double application of Lemma 2.1 implies that

$$\begin{aligned}L^{(k+1)}(\boldsymbol{a})&\le\tau_{k+1}(a_1/b_1,\dots,a_k/b_k)L^{(k+1)}(\boldsymbol{b})\\&\le\left((k-i+2)^{r_i-j}\prod_{m=i+1}^{k}(k-m+2)^{r_m}\right)\left(\prod_{m=1}^{k}\log(2b_1\cdots b_m)\right)\\&\ll_k(k-i+2)^{-j}\left(\prod_{m=i}^{k}(k-m+2)^{r_m}\right)\left(\prod_{m=1}^{i-1}\log y_m\right)\\&\qquad\times(\log y_{i-1}+\log(p_{i,1}\cdots p_{i,j}))^{k-i+1}\\&\lesssim\left(\prod_{m=i}^{k}(k-m+2)^{r_m}\right)\left(\prod_{m=1}^{i-1}\log y_m\right)(\log y_{i-1})^{k-i+1}e^{-U_i-C}\\&\asymp_k e^{-C}\prod_{i=1}^{k}(k-i+2)^{r_i}.\end{aligned}\tag{2.7}$$

The right hand side of (2.7) is much less than $\tau_{k+1}(\boldsymbol{a})=\prod_{m=1}^{k}(k-m+2)^{r_m}$ if $C\to\infty$, in which case there must be many elements of $D_{k+1}(\boldsymbol{a})$ that are close together. The above argument suggests that we should focus on numbers n for which

$$\log\log p_{i,j}-\log\log y_{i-1}\ge\frac{\log(k-i+2)j-U_i-O(1)}{k-i+1}\quad(1\le i\le k,\ R_{i-1}<j\le R_i).\tag{2.8}$$

The number of integers n that satisfy conditions similar to (2.8) was studied by Ford in [Fo07]. Using similar considerations, we find that the probability that an integer n satisfies (2.8) is about

$$\prod_{i=1}^{k}\min\left\{1,\frac{U_iU_{i+1}}{r_i}\right\}\asymp_k\min\left\{1,\frac{(1+\ell_1+\cdots+\ell_{i_0-1})(1+\ell_{i_0+1}+\cdots+\ell_k)}{\ell_{i_0}}\right\},$$

by (2.6). Thus we are led to the refined estimate

$$\frac{H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y})}{x} \approx \frac{\min\left\{1, \dfrac{(1+\ell_1+\cdots+\ell_{i_0-1})(1+\ell_{i_0+1}+\cdots+\ell_k)}{\ell_{i_0}}\right\}}{\sqrt{\log\log y_k}\prod_{i=1}^{k}\left(\dfrac{\log y_i}{\log y_{i-1}}\right)^{Q((k-i+2)^\alpha)}}. \tag{2.9}$$

Finally, we claim that

$$\min\left\{1, \frac{(1+\ell_1+\cdots+\ell_{i_0-1})(1+\ell_{i_0+1}+\cdots+\ell_k)}{\ell_{i_0}}\right\} \asymp_k \beta. \tag{2.10}$$

To see this, fix a small parameter $\delta=\delta(k)$ and observe that if

$$\sum_{i\neq i_1}\ell_i \le \delta\ell_{i_1} = \delta\max_{1\le i\le k}\ell_i,$$

then

$$(k-i_1+2)^\alpha\log(k-i_1+2)\ell_{i_1} = (1+O_k(\delta))(k-i_1+1)\ell_{i_1},$$

by the definition of α. This implies that $|\alpha-\alpha_{k-i_1+1}| \ll_k \delta$. So if $\delta=\delta(k)$ is small enough, then $i_1=i_0$ and (2.10) follows immediately. Consider now the case when $\sum_{i\neq i_1}\ell_i \ge \delta\ell_{i_1}$. We may also assume that $i_1\neq i_0$; else, (2.10) holds trivially. Under these assumptions we have that

$$\beta \ge \min\left\{1, \frac{\sum_{i\neq i_1}\ell_i}{\ell_{i_1}}\right\} \ge \delta$$

and

$$\min\left\{1, \frac{(1+\ell_1+\cdots+\ell_{i_0-1})(1+\ell_{i_0+1}+\cdots+\ell_k)}{\ell_{i_0}}\right\} \ge \min\left\{1, \frac{\ell_{i_1}}{\ell_{i_0}}\right\} = 1,$$

which together prove (2.10) in this last case too. By (2.10), we see that (2.9) agrees with the conclusion of Theorem 1.5.

2.2. **Further analysis and optimality of condition** (1.1)**.** Even though the argument given in Subsection 2.1 gives us Theorem 1.5 heuristically, it does not explain the presence of condition (1.1) in the statement of the theorem. This deficiency stems from the fact that the only piece of information we used about $\mathcal{L}^{(k+1)}(\boldsymbol{a})$ is Lemma 2.1. In order to understand condition (1.1), we need to pay closer attention to the structure of $\mathcal{L}^{(k+1)}(\boldsymbol{a})$. It turns out that when k is large, the rich multiplicative structure and the high dimension of the set $\mathcal{L}^{(k+1)}(\boldsymbol{a})$ lead to many more bounds on its volume $L^{(k+1)}(\boldsymbol{a})$ than just those included in the statement of Lemma 2.1.

Lemma 2.3. *Consider integers $0 = z_0 \le z_1 \le \cdots \le z_k \le z_{k+1} = k$ with $z_i \ge i-1$ for all $i \in \{1,\dots,k\}$. Let $\boldsymbol{a} = (a_1,\dots,a_k) \in \mathbb{N}^k$ such that $\mu^2(a_1\cdots a_k) = 1$. Then we have that*

$$L^{(k+1)}(\boldsymbol{a}) \le \sum_{\substack{d_j|a_j\\1\le j\le k}} \left(\prod_{j=1}^k (z_j-j+1)^{\omega(d_j)}\right) \times \min\left\{\prod_{j=0}^k \log^{z_{j+1}-z_j}(2a_1\cdots a_j), (\log 2)^k \prod_{j=1}^k (k-z_j+1)^{\omega(a_j/d_j)}\right\},$$

with the convention that $0^0 = 1$.

Proof. Given a k-tuple $(d_1,\dots,d_k) \in \mathbb{N}^k$ with $d_1\cdots d_i | a_1\cdots a_i$ for $1 \le i \le k$, we may uniquely write $d_i = d_{i,1}d_{i,2}\cdots d_{i,i}$, $1 \le i \le k$, with $d_{j,j}d_{j+1,j}\cdots d_{k,j}|a_j$ for $1 \le j \le k$. Thus

$$\mathcal{L}^{(k+1)}(\boldsymbol{a}) = \bigcup_{\substack{d_{j,j}d_{j+1,j}\cdots d_{k,j}|a_j\\1\le j\le k}} \prod_{i=1}^k [\log(d_{i,1}d_{i,2}\cdots d_{i,i}/2), \log(d_{i,1}d_{i,2}\cdots d_{i,i})).$$

For $i \in \{1,\dots,k\}$ define m_i as the unique element of $\{0,1,\dots,k\}$ such that $z_{m_i} < i \le z_{m_i+1}$. Note that $i > z_{m_i} \ge m_i - 1$ and thus $m_i \le i$. Set

$$\mathcal{I} = \{(i,j) : 1 \le j \le k, j \le i \le z_j\} = \{(i,j) : 1 \le i \le k, m_i < j \le i\}.$$

Given numbers $d_{i,j}$, $(i,j) \in \mathcal{I}$, with $d_{j,j}\cdots d_{z_j,j}|a_j$, $1 \le j \le k$, we define the set

$$\begin{aligned}
\mathcal{L}(\{d_{i,j} : (i,j) \in \mathcal{I}\}) &= \bigcup_{\substack{d_{i,j},\ (i,j)\notin\mathcal{I}\\ 1\le j\le i\le k\\ d_{z_j+1,j}\cdots d_{k,j}|\frac{a_j}{d_{j,j}\cdots d_{z_j,j}}\ \forall j}} \prod_{i=1}^k [\log(d_{i,1}\cdots d_{i,i}/2), \log(d_{i,1}\cdots d_{i,i})) \\
&= \bigcup_{\substack{d_{z_j+1,j}\cdots d_{k,j}|\frac{a_j}{d_{j,j}\cdots d_{z_j,j}}\\ 1\le j\le k}} \prod_{i=1}^k [\log(d_{i,1}\cdots d_{i,m_i}/2), \log(d_{i,1}\cdots d_{i,m_i})) \\
&\quad + \left(\log(d_{1,m_1+1}\cdots d_{1,1}), \log(d_{2,m_2+1}\cdots d_{2,2}), \dots, \log(d_{k,m_k+1}\cdots d_{k,k})\right).
\end{aligned}$$

The above identity implies that

$$\mathrm{Vol}\left(\mathcal{L}(\{d_{i,j} : (i,j) \in \mathcal{I}\})\right) \le \min\left\{\prod_{i=1}^k \log(2a_1\cdots a_{m_i}), (\log 2)^k \prod_{i=1}^k (k-z_i+1)^{\omega\left(\frac{a_i}{d_{i,i}\cdots d_{z_i,i}}\right)}\right\}.$$

Since

$$\mathcal{L}^{(k+1)}(\boldsymbol{a}) = \bigcup_{\substack{d_{i,j},\ (i,j)\in\mathcal{I}\\ d_{j,j}\cdots d_{z_j,j}|a_j\ \forall j}} \mathcal{L}(\{d_{i,j} : (i,j) \in \mathcal{I}\}),$$

we find that

$$L^{(k+1)}(\boldsymbol{a}) \leq \sum_{\substack{d_{i,j},\,(i,j)\in\mathcal{I} \\ D_j=d_{j,j}\cdots d_{z_j,j}|a_j\ \forall j}} \min\left\{\prod_{i=1}^{k}\log(2a_1\cdots a_{m_i}),(\log 2)^k\prod_{i=1}^{k}(k-z_i+1)^{\omega(a_i/D_i)}\right\}$$

$$= \sum_{\substack{D_j|a_j \\ 1\leq j\leq k}} \left(\prod_{i=1}^{k}(z_i-i+1)^{\omega(D_i)}\right)$$

$$\times \min\left\{\prod_{i=1}^{k}\log(2a_1\cdots a_{m_i}),(\log 2)^k\prod_{i=1}^{k}(k-z_i+1)^{\omega(a_i/D_i)}\right\}.$$

To complete the proof of the lemma note that

$$\prod_{i=1}^{k}\log(2a_1\cdots a_{m_i}) = \prod_{j=0}^{k}\log^{z_{j+1}-z_j}(2a_1\cdots a_j).$$

$\square$

Using the above lemma, we show that condition (1.1) is optimal, that is to say, for every fixed γ such that

$$\frac{1}{\log 2}\log\frac{1}{\log 2} < \gamma < 1-\frac{1}{\log(k+1)}\log\left(\frac{(k+1)\log(k+1)-2\log 2}{k-1}\right), \tag{2.11}$$

there are choices of $y_1 \leq y_2 \leq \cdots y_k$ such that $\alpha = \alpha(k;\boldsymbol{y}) = \gamma$ and

$$H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y}) = o\left(\frac{\beta}{\sqrt{\log\log y_k}}\prod_{i=1}^{k}\left(\frac{\log y_i}{\log y_{i-1}}\right)^{-Q((k-i+2)^{\alpha})}\right) \quad (y_1\to\infty). \tag{2.12}$$

The argument we give is heuristic but, if combined with the results of Sections 4 and 5, it can be made rigorous.

The right inequality in (2.11) is equivalent to

$$\frac{(k+1)\log(k+1)-2\log 2}{(k+1)^{1-\gamma}} < k-1. \tag{2.13}$$

Also, inequalities (1.3) and (2.11) imply that

$$k-(k+1)^{\gamma}\log(k+1) > 0 \quad \text{and} \quad 2^{\gamma}\log 2 - 1 > 0.$$

So if we select $y_1 = y_2 = \cdots = y_{k-1}$ large enough, then there is a unique $y_k \geq y_{k-1}$ such that

$$\ell_k = \frac{1}{2^{\gamma}\log 2-1}\sum_{i=1}^{k-1}(k-i+1-(k-i+2)^{\gamma}\log(k-i+2))\ell_i,$$

that is to say, there is a unique y_k so that the k-tuple $\boldsymbol{y} = (y_1,\ldots,y_k)$ satisfies the relation $\alpha(k,\boldsymbol{y}) = \gamma$. We claim that (2.12) holds and we support this claim with a heuristic argument:

Similarly to Subsection 2.1, we consider $\boldsymbol{a} = (a_1, \ldots, a_k)$ such that $a_i \in \mathcal{P}_*(2y_{i-1}, 2y_i)$. Note that we necessarily have that $a_2 = \cdots a_{k-1} = 1$, since $y_1 = \cdots = y_{k-1}$. Set $r_i = \omega(a_i)$ for all $i \in \{1, \ldots, k\}$ and assume further that $\log a_i \asymp \log y_i$ for $i \in \{1, k\}$, that

$$r_i \sim (k-i+2)^\alpha \ell_i = (k-i+2)^\gamma \ell_i \quad (i \in \{1,k\},\ y_1 \to \infty) \tag{2.14}$$

and that (2.5) holds. We will show that

$$L^{(k+1)}(\boldsymbol{a}) = o\left(\prod_{i=1}^k \log y_i\right) \quad (y_1 \to \infty). \tag{2.15}$$

Indeed, Lemma 2.3, applied with $z_1 = \cdots = z_k = k-1$, implies that

$$\begin{aligned} L^{(k+1)}(\boldsymbol{a}) &\ll_k \sum_{\substack{d_j | a_j \\ 1 \le j \le k}} \left(\prod_{j=1}^k (k-j)^{\omega(d_j)}\right) \min\left\{\log y_k, \prod_{j=1}^k 2^{\omega(a_j/d_j)}\right\} \\ &= \sum_{d_1 | a_1} (k-1)^{\omega(d_1)} \min\left\{\log y_k, 2^{r_k + \omega(a_1/d_1)}\right\} \end{aligned}$$

(note that all summands with $d_k > 1$ vanish and $d_j = a_j = 1$ for $j \in \{2, \ldots, k-1\}$). The main contribution to the sum

$$\sum_{d_1 | a_1} (k-1)^{\omega(d_1)} 2^{r_k + \omega(a_1/d_1)} = (k+1)^{r_1} 2^{r_k} = \prod_{j=1}^k (k-j+2)^{r_j} \asymp_k \prod_{i=1}^k \log y_i$$

comes from integers d_1 such that

$$\omega(d_1) \sim \frac{k-1}{k+1} \cdot r_1 \quad (y_1 \to \infty). \tag{2.16}$$

If d_1 satisfies (2.16), then relations (2.5), (2.13) and (2.14) and the fact that $r_i = 0$ and $\ell_i = O(1)$ for $i \in \{2, \ldots, k-1\}$ imply that

$$\begin{aligned} (r_k + \omega(a_1/d_1)) \log 2 - \log\log y_k &= \frac{2\log 2}{k+1} r_1 + (\log 2) r_k - \ell_1 - \ell_k + o_k(\ell_1) \\ &= (k-1)\ell_1 - \frac{(k+1)\log(k+1) - 2\log 2}{k+1} r_1 + o_k(\ell_1) \to +\infty \end{aligned}$$

as $y_1 \to \infty$. Consequently, for integers d_1 that satisfy (2.16) we have that

$$\min\left\{\log y_k, 2^{r_k + \omega(a_1/d_1)}\right\} = \log y_k = o\left(2^{r_k + \omega(a_1/d_1)}\right) \quad (y_1 \to \infty),$$

which in turn implies that relation (2.15) is indeed true. This yields that, in contrast to the prediction of the arguments in Subsection 2.1, $D_{k+1}(\boldsymbol{a})$ is not well-distributed for such $\boldsymbol{a}$. Hence, in general, relation (2.9) overestimates the size of $H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y})$.

Remark 2.4. The information about $L^{(k+1)}(\boldsymbol{a})$ that is contained in Lemma 2.3 makes its appearance implicitly in the statement of Lemma 6.2. An approach that could potentially extend Theorem 1.5 to the case when condition (1.1) fails is to insert Lemma 2.3 into the proof of the upper bound in Theorem 1.5 (Section 5) and then adjust the lower bound argument accordingly (Sections 6, 7 and 8).

3. Local-to-global estimates

In this section we reduce the counting in $H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y})$ to the estimation of

$$S^{(k+1)}(\boldsymbol{t}) := \sum_{\boldsymbol{a}\in\mathcal{P}_*^k(\boldsymbol{t})} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1\cdots a_k}$$

and prove Theorem 1.7. This reduction has also been carried in the author's thesis [K10b], but we give it here for completeness. The basic ideas behind it can be found in [Fo08a] and [K10a]. However, the details are more complicated, especially in the proof of the upper bound implicit in Theorem 1.7, because of the presence of more parameters. Finally, we employ Theorem 1.7 to deduce Theorem 1.1 in Subsection 3.4.

Remark 3.1. In order to show Theorem 1.7 for some $k\ge 1$, we may assume without loss of generality that $y_1 > C'_k$, where $C'_1, C'_2, \dots, C'_k, \dots$ is an increasing sequence of large constants. Indeed, suppose for the moment that Theorem 1.7 holds for all $k\ge 1$ when $y_1 > C'_k$ and consider the case when $y_1 \le C'_k$. Then either $y_k \le C'_k$, in which case Theorem 1.7 follows immediately, or there exists $l\in\{1,\dots,k-1\}$ such that $y_l \le C'_k < y_{l+1}$. In the latter case let $\boldsymbol{y}' = (y_{l+1},\dots,y_k)$ and $d = \lfloor 2y_1\rfloor\cdots\lfloor 2y_l\rfloor \le 2^l y_1\cdots y_l \le (2C'_k)^k$ and note that

$$H^{(k-l+1)}\left(\frac{x}{d},\boldsymbol{y}',2\boldsymbol{y}'\right) \le H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y}) \le H^{(k-l+1)}(x,\boldsymbol{y}',2\boldsymbol{y}'),$$

Moreover,

$$\frac{x/d}{y_{l+1}\cdots y_k} \ge \frac{x}{2^l y_1\cdots y_k} \ge 2^{k-l} y_k.$$

As a result, the desired bound on $H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y})$ follows by Theorem 1.7 applied to $H^{(k-l+1)}(x,\boldsymbol{y}',2\boldsymbol{y}')$ and $H^{(k-l+1)}(x/d,\boldsymbol{y}',2\boldsymbol{y}')$, which holds since $y_{l+1} > C'_k \ge C'_{k-l}$.

3.1. Auxiliary results.

Before we launch into the proof of Theorem 1.7, we list a few results from number theory and analysis that we shall need. First, we state a standard sieve estimate for easy reference (see for example [HT, Theorem 06]).

Lemma 3.2. *For $4\le 2z\le x$ we have*

$$|\{n\le x : P^-(n) > z\}| \asymp \frac{x}{\log z}.$$

Next, we have the following result, which follows by Lemma 2.3(b) in [K10a].

Lemma 3.3. *Let $f:\mathbb{N}\to[0,+\infty)$ be an arithmetic function that satisfies the inequality $f(ap)\le C_f f(a)$, for all integers a and all primes p with $(a,p)=1$, where C_f is a positive constant depending only on f. Also, let $h\ge 0$ and $3/2\le y\le x\le z^C$ for some $C>0$. Then*

$$\sum_{\substack{a\in\mathcal{P}_*(y,x)\\ a>z}} \frac{f(a)}{a\log^h(P^+(a))} \ll_{C_f,h,C} \exp\left\{-\frac{\log z}{2\log x}\right\} \frac{1}{\log^h x} \sum_{a\in\mathcal{P}_*(y,x)} \frac{f(a)}{a}.$$

Finally, we need a covering lemma which is a slightly different version of Lemma 3.15 in [F]. If r is a positive real number and I is a k-dimensional rectangle, then rI will denote the rectangle which has the same center with I and r times its diameter. More formally, if $\boldsymbol{x_0}$

is the center of I, then $rI := \{r(\boldsymbol{x}-\boldsymbol{x_0})+\boldsymbol{x_0} : \boldsymbol{x}\in I\}$. The lemma is then formulated as follows:

Lemma 3.4. *Let $I_1,...,I_N$ be k-dimensional cubes of the form $[a_1,b_1)\times\cdots\times[a_k,b_k)$ $(b_1-a_1=\cdots=b_k-a_k>0)$. Then there exists a sub-collection $I_{i_1},\dots,I_{i_M}$ of mutually disjoint cubes such that*

$$\bigcup_{n=1}^{N} I_n \subset \bigcup_{m=1}^{M} 3I_{i_m}.$$

3.2. **The lower bound in Theorem 1.7.** We start with the proof of the lower bound implicit in Theorem 1.7, which is simpler. First, we prove a weaker result; then we use Lemma 3.3 to complete the proof. Note that the lemma below is similar to Lemma 2.1 in [Fo08a], Lemma 4.1 in [Fo08b] and Lemma 3.2 in [K10a].

Lemma 3.5. *Let $k\ge 1$, $x\ge 1$ and $3=y_0\le y_1\le y_2\le\cdots\le y_k$ such that $2^k y_1\cdots y_k\le x/y_k$ and $y_1>C'_k$. Then*

$$\frac{H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y})}{x} \gg_k \prod_{i=1}^{k}\left(\frac{\log y_i}{\log y_{i-1}}\right)^{-(k-i+2)} \sum_{\substack{\boldsymbol{a}\in\mathcal{P}_*^k(2\boldsymbol{y})\\ a_i\le y_i^{1/(8k)}\ (1\le i\le k)}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1\cdots a_k}.$$

Proof. Set

$$x' = \frac{x}{2^k y_1\cdots y_k} \ge y_k.$$

Consider integers $n=a_1\cdots a_k p_1\cdots p_k b\le x$ such that the following hold:

(1) $\boldsymbol{a}\in\mathcal{P}_*^k(2\boldsymbol{y})$ and $a_i\le y_i^{1/(8k)}$ for $i=1,\dots,k$;
(2) $p_1,\dots,p_k$ are prime numbers with $(\log(y_1/p_1),\dots,\log(y_k/p_k))\in\mathcal{L}^{(k+1)}(\boldsymbol{a})$;
(3) If $x'\le y_k^2$, then let b be a prime number $>y_k^{1/8}$; if $x'>y_k^2$, then let b be an integer with $P^-(b)>2y_k$.

Note that for every $i\in\{1,\dots,k\}$ all prime factors of a_i lie in $(2y_{i-1},y_i^{1/(8k)}]$. Also, condition (2) in the definition of n is equivalent to the existence of integers $d_1,\dots,d_k$ such that $d_1\cdots d_i|a_1\cdots a_i$ and $y_i/p_i<d_i\le 2y_i/p_i$ for all $i\in\{1,\dots,k\}$. In particular, $\tau_{k+1}(n,\boldsymbol{y},2\boldsymbol{y})\ge 1$. Furthermore, we have that

$$y_i^{7/8}\le\frac{y_i}{a_1\cdots a_i}\le\frac{y_i}{d_i}<p_i\le 2\frac{y_i}{d_i}\le 2y_i.$$

So $(a_1\cdots a_k,p_1\cdots p_k b)=1$ and hence this representation of n, if it exists, is unique up to a possible permutation of $p_1,\dots,p_k$ and the prime factors of b that lie in $(y_1^{7/8},2y_k]$. Since b has at most one such prime factor, n has a bounded number of such representations. Fix $a_1,\dots,a_k$ and $p_1,\dots,p_k$ and note that

$$X := \frac{x}{a_1\cdots a_k p_1\cdots p_k} \ge \frac{x'}{y_k^{1/8}} \ge (x')^{7/8} > 2y_k^{1/8}. \tag{3.1}$$

We start by counting the number of possibilities for b. We consider two cases. First, if $x'>y_k^2$, then $X>4y_k$, by (3.1), provided that C'_k is large enough. So Lemma 3.2 implies that

$$\sum_{b \text{ admissible}} 1 = \sum_{b\le X, P^-(b)>2y_k} 1 \gg_k \frac{X}{\log y_k},$$

by Lemma 3.2. On the other hand, if $x' \le y_k^2$, then

$$X = \frac{x}{a_1\cdots a_k p_1\cdots p_k} \le \frac{x d_1\cdots d_k}{a_1\cdots a_k y_1\cdots y_k} \le 2^k x' \le 2^k y_k^2.$$

The above inequality and (3.1) imply that

$$\sum_{b \text{ admissible}} 1 = \sum_{\substack{y_k^{1/8}<b\le X \\ b \text{ prime}}} 1 \ge \sum_{\substack{X/2<b\le X \\ b \text{ prime}}} 1 \gg \frac{X}{\log X} \gg_k \frac{X}{\log y_k}.$$

In any case, we have that

$$\sum_{b \text{ admissible}} 1 \gg_k \frac{X}{\log y_k}$$

and, consequently,

$$H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y}) \gg_k \frac{x}{\log y_k} \sum_{\substack{\boldsymbol{a}\in\mathcal{P}_*(2\boldsymbol{y}) \\ a_i\le y_i^{1/8k}\ (1\le i\le k)}} \frac{1}{a_1\cdots a_k} \sum_{(\log\frac{y_1}{p_1},\dots,\log\frac{y_k}{p_k})\in\mathcal{L}^{(k+1)}(\boldsymbol{a})} \frac{1}{p_1\cdots p_k}. \tag{3.2}$$

Fix $\boldsymbol{a}\in\mathcal{P}_*^k(2\boldsymbol{y})$ with $a_i\le y_i^{1/(8k)}$ for $i=1,\dots,k$. Let $\{I_r\}_{r=1}^R$ be the collection of cubes $[\log(d_1/2),\log d_1)\times\cdots\times[\log(d_k/2),\log d_k)$ with $d_1\cdots d_i|a_1\cdots a_i$, $1\le i\le k$. Then for $I=[\log(d_1/2),\log d_1)\times\cdots\times[\log(d_k/2),\log d_k)$ in this collection we have that

$$\sum_{(\log\frac{y_1}{p_1},\dots,\log\frac{y_k}{p_k})\in I} \frac{1}{p_1\cdots p_k} = \prod_{i=1}^k \sum_{y_i/d_i<p_i\le 2y_i/d_i} \frac{1}{p_i} \gg_k \frac{1}{\log y_1\cdots\log y_k},$$

because $d_i\le a_1\cdots a_i\le y_i^{1/8}$ for $1\le i\le k$. By Lemma 3.4, there exists a sub-collection $\{I_{r_s}\}_{s=1}^S$ of mutually disjoint cubes so that

$$S(3\log 2)^k \ge \mathrm{Vol}\left(\bigcup_{s=1}^S 3I_{r_s}\right) \ge \mathrm{Vol}\left(\bigcup_{r=1}^R I_r\right) = L^{(k+1)}(\boldsymbol{a}).$$

Hence

$$\begin{aligned}\sum_{(\log\frac{y_1}{p_1},\dots,\log\frac{y_k}{p_k})\in\mathcal{L}^{(k+1)}(\boldsymbol{a})} \frac{1}{p_1\cdots p_k} &\ge \sum_{s=1}^S \sum_{(\log\frac{y_1}{p_1},\dots,\log\frac{y_k}{p_k})\in I_{r_s}} \frac{1}{p_1\cdots p_k} \gg_k \frac{S}{\log y_1\cdots\log y_k} \\ &\gg_k \frac{L^{(k+1)}(\boldsymbol{a})}{\log y_1\cdots\log y_k}.\end{aligned}$$

Combining the above estimate with (3.2) completes the proof of the lemma. □

Having proven the above lemma, it is not so hard to finish the proof of the lower bound of Theorem 1.7. We give the argument below.

Proof of Theorem 1.7 (lower bound). For every fixed $i \in \{1, \dots, k\}$ and integers $a_1, \dots, a_{i-1}$ and $a_{i+1}, \dots, a_k$, the function $a_i \to L^{(k+1)}(\boldsymbol{a})$ satisfies the hypothesis of Lemma 3.3 with $C_f = k - i + 2 \le k + 1$, by Lemma 2.1(b). So if we set

$$\mathcal{P} = \left\{\boldsymbol{a} \in \mathbb{N}^k : a_i \in \mathcal{P}\left(2y_{i-1}, y_i^{1/M}\right) \ (1 \le i \le k)\right\}$$

for some sufficiently large $M = M(k)$, then

$$\sum_{\substack{\boldsymbol{a}\in\mathcal{P}_*^k(2\boldsymbol{y}) \\ a_i \le y_i^{1/(8k)}\ (1\le i\le k)}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1\cdots a_k} \ge \sum_{\substack{\boldsymbol{a}\in\mathcal{P} \\ a_i \le y_i^{1/(8k)}\ (1\le i\le k)}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1\cdots a_k} = \sum_{\boldsymbol{a}\in\mathcal{P}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1\cdots a_k}\left(1 + O_k\left(e^{-\frac{M}{16k}}\right)\right)$$
$$\ge \frac{1}{2}\sum_{\boldsymbol{a}\in\mathcal{P}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1\cdots a_k}.$$

By the above inequality and Lemma 2.1(b), we deduce that

$$S^{(k+1)}(\boldsymbol{y}) \le \sum_{\boldsymbol{a}\in\mathcal{P}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1\cdots a_k} \prod_{i=1}^{k} \sum_{\substack{b_i\in\mathcal{P}_*(y_{i-1},2y_{i-1}) \\ \text{or } b_i\in\mathcal{P}_*(y_i^{1/M},y_i)}} \frac{\tau_{k-i+2}(b_i)}{b_i} \ll_k \sum_{\substack{\boldsymbol{a}\in\mathcal{P}_*^k(2\boldsymbol{y}) \\ a_i \le y_i^{1/(8k)}\ (1\le i\le k)}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1\cdots a_k}.$$

Combining the above estimate with Lemma 3.5 completes the proof of the lower bound in Theorem 1.7. □

3.3. **The upper bound in Theorem 1.7.** In this subsection we complete the proof of Theorem 1.7. Before we proceed to the proof, we need to define some auxiliary notation. For $\boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^k$ and $x \ge 1$ set

$$H_*^{(k+1)}(x, \boldsymbol{y}, \boldsymbol{z}) = |\{n \le x : \mu^2(n) = 1,\ \exists d_1\cdots d_k | n \text{ such that } y_i < d_i \le z_i\ (1 \le i \le k)\}|.$$

Also, for $\boldsymbol{t} \in [1, +\infty)^k$, $\boldsymbol{h} \in [0, +\infty)^k$ and $\epsilon > 0$ define

$$\mathcal{P}_*^k(\boldsymbol{t}; \epsilon) = \left\{\boldsymbol{a} \in \mathbb{N}^k : a_i \in \mathcal{P}_*\left(\max\left\{P^+(a_1\cdots a_{i-1}), \frac{t_{i-1}^{\epsilon}}{a_1\cdots a_{i-1}}\right\}, t_i\right)\ (1 \le i \le k)\right\},$$

where $t_0 = 1$, and

$$S^{(k+1)}(\boldsymbol{t}; \boldsymbol{h}, \epsilon) = \sum_{\boldsymbol{a}\in\mathcal{P}_*^k(\boldsymbol{t};\epsilon)} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1\cdots a_k} \prod_{i=1}^{k} \log^{-h_i}\left(P^+(a_1\cdots a_i) + \frac{t_i^{\epsilon}}{a_1\cdots a_i}\right).$$

Lastly, let

$$\boldsymbol{e}_k = (e_{k,1}, \dots, e_{k,k}) = (\underbrace{1, \dots, 1}_{k-1 \text{ times}}, 2) \in \mathbb{R}^k.$$

Then we have the following estimate.

Lemma 3.6. *Let $\sqrt{C_k'} \le y_1 \le \cdots \le y_k \le x$ with $2^{k+1}y_1\cdots y_k \le x/(2y_k)^{7/8}$. Then*

$$H_*^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y}) - H_*^{(k+1)}(x/2, \boldsymbol{y}, 2\boldsymbol{y}) \ll_k xS^{(k+1)}(2\boldsymbol{y}; \boldsymbol{e}_k, 7/8).$$

Proof. Let $n \in (x/2, x]$ be a square-free integer for which there exist integers $d_i \in (y_i, 2y_i]$, $1 \le i \le k$, with $d_1 \cdots d_k | n$. If we set $d_{k+1} = n/(d_1 \cdots d_k)$ and $y_{k+1} = x/(2^{k+1} y_1 \cdots y_k)$, then we have that $n = d_1 \cdots d_{k+1}$ with $y_i < d_i \le 2^{k+1} y_i$ for $1 \le i \le k+1$. Let $z_1, \dots, z_{k+1}$ be the sequence $y_1, \dots, y_{k+1}$ ordered increasingly. Also, let σ be the unique permutation in S_{k+1} for which $P^+(d_{\sigma(1)}) < \cdots < P^+(d_{\sigma(k+1)})$ and set $p_j = P^+(d_{\sigma(j)})$ for $1 \le j \le k+1$ and $p_0 = 1$. We can write $n = a_1 \cdots a_k p_1 \cdots p_k b$ with $P^-(b) > p_k$ and $a_i \in \mathcal{P}_*(p_{i-1}, p_i)$ for all $1 \le i \le k$. We claim that

$$p_i > Q_i := \max\left\{ P^+(a_1 \cdots a_i), \frac{(2y_i)^{7/8}}{a_1 \cdots a_i} \right\} \quad (1 \le i \le k). \tag{3.3}$$

Indeed, for every $j \in \{1, \dots, k\}$ we have that $y_{\sigma(j)} < d_{\sigma(j)} = p_j d$ for some $d | a_1 \cdots a_j$ and therefore $y_{\sigma(j)} < p_j a_1 \cdots a_j$. Consequently,

$$p_i = \max_{1 \le j \le i} p_j > \max_{1 \le j \le i} \frac{y_{\sigma(j)}}{a_1 \cdots a_j} \ge \frac{\max_{1 \le j \le i} y_{\sigma(j)}}{a_1 \cdots a_i} \ge \frac{z_i}{a_1 \cdots a_i} \ge \frac{(2y_i)^{7/8}}{a_1 \cdots a_i} \quad (1 \le i \le k),$$

by the definition of $z_1, \dots, z_{k+1}$ and our assumption that $y_1 \le \cdots \le y_k \le \frac{1}{2} y_{k+1}^{8/7}$. Moreover,

$$p_i = \max_{1 \le j \le i} p_j > \max_{1 \le j \le i} P^+(a_j) = P^+(a_1 \cdots a_j).$$

So (3.3) follows. In addition,

$$P^+(a_i) < p_i = P^+(d_{\sigma(i)}) \le \max_{1 \le j \le i} P^+(d_j) \le 2y_i \quad (1 \le i \le k),$$

by the choice of σ, and

$$P^-(a_i) > p_{i-1} > Q_{i-1} \quad (2 \le i \le k),$$

by (3.3). In particular, $\boldsymbol{a} = (a_1, \dots, a_k) \in \mathcal{P}_*^k(2\boldsymbol{y}; 7/8)$. Furthermore, note that

$$(d_{\sigma(1)}/p_1) \cdots (d_{\sigma(i)}/p_i) | a_1 \cdots a_i \quad \text{and} \quad \frac{y_{\sigma(i)}}{p_i} < \frac{d_{\sigma(i)}}{p_i} \le \frac{2^{k+1} y_{\sigma(i)}}{p_i} \quad (1 \le i \le k),$$

that is to say, there are numbers $w_1, \dots, w_k \in \{1, 2, 2^2, \dots, 2^k\}$ such that

$$\left(\log \frac{w_1 y_{\sigma(1)}}{p_1}, \dots, \log \frac{w_k y_{\sigma(k)}}{p_k} \right) \in \mathcal{L}^{(k+1)}(\boldsymbol{a}). \tag{3.4}$$

Lastly, observe that $p_{k+1} | b$ and consequently $b \ge p_{k+1} > p_k > Q_k$, by (3.3). Similarly, we have $P^-(b) > p_k > Q_k$. Combining all of the above, we deduce that

$$\begin{aligned} &H_*^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y}) - H_*^{(k+1)}(x/2, \boldsymbol{y}, 2\boldsymbol{y}) \\ &\le \sum_{\sigma \in S_{k+1}} \sum_{\substack{w_i \in \{1,2,\dots,2^k\} \\ 1 \le i \le k}} \sum_{\boldsymbol{a} \in \mathcal{P}_*^k(2\boldsymbol{y}; 7/8)} \sum_{\substack{p_1, \dots, p_k \\ (3.3), (3.4)}} \sum_{\substack{Q_k < b \le x/(a_1 \cdots a_k p_1 \cdots p_k) \\ P^-(b) > Q_k}} 1 \\ &\ll_k \sum_{\sigma \in S_{k+1}} \sum_{\substack{w_i \in \{1,2,\dots,2^k\} \\ 1 \le i \le k}} \sum_{\boldsymbol{a} \in \mathcal{P}_*^k(2\boldsymbol{y}; 7/8)} \sum_{\substack{p_1, \dots, p_k \\ (3.3), (3.4)}} \frac{x}{a_1 \cdots a_k p_1 \cdots p_k \log Q_k}, \end{aligned} \tag{3.5}$$

by Lemma 3.2. We fix σ, $w_1, \dots, w_k$ and $a_1, \dots, a_k$ as above and estimate the sum over the primes $p_1, \dots, p_k$ in the right hand side of (3.5). In order to analyze condition (3.4), consider the collection $\{I_r\}_{r=1}^R$ of cubes of the form $[\log(m_1/2), \log m_1) \times \cdots \times [\log(m_k/2), \log m_k)$ with

$m_1\cdots m_i|a_1\cdots a_i$ for $1\le i\le k$. By Lemma 3.4, there is a sub-collection $\{I_{r_s}\}_{s=1}^S$ of mutually disjoint such cubes for which $\mathcal{L}^{(k+1)}(\boldsymbol{a})\subset\bigcup_{s=1}^S 3I_{r_s}$. Consider $I_{r_s}=[\log(m_1/2),\log m_1)\times\cdots\times[\log(m_k/2),\log m_k)$ in this sub-collection and set

$$U_i=\frac{w_iy_{\sigma(i)}}{2m_i}\quad(1\le i\le k).$$

Then we find that

$$\left(\log\frac{w_1y_{\sigma(1)}}{p_1},\dots,\log\frac{w_ky_{\sigma(k)}}{p_k}\right)\in 3I_{r_s}=[\log(m_1/4),\log(2m_1))\times\cdots\times[\log(m_k/4),\log(2m_k))\tag{3.6}$$

if, and only if, $U_i<p_i\le 8U_i$ for all $i=1,\dots,k$. So

$$\sum_{\substack{p_1,\dots,p_k\\ (3.3),(3.6)}}\frac{1}{p_1\cdots p_k}\le\prod_{i=1}^k\sum_{\substack{U_i<p_i\le 8U_i\\ p_i>Q_i}}\frac{1}{p_i}\ll_k\prod_{i=1}^k\frac{1}{\log(\max\{U_i,Q_i\})}\le\prod_{i=1}^k\frac{1}{\log Q_i}.$$

Therefore we deduce that

$$\sum_{\substack{p_1,\dots,p_k\\ (3.3),(3.4)}}\frac{1}{p_1\cdots p_k}\le\sum_{s=1}^S\sum_{\substack{p_1,\dots,p_k\\ (3.3),(3.6)}}\frac{1}{p_1\cdots p_k}\ll_k\frac{S}{\log Q_1\cdots\log Q_k}\le\frac{L^{(k+1)}(\boldsymbol{a})}{(\log 2)^k\log Q_1\cdots\log Q_k}.$$

Inserting the above estimate into (3.5) completes the proof of the lemma. □

Next, we bound the sum $S^{(k+1)}(\boldsymbol{t};\boldsymbol{h},\epsilon)$ from above in terms of $S^{(k+1)}(\boldsymbol{t})$. This is accomplished by establishing an iterative inequality that simplifies the complicated range of summation $\mathcal{P}_*^k(\boldsymbol{t};\epsilon)$ by gradually reducing it to the much simpler set $\mathcal{P}_*^k(\boldsymbol{t})$ and, at the same time, eliminates the complicated logarithms that appear in the summands of $S^{(k+1)}(\boldsymbol{t};\boldsymbol{h},\epsilon)$. Lemma 3.3 plays a crucial role in the proof of this inequality

Lemma 3.7. *Fix $k\ge1$, $\epsilon>0$ and $\boldsymbol{h}=(h_1,\dots,h_k)\in[0,+\infty)^k$. For $\boldsymbol{t}=(t_1,\dots,t_k)$ with $3\le t_1\le\cdots\le t_k$ we have that*

$$S^{(k+1)}(\boldsymbol{t};\boldsymbol{h},\epsilon)\ll_{k,\boldsymbol{h},\epsilon}\left(\prod_{i=1}^k\log^{-h_i}t_i\right)S^{(k+1)}(\boldsymbol{t}).$$

Proof. Set $\delta=\epsilon/(2k)$ and $t_0=1$. For $l\in\{1,\dots,k\}$ define

$$h_{l,i}=\begin{cases}h_i & \text{if } i\in\{1,\dots,l-1\}\cup\{k\},\\ h_i+k-i+1 & \text{if } l\le i\le k-1,\end{cases}$$

and

$$\mathcal{P}_l(\boldsymbol{t})=\Big\{\boldsymbol{a}\in\mathbb{N}^k: a_i\in\mathcal{P}_*\left(\max\left\{P^+(a_1\cdots a_{i-1}),t_{i-1}^{\epsilon/2+l\delta}/(a_1\cdots a_{i-1})\right\},t_i\right)\ (1\le i\le l),\\ a_i\in\mathcal{P}_*(t_{i-1},t_i)\ (l+1\le i\le k)\Big\}.$$

Also, let $h_{0,i} = h_{1,i}$ for $i \in \{1,\dots,k\}$ and $\mathcal{P}_0(\boldsymbol{t}) = \mathcal{P}_1(\boldsymbol{t})$. Lastly, for $l \in \{0,\dots,k\}$ set $\boldsymbol{h}_l = (h_{l,1},\dots,h_{l,k})$ and

$$\begin{aligned}\widetilde{S}_l^{(k+1)}(\boldsymbol{t};\boldsymbol{h}_l) = \sum_{\boldsymbol{a}\in\mathcal{P}_l(\boldsymbol{t})} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1\cdots a_k} \prod_{i=1}^{l} \log^{-h_{l,i}}\left(P^+(a_1\cdots a_i) + \frac{t_i^{\epsilon/2+l\delta}}{a_1\cdots a_i}\right) \\ \times \prod_{i=l+1}^{k} \log^{-h_{l,i}}\left(P^+(a_1\cdots a_l) + \frac{t_i^{\epsilon/2+l\delta}}{a_1\cdots a_l}\right).\end{aligned}$$

Note that

$$\widetilde{S}_k^{(k+1)}(\boldsymbol{t};\boldsymbol{h}_k) = S^{(k+1)}(\boldsymbol{t};\boldsymbol{h},\epsilon) \tag{3.7}$$

and

$$\begin{aligned}\widetilde{S}_0^{(k+1)}(\boldsymbol{t};\boldsymbol{h}_0) &\asymp_{k,\epsilon,\boldsymbol{h}} \left(\prod_{i=1}^{k}(\log t_i)^{-h_{0,i}}\right) S^{(k+1)}(\boldsymbol{t}) \\ &= \left(\prod_{i=1}^{k-1}(\log t_i)^{-(h_i+k-i+1)}\right)(\log t_k)^{-h_k} S^{(k+1)}(\boldsymbol{t}).\end{aligned} \tag{3.8}$$

We claim that

$$\widetilde{S}_l^{(k+1)}(\boldsymbol{t};\boldsymbol{h}_l) \ll_{k,\boldsymbol{h},\epsilon} (\log 2t_{l-1})^{k-l+2}\, \widetilde{S}_{l-1}^{(k+1)}(\boldsymbol{t};\boldsymbol{h}_{l-1}) \quad (1 \le l \le k). \tag{3.9}$$

Clearly, if we prove (3.9), then the lemma will follow immediately by iterating (3.9) and combining the resulting inequality with relations (3.7) and (3.8). So we fix $l \in \{1,\dots,k\}$ and proceed to the proof of (3.9). Consider integers $a_1,\dots,a_{l-1}$ such that

$$a_i \in \mathcal{P}_*\left(\max\left\{P^+(a_1\cdots a_{i-1}), \frac{t_{i-1}^{\epsilon/2+l\delta}}{a_1\cdots a_{i-1}}\right\}, t_i\right) \quad (1 \le i \le l-1)$$

and $a_{l+1},\dots,a_k$ such that

$$a_i \in \mathcal{P}_*(t_{i-1},t_i) \quad (l+1 \le i \le k)$$

and set

$$t'_{l-1} = \max\left\{P^+(a_1\cdots a_{l-1}), \frac{t_{l-1}^{\epsilon/2+l\delta}}{a_1\cdots a_{l-1}}\right\}.$$

Observe that in order to show (3.9) it suffices to prove that

$$\begin{aligned}T := \sum_{a_l\in\mathcal{P}_*(t'_{l-1},t_l)} \frac{L^{(k+1)}(\boldsymbol{a})}{a_l} \prod_{i=l}^{k} \log^{-h_{l,i}}\left(P^+(a_1\cdots a_l) + \frac{t_i^{\epsilon/2+l\delta}}{a_1\cdots a_l}\right) \\ \ll_{k,\boldsymbol{h},\epsilon} \sum_{a_l\in\mathcal{P}_*(t'_{l-1},t_l)} \frac{L^{(k+1)}(\boldsymbol{a})}{a_l} \prod_{i=l}^{k} \log^{-h_{l,i}}\left(P^+(a_1\cdots a_{l-1}) + \frac{t_i^{\epsilon/2+(l-1)\delta}}{a_1\cdots a_{l-1}}\right).\end{aligned} \tag{3.10}$$

Indeed, if (3.10) holds, then Lemma 2.1(b) and the relation

$$\sum_{a\in\mathcal{P}_*(t'_{l-1},t_{l-1})}\frac{\tau_{k-l+2}(a)}{a}=\prod_{t'_{l-1}<p\le t_{l-1}}\left(1+\frac{k-l+2}{p}\right)\ll_k\left(\frac{\log 2t_{l-1}}{\log 2t'_{l-1}}\right)^{k-l+2}$$

imply that

$$T\ll_{k,\boldsymbol{h},\epsilon}\left(\frac{\log 2t_{l-1}}{\log 2t'_{l-1}}\right)^{k-l+2}\sum_{a_l\in\mathcal{P}_*(t_{l-1},t_l)}\frac{L^{(k+1)}(\boldsymbol{a})}{a_l}\prod_{i=l}^{k}\log^{-h_{l,i}}\left(P^+(a_1\cdots a_{l-1})+\frac{t_i^{\epsilon/2+(l-1)\delta}}{a_1\cdots a_{l-1}}\right)$$

thus completing the proof of (3.9). To prove (3.10) we decompose T into the sums

$$T_m:=\sum_{\substack{a_l\in\mathcal{P}_*(t'_{l-1},t_l)\\ a_l\in I_m}}\frac{L^{(k+1)}(\boldsymbol{a})}{a_l}\prod_{i=l}^{k}\log^{-h_{l,i}}\left(P^+(a_1\cdots a_l)+\frac{t_i^{\epsilon/2+l\delta}}{a_1\cdots a_l}\right)\quad(l\le m\le k+1),$$

where $I_l=(0,t_l^\delta]$, $I_m=(t_{m-1}^\delta,t_m^\delta]$ if $m\in\{l+1,\dots,k\}$ and $I_{k+1}=(t_k^\delta,+\infty)$. First, we estimate T_l. If $a_l\in I_l$, then

$$P^+(a_1\cdots a_l)+\frac{t_i^{\epsilon/2+l\delta}}{a_1\cdots a_l}\ge P^+(a_1\cdots a_{l-1})+\frac{t_i^{\epsilon/2+(l-1)\delta}}{a_1\cdots a_{l-1}}\quad(l\le i\le k)$$

and thus we immediately deduce that

$$T_l\le\sum_{a_l\in\mathcal{P}_*(t'_{l-1},t_l)}\frac{L^{(k+1)}(\boldsymbol{a})}{a_l}\prod_{i=l}^{k}\log^{-h_{l,i}}\left(P^+(a_1\cdots a_{l-1})+\frac{t_i^{\epsilon/2+(l-1)\delta}}{a_1\cdots a_{l-1}}\right).\tag{3.11}$$

Next, we fix $m\in\{l+1,\dots,k+1\}$ and bound T_m. For every $a_l\in I_m$ we have that

$$P^+(a_1\cdots a_l)+\frac{t_i^{\epsilon/2+l\delta}}{a_1\cdots a_l}\ge\begin{cases}P^+(a_l) & \text{if } l\le i<m,\\ P^+(a_1\cdots a_{l-1})+\dfrac{t_i^{\epsilon/2+(l-1)\delta}}{a_1\cdots a_{l-1}} & \text{if } m\le i\le k.\end{cases}$$

Moreover, the function $a_l \to L^{(k+1)}(\boldsymbol{a})$ satisfies the hypothesis of Lemma 3.3 with $C_f = k-l+2$, by Lemma 2.1(b). Hence

$$\begin{aligned}
T_m &\le \left(\prod_{i=m}^{k} \log^{-h_{l,i}}\left(P^+(a_1\cdots a_{l-1}) + \frac{t_i^{\epsilon/2+(l-1)\delta}}{a_1\cdots a_{l-1}}\right)\right) \sum_{\substack{a_l\in\mathcal{P}_*(t'_{l-1},t_l)\\ a_l>t_{m-1}^{\delta}}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_l(\log P^+(a_l))^{h_{l,l}+\cdots+h_{l,m-1}}}\\
&\ll_{k,\boldsymbol{h},\epsilon} \left(\prod_{i=l}^{k} \log^{-h_{l,i}}\left(P^+(a_1\cdots a_{l-1}) + \frac{t_i^{\epsilon/2+(l-1)\delta}}{a_1\cdots a_{l-1}}\right)\right)\left(\prod_{i=l}^{m-1}\log^{h_{l,i}} t_i\right)\\
&\qquad \times \exp\left\{-\frac{\delta\log t_{m-1}}{2\log t_l}\right\}(\log t_l)^{-(h_{l,l}+\cdots+h_{l,m-1})}\sum_{a_l\in\mathcal{P}_*(t'_{l-1},t_l)}\frac{L^{(k+1)}(\boldsymbol{a})}{a_l}\\
&\ll_{k,\boldsymbol{h},\epsilon} \sum_{a_l\in\mathcal{P}_*(t'_{l-1},t_l)}\frac{L^{(k+1)}(\boldsymbol{a})}{a_l}\prod_{i=l}^{k}\log^{-h_{l,i}}\left(P^+(a_1\cdots a_{l-1}) + \frac{t_i^{\epsilon/2+(l-1)\delta}}{a_1\cdots a_{l-1}}\right).
\end{aligned}$$

Combining the above estimate with (3.11) shows (3.10). This completes the proof of (3.9) and hence of the lemma. □

Before we prove the upper bound in Theorem 1.7, we need one last intermediate result.

Lemma 3.8. *Let $1 \le l \le k-1$ and $3 \le t_1 \le \cdots \le t_k$. Then*

$$S^{(k-l+1)}(t_{l+1},\dots,t_k) \le (\log 2)^{-l} S^{(k+1)}(t_1,\dots,t_k)$$

and

$$S^{(k+1)}(t_1,\dots,t_k) \gg_k \log t_k.$$

Proof. Note that

$$\begin{aligned}
\mathcal{L}^{(k+1)}(\boldsymbol{a}) &\supset \bigcup_{\substack{d_1\cdots d_i|a_1\cdots a_i\ (1\le i\le k)\\ d_i=1\ (1\le i\le l)}} [\log(d_1/2),\log d_1)\times\cdots\times[\log(d_k/2),\log d_k)\\
&= [-\log 2, 0)^l \times \mathcal{L}^{(k-l+1)}(a_1\cdots a_{l+1}, a_{l+2},\dots,a_k)
\end{aligned}$$

and, consequently,

$$L^{(k+1)}(\boldsymbol{a}) \ge (\log 2)^l L^{(k-l+1)}(a_1\cdots a_{l+1}, a_{l+2},\dots,a_k).$$

Summing over $\boldsymbol{a}\in\mathcal{P}_*^k(\boldsymbol{t})$ then proves the first part of the lemma.

For the second part, note that

$$S^{(k+1)}(\boldsymbol{a}) \ge (\log 2)^k \sum_{\boldsymbol{a}\in\mathcal{P}_*^k(\boldsymbol{t})}\frac{1}{a_1\cdots a_k} \asymp_k \log t_k.$$

□

We are now in position to show the upper bound in Theorem 1.7. In fact, we shall prove a slightly stronger estimate, which will be useful in the proof of Theorem 1.5.

Theorem 3.9. *Fix $k \ge 1$. Let $x \ge 1$ and $C'_k \le y_1 \le \cdots \le y_k$ with $2^k y_1 \cdots y_k \le x/y_k$. There exists a constant c_k such that*

$$\frac{H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y})}{x} \ll_k \left(\prod_{i=1}^{k} \log^{-e_{k,i}} y_i\right) \sum_{\substack{\boldsymbol{a}\in\mathcal{P}_*^k(\boldsymbol{y}) \\ a_i \le y_i^{c_k}\ (1\le i\le k)}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1\cdots a_k}.$$

Proof. Observe that it suffices to show that

$$H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y}) \ll_k x \left(\prod_{i=1}^{k} \log^{-e_{k,i}} y_i\right) T, \tag{3.12}$$

where

$$T := \max\{S^{(k+1)}(\boldsymbol{t}) : 1 \le t_1 \le \cdots \le t_k,\ \sqrt{y_i} \le t_i \le 2y_i\ (1\le i\le k)\}.$$

Indeed, assume for the moment that (3.12) holds. Note that

$$T \ll_k S^{(k+1)}(\boldsymbol{y}),$$

by Lemma 2.1(b) and inequality (1.4). Also, for every $i \in \{1,\dots,k\}$, we have that

$$\sum_{\substack{\boldsymbol{a}\in\mathcal{P}_*^k(\boldsymbol{y}) \\ a_i > y_i^{c_k}}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1\cdots a_k} \ll_k e^{-c_k/2} S^{(k+1)}(\boldsymbol{y}),$$

by Lemma 3.3 applied to the arithmetic function $a_i \to L^{(k+1)}(\boldsymbol{a})$. Hence, if c_k is large enough, we find that

$$T \ll_k S^{(k+1)}(\boldsymbol{y}) \le 2 \sum_{\substack{\boldsymbol{a}\in\mathcal{P}_*^k(\boldsymbol{y}) \\ a_i \le y_i^{c_k}\ (1\le i\le k)}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1\cdots a_k}$$

which, together with (3.12), completes the proof of the theorem.

In order to prove (3.12), we first reduce the counting in $H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y})$ to square-free integers. Let $n \le x$ be an integer counted by $H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y})$ and write $n = ab$ with a being square-full, b square-free and $(a,b)=1$. The number of $n \le x$ with $a > (\log y_k)^{2k}$ is at most

$$x \sum_{\substack{a > (\log y_k)^{2k} \\ a \text{ square-full}}} \frac{1}{a} \ll \frac{x}{(\log y_k)^k}.$$

Assume now that

$$a \in I_m := \{a \in \mathbb{N} \cap ((\log y_{m-1})^{2k}, (\log y_m)^{2k}] : a \text{ square} - \text{full}\}$$

for some $m \in \{1,\dots,k\}$, where for the convenience of notation we have set $y_0 = 1$. Then we may uniquely write $d_i = f_i e_i$, $m \le i \le k$, with $f_m \cdots f_k | a$ and $e_m \cdots e_k | b$. Therefore

$$\begin{aligned} H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y}) \le \sum_{m=1}^{k} \sum_{a\in I_m} \sum_{f_m\cdots f_k | a} H_*^{(k-m+2)}\left(\frac{x}{a}, \left(\frac{y_m}{f_m},\dots,\frac{y_k}{f_k}\right), 2\left(\frac{y_m}{f_m},\dots,\frac{y_k}{f_k}\right)\right) \\ + O\left(\frac{x}{(\log y_k)^k}\right). \end{aligned} \tag{3.13}$$

Fix $m \in \{1, \dots, k\}$, $a \in I_m$ and $f_m, \dots, f_k$ with $f_m \cdots f_k | a$. Let $z_m, \dots, z_k$ be the sequence $y_m/f_m, \dots, y_k/f_k$ in increasing order and set $\boldsymbol{z}' = (z_m, \dots, z_k)$. Since $y_m \le \cdots \le y_k$ and

$$\frac{y_i}{f_i} \ge \frac{y_i}{a} \ge \frac{y_i}{(\log y_m)^k} \ge \sqrt{y_i} \quad (m \le i \le k),$$

we have that

$$\sqrt{y_i} \le z_i \le y_i \quad (m \le i \le k). \tag{3.14}$$

Next, observe that

(3.15)
$$H_*^{(k-m+2)}\left(\frac{x}{a}, \boldsymbol{z}', 2\boldsymbol{z}'\right) \le \sum_{\substack{r \in \mathbb{N} \\ 2^r \le (\log y_k)^k}} \left(H_*^{(k-m+2)}\left(\frac{x}{2^{r-1}a}, \boldsymbol{z}', 2\boldsymbol{z}'\right) - H_*^{(k-m+2)}\left(\frac{x}{2^r a}, \boldsymbol{z}', 2\boldsymbol{z}'\right)\right) + \frac{2x}{a(\log y_k)^k}.$$

For r with $2^r \le (\log y_k)^k$ we have that

$$\frac{x/(2^{r-1}a)}{2^{k-m+2} z_m \cdots z_k} \ge \frac{x}{2^k y_1 \cdots y_k} \frac{1}{2^r a} \ge \frac{y_k}{(\log y_k)^{3k}} \ge (2z_k)^{7/8}.$$

Thus Lemma 3.6 (applied with $k-m+1$ in place of k, $x/(2^{r-1}a)$ in place of x and $z_m, \dots, z_k$ in place of $y_1, \dots, y_k$), Lemmas 3.7 and 3.8 and relation (3.14) yield

(3.16)
$$\begin{aligned} H_*^{(k-m+2)}\left(\frac{x}{2^{r-1}a}, \boldsymbol{z}', 2\boldsymbol{z}'\right) - H_*^{(k-m+2)}\left(\frac{x}{2^r a}, \boldsymbol{z}', 2\boldsymbol{z}'\right) &\ll_k \frac{x}{2^r a} \left(\prod_{i=m}^k (\log z_i)^{-e_{k,i}} \right) S^{(k-m+2)}(2\boldsymbol{z}') \\ &\ll_k \frac{x}{2^r a} \left(\prod_{i=m}^k (\log y_i)^{-e_{k,i}} \right) T. \end{aligned}$$

Since $T \gg_k \log y_k$ by Lemma 3.8, inequalities (3.15) and (3.16) yield

$$H_*^{(k-m+2)}\left(\frac{x}{a}, \boldsymbol{z}', 2\boldsymbol{z}'\right) \ll_k \frac{x}{a} \left(\prod_{i=m}^k \log^{-e_{k,i}} y_i \right) T + \frac{x}{a(\log y_k)^k} \ll_k \frac{x}{a} \left(\prod_{i=m}^k \log^{-e_{k,i}} y_i \right) T.$$

So

$$\sum_{a \in I_m} \sum_{f_m \cdots f_k | a} H_*^{(k-m+2)}\left(\frac{x}{a}, \boldsymbol{z}', 2\boldsymbol{z}'\right) \ll_k \frac{xT}{\prod_{i=m}^k (\log y_i)^{e_{k,i}}} \sum_{a \in I_m} \frac{\tau_{k-m+2}(a)}{a} \ll_k \frac{xT}{\prod_{i=1}^k (\log y_i)^{e_{k,i}}}.$$

Inserting the above estimate into (3.13) and using the inequality $T \gg_k \log y_k$ completes the proof of the theorem. $\square$

3.4. **Proof of Theorem 1.1.** In this subsection we prove Theorem 1.1. Consider real numbers $3 = N_0 \le N_1 \le \cdots \le N_{k+1}$. Using an inductive argument, similar to the one given in Remark 3.1, we may assume without loss of generality that $N_1 \ge 4(C_k')^2$. Set $\boldsymbol{N} = (N_1, \dots, N_k)$ and note that

(3.17)
$$A_{k+1}(N_1, \dots, N_{k+1}) \ge H^{(k+1)}\left(\frac{N_1 \cdots N_{k+1}}{2^{k^2}}, \frac{\boldsymbol{N}}{2^k}, \frac{\boldsymbol{N}}{2^{k-1}}\right) \asymp_k H^{(k+1)}\left(N_1 \cdots N_{k+1}, \frac{\boldsymbol{N}}{2}, \boldsymbol{N}\right),$$

by Corollary 1.8. Also, we have that

(3.18)
$$A_{k+1}(N_1, \dots, N_{k+1}) \le \sum_{\substack{1 \le 2^{m_i} \le N_i \\ 1 \le i \le k}} H^{(k+1)}\left(\frac{N_1 \cdots N_{k+1}}{2^{m_1 + \cdots + m_k}}, \left(\frac{N_1}{2^{m_1+1}}, \dots, \frac{N_k}{2^{m_k+1}}\right), \left(\frac{N_1}{2^{m_1}}, \dots, \frac{N_k}{2^{m_k}}\right)\right).$$

For $i \in \{0, 1, \dots, k\}$ let $\mathcal{M}_i$ be the set of vectors $\boldsymbol{m} \in (\mathbb{N} \cup \{0\})^k$ such that $2^{m_j} \le \sqrt{N_j}$ for $i < j \le k$ and $\sqrt{N_i} < 2^{m_i} \le N_i$ and set

$$T_i = \sum_{\boldsymbol{m} \in \mathcal{M}_i} H^{(k+1)}\left(\frac{N_1 \cdots N_{k+1}}{2^{m_1 + \cdots + m_k}}, \left(\frac{N_1}{2^{m_1+1}}, \dots, \frac{N_k}{2^{m_k+1}}\right), \left(\frac{N_1}{2^{m_1}}, \dots, \frac{N_k}{2^{m_k}}\right)\right).$$

We have that

$$A_{k+1}(N_1, \dots, N_{k+1}) \le \sum_{i=0}^{k} T_i, \tag{3.19}$$

by (3.18). We fix $i \in \{0, 1, \dots, k\}$ and proceed to the estimation of T_i. Consider $\boldsymbol{m} \in \mathcal{M}_i$ and let $\boldsymbol{N}' = (N_{i+1}', \dots, N_k')$ be the vector whose coordinates are the sequence $\{N_j/2^{m_j+1}\}_{j=i+1}^k$ in increasing order. We have that $\sqrt{N_j} \le 2N_j' \le N_j$ for all $i+1 \le j \le k$. Thus

$$\begin{aligned} &H^{(k+1)}\left(\frac{N_1 \cdots N_{k+1}}{2^{m_1 + \cdots + m_k}}, \left(\frac{N_1}{2^{m_1+1}}, \dots, \frac{N_k}{2^{m_k+1}}\right), \left(\frac{N_1}{2^{m_1}}, \dots, \frac{N_k}{2^{m_k}}\right)\right) \\ &\le H^{(k-i+1)}\left(\frac{N_1 \cdots N_{k+1}}{2^{m_1 + \cdots + m_k}}, \boldsymbol{N}', 2\boldsymbol{N}'\right) \asymp_k \frac{N_1 \cdots N_{k+1}}{2^{m_1 + \cdots + m_k}} S^{(k-i+1)}(\boldsymbol{N}') \prod_{j=i+1}^{k} (\log N_j)^{-e_{k,j}}, \end{aligned} \tag{3.20}$$

by Theorem 1.7, with the notational convention that $S^{(1)}(\emptyset) = 1$. Furthermore, we have that

$$\begin{aligned} S^{(k-i+1)}(\boldsymbol{N}') &\le (\log 2)^{-i} S^{(k+1)}(\sqrt{N_1}, \dots, \sqrt{N_i}, \boldsymbol{N}') \\ &\asymp_k S^{(k+1)}(N_1, \dots, N_k) \asymp_k \frac{H^{(k+1)}(N_1 \cdots N_{k+1}, \boldsymbol{N}/2, \boldsymbol{N})}{N_1 \cdots N_{k+1}} \prod_{j=1}^{k} (\log N_j)^{e_{k,j}}, \end{aligned} \tag{3.21}$$

by Lemma 3.8, Corolllary 1.6 and Theorem 1.7. Combining (3.20) and (3.21) we deduce that

$$H^{(k+1)}\left(\frac{N_1 \cdots N_{k+1}}{2^{m_1 + \cdots + m_k}}, \boldsymbol{N}', 2\boldsymbol{N}'\right) \ll_k \frac{H^{(k+1)}(N_1 \cdots N_{k+1}, \boldsymbol{N}/2, \boldsymbol{N})}{2^{m_1 + \cdots + m_k}} (\log N_i)^{k+1}.$$

Summing the above inequality over $\boldsymbol{m} \in \mathcal{M}_i$ gives us that

$$T_i \ll_k H^{(k+1)}\left(N_1 \cdots N_{k+1}, \frac{\boldsymbol{N}}{2}, \boldsymbol{N}\right) \frac{(\log N_i)^{k+1}}{\sqrt{N_i}},$$

which together with (3.17) and (3.19) completes the proof of Theorem 1.1.

4. Linear constraints on a Poisson distribution

A k-dimensional Poisson distribution with parameters $z_1, \dots, z_k$ is a probability distribution on the lattice $(\mathbb{N} \cup \{0\})^k$ that assigns to each lattice point $(r_1, \dots, r_k)$ the probability $\prod_{i=1}^k e^{-z_i} z_i^{r_i}/r_i!$. Our goal in this section is to estimate the probability that lattice points obeying such a distribution lie close to a hyperplane and other related quantities. Throughout this entire section we fix positive real numbers $\lambda_1, \dots, \lambda_k$ and we set $\Lambda = \max_{1 \le i \le k} \lambda_i$. Given $R \ge \Lambda$, let

$$\mathcal{H}^k(R) = \left\{(r_1, \dots, r_k) \in (\mathbb{N} \cup \{0\})^k : R - \Lambda < \sum_{i=1}^k \lambda_i r_i \le R\right\}$$

$$\mathcal{H}_-^k(R) = \left\{(r_1, \dots, r_k) \in (\mathbb{N} \cup \{0\})^k : \sum_{i=1}^k \lambda_i r_i \le R\right\}$$

and

$$\mathcal{H}_+^k(R) = \left\{(r_1, \dots, r_k) \in (\mathbb{N} \cup \{0\})^k : \sum_{i=1}^k \lambda_i r_i \ge R\right\}.$$

Also, define the number $\alpha(R) = \alpha(R; k, \boldsymbol{z}, \boldsymbol{\lambda})$ implicitly via the equation

$$\sum_{i=1}^k \lambda_i e^{\alpha(R)\lambda_i} z_i = R$$

and set

$$\mathcal{H}^k(R, \delta) = \left\{\boldsymbol{r} \in \mathcal{H}^k(R) : \left|r_i - e^{\alpha(R)\lambda_i} z_i\right| \le \frac{\Lambda}{\lambda_i} \max\left\{k, \delta\sqrt{e^{\alpha(R)\lambda_i} z_i}\right\} \ (1 \le i \le k)\right\}.$$

Remark 4.1. The motivation for the definition of $\alpha(R)$ may be briefly summarized as follows: By Stirling's formula, we have that

$$\prod_{i=1}^k \frac{z_i^{r_i}}{r_i!} \sim_k \prod_{i=1}^k \frac{1}{\sqrt{2\pi r_i}} \left(\frac{z_i e}{r_i}\right)^{r_i}. \tag{4.1}$$

Using Lagrange multipliers, we see that when $\boldsymbol{r}$ ranges over $\mathcal{H}^k(R)$, the maximum of the right hand side in (4.1) occurs when $r_i = e^{\alpha(R)\lambda_i} z_i + O_{k,\boldsymbol{\lambda}}(1)$ for all $i \in \{1, \dots, k\}$.

Lemma 4.2. *Let $k \in \mathbb{N}$, $0 < \delta \le 1$, $z_1, \dots, z_k \ge 1$ and $\lambda_1, \dots, \lambda_k > 0$. There is a constant $c = c(k, \boldsymbol{\lambda})$ such that the following hold:*

(1) *If* $R\ge\max\{\Lambda,\delta(z_1+\cdots+z_k)\}$, *then*

$$\mathbf{Prob}(\mathcal{H}^k(R,\delta))\gg_{k,\boldsymbol{\lambda},\delta}\frac{e^{-c|\alpha(R)|}}{\sqrt{R}}\prod_{i=1}^k\exp\{-Q(e^{\alpha(R)\lambda_i})z_i\}.$$

(2) *If* $R\ge\Lambda$, *then*

$$\mathbf{Prob}(\mathcal{H}^k(R))\ll_{k,\boldsymbol{\lambda}}\frac{e^{c|\alpha(R)|}}{\sqrt{R}}\prod_{i=1}^k\exp\{-Q(e^{\alpha(R)\lambda_i})z_i\}.$$

Proof. By Stirling's formula, we have that

$$\prod_{i=1}^k e^{-z_i}\frac{z_i^{r_i}}{r_i!}\asymp_k\left(\prod_{i=1}^k\frac{\sqrt{r_i+1}}{z_i}\right)e^{F(\boldsymbol{r})},\tag{4.2}$$

where

$$F(\boldsymbol{r})=-(z_1+\cdots+z_k)+\sum_{i=1}^k(r_i+1)\log\frac{z_ie}{r_i+1}.\tag{4.3}$$

Set $r_i^*=e^{\lambda_i\alpha(R)}z_i-1$ for $i\in\{1,\dots,k\}$. Without loss of generality, we assume that $r_k^*+1=\max_{1\le i\le k}(r_i^*+1)$, so that

$$r_k^*+1\asymp_{k,\boldsymbol{\lambda}}R.\tag{4.4}$$

In order to prove part (a) of the lemma, we shall employ quadratic approximation to $F(\boldsymbol{r})$ around the point $\boldsymbol{r}^*$. However, for part (b) we need to be more careful: we shall reparametrize the set $\mathcal{H}^k(R)$ first and then use the saddle point method. We give the details of the proof below.

(a) Since $R\ge\delta(z_1+\cdots+z_k)$, then (4.4) yields

$$e^{\lambda_k\alpha(R)}z_k\gg_{k,\lambda}R\ge\delta z_k$$

and thus $\alpha(R)\ge -C$ for some constant $C=C(k,\boldsymbol{\lambda},\delta)$. In turn, this implies that $r_i^*+1\gg_{k,\boldsymbol{\lambda},\delta}z_i\ge1$ for all $i\in\{1,\dots,k\}$. By Taylor's theorem, for every $\boldsymbol{r}\in\mathcal{H}^k(R,\delta)$ there is a vector $\boldsymbol{\xi}\in\mathbb{R}^k$ that lies on the line segment connecting $\boldsymbol{r}$ and $\boldsymbol{r}^*$ and satisfies

$$\begin{aligned}F(\boldsymbol{r})&=F(\boldsymbol{r}^*)+\sum_{i=1}^k\frac{\partial F(\boldsymbol{r}^*)}{\partial x_i}(r_i-r_i^*)+\frac12\sum_{1\le i,j\le k}\frac{\partial^2F(\boldsymbol{\xi})}{\partial x_i\partial x_j}(r_i-r_i^*)(r_j-r_j^*)\\&=F(\boldsymbol{r}^*)-\sum_{i=1}^k\frac{(r_i-r_i^*)^2}{2\xi_i+2}+O_{k,\boldsymbol{\lambda}}(|\alpha(R)|)=F(\boldsymbol{r}^*)+O_{k,\boldsymbol{\lambda}}(1+|\alpha(R)|).\end{aligned}$$

Since we also have that

$$\begin{aligned}|\mathcal{H}^k(R,\delta)|&\ge\left|\left\{\boldsymbol{r}\in\mathcal{H}^k(R):|r_i-r_i^*-1|\le\frac{\Lambda}{k\lambda_i}\max\left\{k,\delta\sqrt{r_i^*+1}-1\right\}\ (1\le i\le k-1)\right\}\right|\\&\asymp_{k,\boldsymbol{\lambda},\delta}\prod_{i=1}^{k-1}\sqrt{r_i^*+1}\asymp_{k,\boldsymbol{\lambda}}\sqrt{\frac{(r_1^*+1)\cdots(r_k^*+1)}{R}},\end{aligned}$$

by our assumption that $r_k^*+1=\max_{1\le i\le k}(r_i^*+1)$, and

$$F(\boldsymbol{r}^*)=-\sum_{i=1}^{k}Q(e^{\lambda_i\alpha(R)})z_i, \tag{4.5}$$

the desired lower bound on $\mathbf{Prob}(\mathcal{H}^k(R,\delta))$ follows.

(b) Let

$$\mathcal{R}=\{\tilde{\boldsymbol{r}}=(r_1,\dots,r_{k-1})\in(\mathbb{N}\cup\{0\})^{k-1}:\lambda_1r_1+\cdots+\lambda_{k-1}r_{k-1}\le R\}$$

and, for $\tilde{\boldsymbol{r}}\in\mathcal{R}$, set

$$f(\tilde{\boldsymbol{r}})=\frac{1}{\lambda_k}\left(R-\sum_{i=1}^{k-1}\lambda_ir_i\right)\quad\text{and}\quad G(\tilde{\boldsymbol{r}})=F(\tilde{\boldsymbol{r}},f(\tilde{\boldsymbol{r}})),$$

where F is defined by (4.3). Given $\tilde{\boldsymbol{r}}\in\mathcal{R}$, there is a positive but bounded number of integers r_k such that $(r_1,\dots,r_k)\in\mathcal{H}^k(R)$: Indeed, we have that $(r_1,\dots,r_k)\in\mathcal{H}^k(R)$ if, and only if,

$$r_k\ge0\quad\text{and}\quad f(\tilde{\boldsymbol{r}})-\Lambda/\lambda_k<r_k\le f(\tilde{\boldsymbol{r}}). \tag{4.6}$$

Also, relation (4.6) and the Mean Value Theorem imply that there is some $\xi\in(r_k+1,f(\tilde{\boldsymbol{r}})+1)$ such that

$$(r_k+1)\log\frac{z_ke}{r_k+1}-(f(\tilde{\boldsymbol{r}})+1)\log\frac{z_ke}{f(\tilde{\boldsymbol{r}})+1}=(f(\tilde{\boldsymbol{r}})-r_k)\log\frac{\xi}{z_k}.$$

We have that

$$\log\frac{\xi}{z_k}\le\log\frac{R/\lambda_k+1}{z_k}=\log\frac{r_k^*+1}{z_k}+O_{k,\boldsymbol{\lambda}}(1)\le\lambda_k|\alpha(R)|+O_{k,\boldsymbol{\lambda}}(1),$$

by (4.4). So (4.2) yields that

$$\mathbf{Prob}(\mathcal{H}^k(R))\ll_{k,\boldsymbol{\lambda}}e^{\Lambda|\alpha(R)|}\sum_{\tilde{\boldsymbol{r}}\in\mathcal{R}}\left(\prod_{i=1}^{k-1}\frac{\sqrt{r_i+1}}{z_i}\right)\frac{\sqrt{f(\tilde{\boldsymbol{r}})+1}}{z_k}e^{G(\tilde{\boldsymbol{r}})}.$$

Since we also have that $f(\tilde{\boldsymbol{r}})+1\le R/\lambda_k+1\asymp_{k,\boldsymbol{\lambda}}r_k^*+1$, by (4.4), we deduce that

$$\mathbf{Prob}(\mathcal{H}^k(R))\ll_{k,\boldsymbol{\lambda}}\frac{e^{O_{k,\boldsymbol{\lambda}}(|\alpha(R)|)}}{\sqrt{R}}\sum_{\tilde{\boldsymbol{r}}\in\mathcal{R}}\left(\prod_{i=1}^{k-1}\frac{\sqrt{r_i+1}}{z_i}\right)e^{G(\tilde{\boldsymbol{r}})}. \tag{4.7}$$

In order to estimate the right hand side of (4.7), we shall use quadratic approximation to $G(\tilde{\boldsymbol{r}})$ around the point $\tilde{\boldsymbol{r}}^*=(r_1^*,\dots,r_{k-1}^*)$. We have that

$$\frac{\partial G(\tilde{\boldsymbol{r}}^*)}{\partial x_i}=\log\frac{z_i}{r_i^*+1}+\frac{\lambda_i}{\lambda_k}\log\frac{f(\tilde{\boldsymbol{r}}^*)+1}{z_k}=\frac{\lambda_i}{\lambda_k}\log\frac{f(\tilde{\boldsymbol{r}}^*)+1}{r_k^*+1}\ll_{k,\boldsymbol{\lambda}}\frac{1}{R}\quad(1\le i\le k-1),$$

by (4.4) and (4.6). Also,

$$\frac{\partial^2G}{\partial x_i\partial x_j}(\tilde{\boldsymbol{r}})=-\frac{\delta_{i,j}}{r_i+1}-\frac{\lambda_i\lambda_j}{\lambda_k^2(f(\tilde{\boldsymbol{r}})+1)}\quad(1\le i,j\le k-1),$$

where $\delta_{i,j}$ is the standard Kronecker symbol. So for every $\tilde{\boldsymbol{r}} \in \mathcal{R}$ there is a vector $\boldsymbol{\xi} = (\xi_1, \dots, \xi_{k-1})$ that lies on the line segment connecting $\tilde{\boldsymbol{r}}$ and $\tilde{\boldsymbol{r}}^*$ and satisfies

$$\begin{aligned} G(\tilde{\boldsymbol{r}}) &= G(\tilde{\boldsymbol{r}}^*) + O_{k,\boldsymbol{\lambda}}\left(\frac{1}{R}\sum_{i=1}^{k-1}|r_i - r_i^*|\right) - \sum_{i=1}^{k-1}\frac{(r_i - r_i^*)^2}{2(\xi_i+1)} - \frac{1}{2}\left(\sum_{i=1}^{k-1}\frac{\lambda_i(r_i - r_i^*)}{\lambda_k\sqrt{f(\boldsymbol{\xi})+1}}\right)^2 \\ &= G(\tilde{\boldsymbol{r}}^*) + O_{k,\boldsymbol{\lambda}}(1) - \sum_{i=1}^{k-1}\frac{(r_i - r_i^*)^2}{2\xi_i+2} - \frac{(f(\tilde{\boldsymbol{r}}) - f(\tilde{\boldsymbol{r}}^*))^2}{2f(\boldsymbol{\xi})+2}. \end{aligned} \tag{4.8}$$

Next, we split the set $\mathcal{R}$ into certain subsets. Let

$$\mathcal{R}_1 = \{\tilde{\boldsymbol{r}} \in \mathcal{R} : f(\tilde{\boldsymbol{r}}) + 1 > (1+\eta)(r_k^*+1) - \Lambda/\lambda_k\},$$

$$\mathcal{R}_2 = \{\tilde{\boldsymbol{r}} \in \mathcal{R}\setminus\mathcal{R}_1 : r_i \le 3r_i^* + 4 \ (1 \le i \le k-1)\},$$

where $\eta = -1 + 2^{\lambda_k/\Lambda} > 0$, and for $I \subset \{1, \dots, k-1\}$ set

$$\mathcal{R}_3(I) = \{\tilde{\boldsymbol{r}} \in \mathcal{R}\setminus\mathcal{R}_1 : r_i > 3r_i^* + 4 \ (i \in I), r_i \le 3r_i^* + 4 \ (i \notin I, 1 \le i \le k-1)\}.$$

If $\tilde{\boldsymbol{r}} \in \mathcal{R}_1$, then (4.8) implies that

$$G(\tilde{\boldsymbol{r}}) \le G(\tilde{\boldsymbol{r}}^*) + O_{k,\boldsymbol{\lambda}}(1) - \frac{(\eta r_k^* - O_{k,\boldsymbol{\lambda}}(1))^2}{2R/\lambda_k} \le G(\tilde{\boldsymbol{r}}_*) + O_{k,\boldsymbol{\lambda}}(1) - c_0 R$$

for some positive constant $c_0 = c_0(k, \boldsymbol{\lambda})$, by (4.4). Therefore

$$\sum_{\tilde{\boldsymbol{r}}\in\mathcal{R}_1}\left(\prod_{i=1}^{k-1}\frac{\sqrt{r_i+1}}{z_i}\right)e^{G(\tilde{\boldsymbol{r}})} \ll_{k,\boldsymbol{\lambda}} R^{3(k-1)/2}e^{G(\tilde{\boldsymbol{r}}^*)-c_0R} \ll_{k,\boldsymbol{\lambda}} e^{G(\tilde{\boldsymbol{r}}^*)}. \tag{4.9}$$

Next, if $\tilde{\boldsymbol{r}} \in \mathcal{R}_2$, then for any $\boldsymbol{\xi}$ that lies on the line segment connecting $\tilde{\boldsymbol{r}}$ and $\tilde{\boldsymbol{r}}^*$ we have that $\xi_i \le 3r_i^* + 4$. Consequently,

$$G(\tilde{\boldsymbol{r}}) \le G(\tilde{\boldsymbol{r}}^*) + O_{k,\boldsymbol{\lambda}}(1) - \sum_{i=1}^{k-1}\frac{(r_i - r_i^*)^2}{6r_i^*+10},$$

by (4.8). So we deduce that

$$\begin{aligned} \sum_{\tilde{\boldsymbol{r}}\in\mathcal{R}_2}\left(\prod_{i=1}^{k-1}\frac{\sqrt{r_i+1}}{z_i}\right)e^{G(\tilde{\boldsymbol{r}})} &\ll_{k,\boldsymbol{\lambda}} \left(\prod_{i=1}^{k-1}\frac{\sqrt{r_i^*+2}}{z_i}\right)e^{G(\tilde{\boldsymbol{r}}^*)}\prod_{i=1}^{k-1}\sqrt{r_i^*+2} \\ &= e^{O_{k,\boldsymbol{\lambda}}(1+|\alpha(R)|)+G(\tilde{\boldsymbol{r}}^*)}. \end{aligned} \tag{4.10}$$

Lastly, fix some non-empty set $I \subset \{1, \dots, k-1\}$ and $i \in I$ and consider $\tilde{\boldsymbol{r}} \in \mathcal{R}_3(I)$. Set

$$\tilde{\boldsymbol{r}}_i = (r_1, \dots, r_{i-1}, r_i - 1, r_{i+1}, \dots, r_k).$$

Then for every vector $\boldsymbol{s}$ that lies in the line segment connecting $\tilde{\boldsymbol{r}}$ and $\tilde{\boldsymbol{r}}_i$ we have that

$$\frac{\partial G}{\partial x_i}(\boldsymbol{s}) \le \log\frac{z_i}{r_i-1} + \frac{\lambda_i}{\lambda_k}\log\frac{f(\tilde{\boldsymbol{r}}_i)+1}{z_k} \le \log\frac{z_i}{3r_i^*+3} + \frac{\lambda_i}{\lambda_k}\log\frac{(1+\eta)(r_k^*+1)}{z_k} \le -\log\frac{3}{2}.$$

So by the Mean Value Theorem we find that $e^{G(\tilde{\boldsymbol{r}})} \le \frac{2}{3} e^{G(\tilde{\boldsymbol{r}}_i)}$ and, consequently,

$$\sum_{\tilde{\boldsymbol{r}} \in \mathcal{R}_3(I)} \left(\prod_{i=1}^{k-1} \frac{\sqrt{r_i+1}}{z_i} \right) e^{G(\tilde{\boldsymbol{r}})} \ll_{k,\boldsymbol{\lambda}} \sum_{\tilde{\boldsymbol{r}} \in \mathcal{R}_3(I\setminus\{i\}) \cup \mathcal{R}_1} \left(\prod_{i=1}^{k-1} \frac{\sqrt{r_i+1}}{z_i} \right) e^{G(\tilde{\boldsymbol{r}})}.$$

Iterating the above inequality yields that

$$\sum_{\tilde{\boldsymbol{r}} \in \mathcal{R}_3(I)} \left(\prod_{i=1}^{k-1} \frac{\sqrt{r_i+1}}{z_i} \right) e^{G(\tilde{\boldsymbol{r}})} \ll_{k,\boldsymbol{\lambda}} \sum_{\tilde{\boldsymbol{r}} \in \mathcal{R}_1 \cup \mathcal{R}_2} \left(\prod_{i=1}^{k-1} \frac{\sqrt{r_i+1}}{z_i} \right) e^{G(\tilde{\boldsymbol{r}})},$$

since $\mathcal{R}_3(\emptyset) = \mathcal{R}_2$. Combining the above estimate with relations (4.7), (4.9) and (4.10) shows that

$$\mathbf{Prob}(\mathcal{H}^k(R)) \le \frac{e^{O_{k,\boldsymbol{\lambda}}(1+|\alpha(R)|)+G(\tilde{\boldsymbol{r}}^*)}}{\sqrt{R}}.$$

To complete the proof of the lemma, note that

$$|G(\tilde{\boldsymbol{r}}^*) - F(\boldsymbol{r}^*)| = |F(\tilde{\boldsymbol{r}}^*, f(\tilde{\boldsymbol{r}}^*)) - F(\tilde{\boldsymbol{r}}^*, r_k^*)| \ll_{k,\boldsymbol{\lambda}} 1 + |\alpha(R)|,$$

which together with (4.5) implies that

$$G(\boldsymbol{r}^*) = -\sum_{i=1}^{k} Q(e^{\lambda_i \alpha(R)}) z_i + O_{k,\boldsymbol{\lambda}}(1+|\alpha(R)|).$$

□

Finally, as a consequence of Lemma 4.2, we have the following estimates.

Lemma 4.3. *Let* $k \in \mathbb{N}$, $C \ge 0$, $z_1, \dots, z_k \ge 1$, $\lambda_1, \dots, \lambda_k > 0$ *and* $\mu_1, \dots, \mu_k > 0$ *such that* $Z = \mu_1 z_1 + \cdots + \mu_k z_k \ge \Lambda$.

(a) *If* $\lambda_i < \mu_i$ *for all* $i \in \{1, \dots, k\}$, *then*

$$\sum_{\boldsymbol{r} \in \mathcal{H}_+^k(Z)} \left(1 + \sum_{i=1}^{k} \lambda_i r_i - Z \right)^C \prod_{i=1}^{k} \frac{e^{-z_i} z_i^{r_i}}{r_i!} \ll_{k,\boldsymbol{\lambda},\boldsymbol{\mu},C} \mathbf{Prob}(\mathcal{H}^k(Z)).$$

(b) *If* $\log(\mu_i/\lambda_i) < \lambda_i$ *for all* $i \in \{1, \dots, k\}$, *then*

$$\sum_{\boldsymbol{r} \in \mathcal{H}_-^k(Z)} \left(1 + Z - \sum_{i=1}^{k} \lambda_i r_i \right)^C \prod_{i=1}^{k} \frac{e^{-z_i} (e^{\lambda_i} z_i)^{r_i}}{r_i!} \ll_{k,\boldsymbol{\lambda},\boldsymbol{\mu},C} e^Z \mathbf{Prob}(\mathcal{H}^k(Z)).$$

Proof. (a) Let S_+ be the sum in question. If we set

$$G(R) = -\sum_{i=1}^{k} Q(e^{\alpha(R)\lambda_i}) z_i = -R\,\alpha(R) + \sum_{i=1}^{k} \left(e^{\alpha(R)\lambda_i} - 1 \right) z_i,$$

then Lemma 4.2(b) implies that

$$S_+ \ll_{k,C,\boldsymbol{\lambda},\boldsymbol{\mu}} \sum_{n=0}^{\infty} (1+n)^C \frac{\exp\{c|\alpha(Z+n\Lambda)| + G(Z+n\Lambda)\}}{\sqrt{Z+n\Lambda}}. \tag{4.11}$$

Differentiating implicitly the defining equation of $\alpha(R)$, we find that there are positive constants $c_1 = c_1(k, \boldsymbol{\lambda})$ and $c_2 = c_2(k, \boldsymbol{\lambda})$ such that

$$\frac{c_1}{R} \le \alpha'(R) = \left(\sum_{i=1}^{k} \lambda_i^2 e^{\alpha(R)\lambda_i} z_i \right)^{-1} \le \frac{c_2}{R} \quad (R \ge \Lambda).$$

Also, we have that

$$\alpha(Z) \ge \min_{1 \le i \le k} \frac{1}{\lambda_i} \log\left(\frac{\mu_i}{\lambda_i}\right) > 0, \tag{4.12}$$

by the definition of $\alpha(Z)$ and our assumption that $\lambda_i < \mu_i$ for all i. So

$$G'(R) = -\alpha(R) \le -\alpha(Z) < 0 \quad (R \ge Z).$$

Combining the above remarks, we see that the summands in the right hand side of (4.11) decay exponentially. Hence

$$S_+ \ll_{k,C,\boldsymbol{\lambda},\boldsymbol{\mu}} \frac{\exp\{c\alpha(Z) + G(Z)\}}{\sqrt{Z}},$$

which together with Lemma 4.2(a) implies that

$$S_+ \ll_{k,C,\boldsymbol{\lambda},\boldsymbol{\mu}} e^{2c\alpha(Z)} \mathbf{Prob}(\mathcal{H}^k(Z)).$$

To complete the proof, note that

$$\alpha(Z) \le \max_{1 \le i \le k} \frac{1}{\lambda_i} \log\left(\frac{\mu_i}{\lambda_i}\right) \ll_{k,\boldsymbol{\lambda},\boldsymbol{\mu}} 1, \tag{4.13}$$

by the definition of $\alpha(Z)$.

(b) We argue as in part (a). Let S_- be the sum we want to estimate. Then

$$S_- \ll_{k,C,\boldsymbol{\lambda},\boldsymbol{\mu}} \sum_{0 \le n \le Z/\Lambda - 1} (1+n)^C \frac{\exp\{c|\alpha(Z - n\Lambda)| + H(Z - n\Lambda)\}}{\sqrt{Z - n\Lambda}},$$

where $H(R) = R + G(R)$, by Lemma 4.2(b). We have that

$$H'(R) = 1 - \alpha(R) \ge 1 - \alpha(Z) > 0 \quad (R \ge Z),$$

by the first inequality in (4.13) and our assumption that $\log(\mu_i/\lambda_i) < \lambda_i$ for all i. Thus

$$S_- \ll_{k,C,\boldsymbol{\lambda},\boldsymbol{\mu}} \frac{\exp\{c|\alpha(Z)| + H(Z)\}}{\sqrt{Z}} \ll_{k,\boldsymbol{\lambda},\boldsymbol{\mu}} e^{Z + 2c|\alpha(Z)|} \mathbf{Prob}(\mathcal{H}^k(Z)) \ll_{k,\boldsymbol{\lambda}} e^Z \mathbf{Prob}(\mathcal{H}^k(Z)),$$

by Lemma 4.2(a) and relations (4.12) and (4.13). This completes the proof of the lemma. $\square$

5. The upper bound in Theorem 1.5

5.1. **Outline of the proof.** In this subsection we give the key steps of the proof of the upper bound in Theorem 1.5 with most of the technical details omitted. Observe that, in view of Corollary 1.8, we may assume that the numbers $\ell_1, \dots, \ell_k$ are sufficiently large. Our starting point is Theorem 3.9. We break the sum

$$\sum_{\substack{\boldsymbol{a} \in \mathcal{P}_*^k(\boldsymbol{y}) \\ a_i \le y_i^{c_k} \ (1 \le i \le k)}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1 \cdots a_k}$$

into pieces according to the number of prime factors of the variables $a_1, \dots, a_k$. More precisely, set $\omega_k(a) = |\{p|n : p > k\}|$ and

$$S_{\boldsymbol{r}}^{(k+1)}(\boldsymbol{y}) = \sum_{\substack{\boldsymbol{a} \in \mathcal{P}_*^k(\boldsymbol{y}) \\ \omega_k(a_i) = r_i, a_i \le y_i^{c_k} \\ 1 \le i \le k}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1 \cdots a_k} \quad (\boldsymbol{r} \in (\mathbb{N} \cup \{0\})^k).$$

Also, for each fixed $i \in \{1, \dots, k\}$ define a sequence of prime numbers $\lambda_{i,1}, \lambda_{i,2}, \dots,$ as follows. Set $\rho_m = (m+1)^{1/m}$ for $m \in \mathbb{N}$, $\lambda_{i,0} = \max\{k, y_{i-1}\}$ and define inductively $\lambda_{i,j}$ as the largest element of the set $\{p \text{ prime} : \lambda_{i,0} < p \le y_i\}$ such that

$$\sum_{\lambda_{i,j-1} < p \le \lambda_{i,j}} \frac{1}{p} \le \log(\rho_{k-i+1}). \tag{5.1}$$

Notice that the sequence $\{\lambda_{i,j}\}_{j \in \mathbb{N}}$ eventually becomes constant. Let v_i be the smallest integer satisfying $\lambda_{i,v_i} = \lambda_{i,v_i+1}$. Set

$$D_{i,j} = \{p \text{ prime} : \lambda_{i,j-1} < p \le \lambda_{i,j}\} \quad (1 \le i \le k, 1 \le j \le v_i)$$

and observe that

$$\bigcup_{j=1}^{v_i} D_{i,j} = \{p \text{ prime} : \max\{y_{i-1}, k\} < p \le y_i\} \quad (1 \le i \le k). \tag{5.2}$$

Also, we have the following estimate.

Lemma 5.1. *There exists some positive number L_k such that*

$$(\rho_{k-i+1})^{j-L_k} \le \frac{\log \lambda_{i,j}}{\log y_{i-1}} \le (\rho_{k-i+1})^{j+L_k} \quad (1 \le i \le k, 1 \le j \le v_i).$$

Consequently, we have that

$$v_i = \frac{\ell_i}{\log(\rho_{k-i+1})} + O_k(1) \quad (1 \le i \le k).$$

Proof. The proof is similar to the proof of Lemma 4.6 in [Fo08b] and Lemma 3.4 in [K10a]. □

Set $\boldsymbol{v} = (v_1, \dots, v_k)$,

$$\Delta_r = \{(\xi_1, \dots, \xi_r) \in \mathbb{R}^r : 0 \le \xi_1 \le \cdots \le \xi_r \le 1\}$$

and for $i \in \{1, \dots, k\}$ and $\boldsymbol{\xi}_i = (\xi_{i,1}, \dots, \xi_{i,r_i}) \in \Delta_{r_i}$ define

$$F_i(\boldsymbol{\xi}_i) = \left(\min_{0 \le j \le r_i} \rho_{k-i+1}^{-j} (1 + \rho_{k-i+1}^{v_i \xi_{i,1}} + \cdots + \rho_{k-i+1}^{v_i \xi_{i,j}}) \right)^{k-i+1}.$$

We shall bound $S_{\boldsymbol{r}}^{(k+1)}(\boldsymbol{y})$ in terms of

$$U_{\boldsymbol{r}}^{(k+1)}(\boldsymbol{v}) = \underset{\substack{\boldsymbol{\xi}_i \in \Delta_{r_i} \\ F_i(\boldsymbol{\xi}_i) \le C_k (k-i+2)^{v_i - r_i} \\ 1 \le i \le k}}{\int \cdots \int} \min_{1 \le i \le k} \left\{ F_i(\boldsymbol{\xi}_i) \prod_{m=1}^{i-1} (k-m+2)^{v_m - r_m} \right\} d\boldsymbol{\xi}_1 \cdots d\boldsymbol{\xi}_k,$$

where C_k is a sufficiently large constant.

Lemma 5.2. *If y_1 is large enough, then*

$$S_{\boldsymbol{r}}^{(k+1)}(\boldsymbol{y}) \ll_k U_{\boldsymbol{r}}^{(k+1)}(\boldsymbol{v}) \prod_{i=1}^{k} \left(v_i (k-i+2) \log(\rho_{k-i+1}) \right)^{r_i}.$$

Lemma 5.2 will be proven in Subsection 5.2. Next, we give an upper bound on $U_{\boldsymbol{r}}^{(k+1)}(\boldsymbol{v})$, but first we need to introduce some notation. For $\boldsymbol{r} \in (\mathbb{N} \cup \{0\})^k$, $1 \le i \le k+1$ and $1 \le j \le k+1$ set

$$B_{i,j} = \begin{cases} -\displaystyle\sum_{m=j}^{i-1} (r_m - v_m) \log(k-m+2) & \text{if } 1 \le j < i, \\ 0 & \text{if } j = i, \\ \displaystyle\sum_{m=i}^{j-1} (r_m - v_m) \log(k-m+2) & \text{if } i < j. \end{cases}$$

Observe that

$$B_{i,m} + B_{m,j} = B_{i,j} \quad (1 \le i, m, j \le k+1). \tag{5.3}$$

For $j \in \{1, \dots, k+1\}$ set

$$\mathcal{R}_j = \{\boldsymbol{r} \in (\mathbb{N} \cup \{0\})^k : B_{i,j} \ge 0 \ (1 \le i \le k+1)\}.$$

Then

$$\bigcup_{j=1}^{k+1} \mathcal{R}_j = (\mathbb{N} \cup \{0\})^k.$$

Indeed, for every $\boldsymbol{r} \in (\mathbb{N} \cup \{0\})^k$ there is some $j \in \{1, \dots, k+1\}$ such that $B_{1,j} \ge B_{1,i}$ for all $i \in \{1, \dots, k+1\}$. So $\boldsymbol{r} \in \mathcal{R}_j$, by (5.3).

The following estimate will be shown in Subsection 5.2.

Lemma 5.3. *Let $j \in \{1, \dots, k+1\}$ and $\boldsymbol{r} \in \mathcal{R}_j$. Then*

$$U_{\boldsymbol{r}}^{(k+1)}(\boldsymbol{v}) \ll_k \min\left\{1, \frac{(1+B_{i_0,j})(1+B_{i_0+1,j})}{r_{i_0}+1}\right\} \frac{\prod_{m=1}^{j-1}(k-m+2)^{v_m-r_m}}{r_1! \cdots r_k!}.$$

By Lemma 5.3 and the results of Section 4, we obtain the following estimate, which will be proven in Subsection 5.2.

Lemma 5.4. *We have that*

$$\sum_{\boldsymbol{r} \in (\mathbb{N}\cup\{0\})^k} S_{\boldsymbol{r}}^{(k+1)}(\boldsymbol{y}) \ll_k \frac{\beta}{\sqrt{\log\log y_k}} \prod_{i=1}^{k} \left(\frac{\log y_i}{\log y_{i-1}}\right)^{k-i+2-Q((k-i+2)^\alpha)}.$$

The upper bound in Theorem 1.5 now follows immediately by Theorem 3.9 and Lemma 5.4.

5.2. **Completion of the proof.** In this subsection we give the proofs of Lemmas 5.2, 5.3 and 5.4.

Proof of Lemma 5.2. Let $a_1 = a_1' p_{1,1} \cdots p_{1,r_1} \le y_1^{c_k}$ with $a_1' \in \mathcal{P}_*(1,k)$ and $k < p_{1,1} < \cdots < p_{1,r_1} \le y_1$. Also, for $m \in \{2, \dots, k\}$ let $a_m = p_{m,1} \cdots p_{m,r_m} \le y_m^{c_k}$ with $y_{m-1} < p_{m,1} < \cdots < p_{m,r_m}$. For each $m \in \{1, \dots, k\}$ let $b_m = p_{m,1} \cdots p_{m,r_m}$. Also, for $1 \le m \le k$ and $1 \le i \le r_m$ define $n_{m,i} \in \{1, \dots, v_m\}$ by $p_{m,i} \in D_{m,n_{m,i}}$ and put $\boldsymbol{n}_m = (n_{m,1}, \dots, n_{m,r_m})$. For every $i \in \{1, \dots, k\}$ Lemma 2.1(b) implies that

$$\begin{aligned} L^{(k+1)}(\boldsymbol{a}) &\le \tau_{k+1}(a_1', \underbrace{1, \dots, 1}_{i-1 \text{ times}}, b_{i+1}, \dots, b_k) L^{(k+1)}(b_1, \dots, b_i, \underbrace{1, \dots, 1}_{k-i \text{ times}}) \\ &= \tau_{k+1}(a_1') \left(\prod_{m=i+1}^{k} (k-m+2)^{r_m}\right) L^{(k+1)}(b_1, \dots, b_i, \underbrace{1, \dots, 1}_{k-i \text{ times}}). \end{aligned}$$

Moreover, Lemmas 2.1 and 5.1 together with our assumption that $a_i \le y_i^{c_k}$ for $1 \le i \le k$ imply that for every $j \in \{0, 1, \dots, r_i\}$ we have

$$\begin{aligned} &L^{(k+1)}(b_1, \dots, b_i, \underbrace{1, \dots, 1}_{k-i \text{ times}}) \\ &\quad\le (k-i+2)^{r_i-j} L^{(k+1)}(b_1, \dots, b_{i-1}, p_{i,1} \cdots p_{i,j}, \underbrace{1, \dots, 1}_{k-i \text{ times}}) \\ &\quad\le (k-i+2)^{r_i-j} \left(\prod_{m=1}^{i-1} \log(2b_1 \cdots b_m)\right) \left(\log(2b_1 \cdots b_{i-1}) + \log(p_{i,1} \cdots p_{i,j})\right)^{k-i+1} \\ &\quad\ll_k (k-i+2)^{r_i-j} \left(\prod_{m=1}^{i-1} \log y_m\right) \left(\log y_{i-1} \left(1 + \rho_{k-i+1}^{n_{i,1}} + \cdots + \rho_{k-i+1}^{n_{i,j}}\right)\right)^{k-i+1} \\ &\quad\asymp_k (k-i+2)^{r_i} \left(\prod_{m=1}^{i-1} (k-m+2)^{v_m}\right) \left(\rho_{k-i+1}^{-j} \left(1 + \rho_{k-i+1}^{n_{i,1}} + \cdots + \rho_{k-i+1}^{n_{i,j}}\right)\right)^{k-i+1}. \end{aligned}$$

So if we set

$$G_i(\boldsymbol{n}_i) = \left(\min_{0\le j\le r_i} \rho_{k-i+1}^{-j} \left(1+\rho_{k-i+1}^{n_{i,1}}+\cdots+\rho_{k-i+1}^{n_{i,j}}\right)\right)^{k-i+1}$$

and

$$G(\boldsymbol{n}_1,\dots,\boldsymbol{n}_k) = \min_{1\le i\le k} \left\{ G_i(\boldsymbol{n}_i) \prod_{m=1}^{i-1} (k-m+2)^{v_m-r_m} \right\},$$

then we find that

$$L^{(k+1)}(\boldsymbol{a}) \ll_k \tau_{k+1}(a_1')G(\boldsymbol{n}_1,\dots,\boldsymbol{n}_k)\prod_{i=1}^k (k-i+2)^{r_i}.$$

Next, note that

$$\begin{aligned} G_i(\boldsymbol{n}_i) &\le (k-i+2)^{-r_i}\left(1+\rho_{k-i+1}^{n_{i,1}}+\cdots+\rho_{k-i+1}^{n_{i,r_i}}\right)^{k-i+1} \\ &\asymp_k (k-i+2)^{-r_i}\left(\frac{\log 2a_i}{\log y_{i-1}}\right)^{k-i+1} \ll_k (k-i+2)^{v_i-r_i}, \end{aligned} \tag{5.4}$$

by Lemma 5.1 and our assumption that $a_i \le y_i^{c_k}$. Also,

$$\sum_{a_1'\in\mathcal{P}_*(1,k)} \frac{\tau_{k+1}(a_1')}{a_1'} \ll_k 1.$$

So if $\mathcal{N}$ denotes the set of k-tuples $\boldsymbol{n}=(\boldsymbol{n}_1,\dots,\boldsymbol{n}_k)$ satisfying $1\le n_{m,1}\le\cdots\le n_{m,r_m}\le v_m$ for $1\le m\le k$ and inequality (5.4), then

$$S_{\boldsymbol{r}}^{(k+1)}(\boldsymbol{y}) \ll_k \sum_{\boldsymbol{n}\in\mathcal{N}} G(\boldsymbol{n}) \prod_{i=1}^k \left((k-i+2)^{r_i} \sum_{\substack{p_{i,1}<\cdots<p_{i,r_i}\\ p_{i,j}\in D_{i,n_{i,j}}\\ 1\le j\le r_i}} \frac{1}{p_{i,1}\cdots p_{i,r_i}} \right). \tag{5.5}$$

Fix $i\in\{1,\dots,k\}$. Let $g_{i,s} = |\{1\le j\le r_i : n_{i,j}=s\}|$ for $s\in\{1,\dots,v_i\}$. By (5.1), the sum over $p_{i,1},\dots,p_{i,r_i}$ in (5.5) is at most

$$\prod_{s=1}^{v_i} \frac{1}{g_{i,s}!}\left(\sum_{p\in D_{i,s}}\frac{1}{p}\right)^{g_{i,s}} \le \frac{(\log(\rho_{k-i+1}))^{r_i}}{g_{i,1}!\cdots g_{i,v_i}!} = (v_i\log(\rho_{k-i+1}))^{r_i}\,\mathrm{Vol}(I(\boldsymbol{n}_i)), \tag{5.6}$$

where

$$I(\boldsymbol{n}_i) := \{\boldsymbol{\xi}_i\in\Delta_{r_i} : n_{i,j}-1\le v_i\xi_{i,j}<n_{i,j}\ (1\le j\le r_i)\}.$$

By (5.5) and (5.6) we deduce that

$$S_{\boldsymbol{r}}^{(k+1)}(\boldsymbol{y}) \ll_k \left(\prod_{i=1}^k (v_i(k-i+2)\log(\rho_{k-i+1}))^{r_i}\right) \sum_{\boldsymbol{n}\in\mathcal{N}} G(\boldsymbol{n})\,\mathrm{Vol}\left(I(\boldsymbol{n}_1)\times\cdots\times I(\boldsymbol{n}_k)\right). \tag{5.7}$$

Finally, note that the definition of $I(\boldsymbol{n}_i)$ and (5.4) imply that

$$G_i(\boldsymbol{n}_i) \le (k-i+2)F_i(\boldsymbol{\xi}_i) \le (k-i+2)G_i(\boldsymbol{n}_i) \le C_k(k-i+2)^{v_i-r_i+1} \quad (\boldsymbol{\xi}_i\in I(\boldsymbol{n}_i))$$

for some sufficiently large constant C_k and, consequently,

$$\sum_{\boldsymbol{n}\in\mathcal{N}} G(\boldsymbol{n})\,\mathrm{Vol}\,(I(\boldsymbol{n}_1)\times\cdots\times I(\boldsymbol{n}_k)) \ll_k U_{\boldsymbol{r}}^{(k+1)}(\boldsymbol{v}).$$

Inserting the above estimate into (5.7) completes the proof of the lemma. □

Our next goal is to show Lemma 5.3. First, we state an auxiliary result.

Lemma 5.5. *Let $\mu > 1$, $A \ge 0$, $r, v \in \mathbb{N}$ and $\gamma \ge 0$. Consider the set $\mathcal{T}_\mu(r,v,\gamma)$ of all vectors $(\xi_1,\ldots,\xi_r)\in\Delta_r$ such that $\mu^{v\xi_1}+\cdots+\mu^{v\xi_j}\ge\mu^{j-\gamma}$ for $1\le j\le r$. If $\gamma\ge r-v-A$, then*

$$\mathrm{Vol}\,(\mathcal{T}_\mu(r,v,\gamma)) \ll_{\mu,A} \frac{1}{r!}\min\left\{1,\frac{(\gamma-r+v+A+1)(\gamma+1)}{r}\right\}.$$

Proof. If $1\le r\le 2v$, then the result follows by Lemma 5.3 in [K10a] (see also Lemma 4.4 in [Fo08a]) and the trivial bound $\mathrm{Vol}(\mathcal{T}_\mu(r,v,\gamma))\le\mathrm{Vol}(\Delta_r)=1/r!$. If $r>2v$, then we have that $\gamma\ge r-v-A\ge r/2-A$ and, consequently,

$$\frac{(\gamma-r+v+A+1)(\gamma+1)}{r}\gg_A 1.$$

So the lemma holds in this case too by the trivial estimate $\mathrm{Vol}\,(\mathcal{T}_\mu(r,v,\gamma))\le 1/r!$. □

Proof of Lemma 5.3. Let $j\in\{1,\ldots,k+1\}$ and $\boldsymbol{r}\in\mathcal{R}_j$. For each $i\in\{1,\ldots,k\}$, let $\mathcal{T}_i$ be the set of $\boldsymbol{\xi}=(\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_k)\in\Delta_{r_1}\times\cdots\times\Delta_{r_k}$ such that

$$\min_{1\le s\le k}\left\{F_s(\boldsymbol{\xi}_s)\prod_{m=1}^{s-1}(k-m+2)^{v_m-r_m}\right\}=\min_{1\le s\le k}\{F_s(\boldsymbol{\xi}_s)e^{-B_{1,s}}\}=F_i(\boldsymbol{\xi}_i)e^{-B_{1,i}} \tag{5.8}$$

and

$$F_s(\boldsymbol{\xi}_s)\le C_k(k-s+2)^{v_s-r_s}\quad(1\le s\le k).$$

Then for every $\boldsymbol{\xi}\in\mathcal{T}_i$ we have that

$$F_i(\boldsymbol{\xi}_i)e^{-B_{1,i}}\le\min_{1\le s\le k}\left\{\min\{C_k(k-s+2)^{v_s-r_s},1\}e^{-B_{1,s}}\right\},$$

which, together with (5.3), implies that

$$F_i(\boldsymbol{\xi}_i)\le C_ke^{B_{1,i}}\min_{1\le s\le k}e^{-\max\{B_{1,s},B_{1,s+1}\}}=C_ke^{-\max\{B_{i,1},\ldots,B_{i,k+1}\}}.$$

Relation (5.3) and our assumption that $\boldsymbol{r}\in\mathcal{R}_j$ imply that $B_{i,j}=B_{i,s}+B_{s,j}\ge B_{i,s}$ for all $s\in\{1,\ldots,k+1\}$, that is to say, $\max\{B_{i,1},\ldots,B_{i,k+1}\}=B_{i,j}$ and, consequently,

$$F_i(\boldsymbol{\xi}_i)\le C_ke^{-B_{i,j}}.$$

For $i\in\{1,\ldots,k\}$ and $n\ge B_{i,j}\ge\max\{B_{i,i_0},B_{i,i_0+1},0\}$, define $\mathcal{T}_i(n)$ to be the set of $(\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_k)\in\mathcal{T}_i$ such that

$$C_ke^{-n}<F_i(\boldsymbol{\xi}_i)\le C_ke^{-n+1}.$$

Then for $(\xi_1,\ldots,\xi_k)\in\mathcal{T}_i(n)$ relations (5.3) and (5.8) imply that

$$F_{i_0}(\boldsymbol{\xi}_{i_0})\ge e^{B_{i,i_0}}F_i(\boldsymbol{\xi}_i)>C_ke^{B_{i,i_0}-n}.$$

Hence, for every $j \in \{1, \ldots, r_{i_0}\}$, we have that

$$\rho_{k-i_0+1}^{-j}\left(\rho_{k-i_0+1}^{v_{i_0}\xi_{i_0,1}} + \cdots + \rho_{k-i_0+1}^{v_{i_0}\xi_{i_0,j}}\right) \geq \max\left\{\left(F_{i_0}(\boldsymbol{\xi}_{i_0})\right)^{1/(k-i_0+1)} - \rho_{k-i_0+1}^{-j}, \rho_{k-i_0+1}^{-j}\right\}$$
$$\geq \frac{1}{2}\left(F_{i_0}(\boldsymbol{\xi}_{i_0})\right)^{1/(k-i_0+1)} \geq \left(\rho_{k-i_0+1}\right)^{-\frac{n-B_{i,i_0}}{\log(k-i_0+2)}},$$

provided that C_k is large enough. So Lemma 5.5 gives us that

$$\begin{aligned}
U_{\boldsymbol{r}}^{(k+1)}(\boldsymbol{v}) &\leq \sum_{i=1}^{k} \int\limits_{\mathcal{T}_i} e^{B_{i,1}} F_i(\boldsymbol{\xi}_i) d\boldsymbol{\xi} \\
&\leq C_k \sum_{i=1}^{k} \sum_{n \geq B_{i,j}} e^{B_{i,1}-n+1} \left(\prod_{\substack{1 \leq j \leq k \\ j \neq i_0}} \frac{1}{r_j!}\right) \mathrm{Vol}\left(\mathcal{T}_{\rho_{k-i_0+1}}\left(r_{i_0}, v_{i_0}, \frac{n-B_{i,i_0}}{\log(k-i_0+2)}\right)\right) \\
&\ll_k \sum_{i=1}^{k} \frac{e^{B_{i,1}}}{r_1! \cdots r_k!} \sum_{n \geq B_{i,j}} \frac{1}{e^n} \min\left\{1, \frac{(n-B_{i,i_0}+1)(n-B_{i,i_0+1}+1)}{r_{i_0}+1}\right\} \\
&\ll_k \sum_{i=1}^{k} \frac{e^{B_{i,1}}}{r_1! \cdots r_k!} \frac{1}{e^{B_{i,j}}} \min\left\{1, \frac{(B_{i,j}-B_{i,i_0}+1)(B_{i,j}-B_{i,i_0+1}+1)}{r_{i_0}+1}\right\} \\
&= \frac{k e^{B_{j,1}}}{r_1! \cdots r_k!} \min\left\{1, \frac{(B_{i_0,j}+1)(B_{i_0+1,j}+1)}{r_{i_0}+1}\right\},
\end{aligned}$$

which completes the proof of the lemma. $\square$

We conclude this section with the proof of Lemma 5.4.

Proof of Lemma 5.4. Lemmas 5.2 and 5.3 imply that

$$\begin{aligned}
\sum_{\boldsymbol{r} \in (\mathbb{N} \cup \{0\})^k} S_{\boldsymbol{r}}^{(k+1)}(\boldsymbol{y}) &\ll_k \sum_{j=1}^{k+1} \sum_{\boldsymbol{r} \in \mathcal{R}_j} \left(\prod_{m=1}^{j-1} (k-m+2)^{v_m}\right) \left(\prod_{m=j}^{k} (k-m+2)^{r_m}\right) \\
&\quad \times \min\left\{1, \frac{(1+B_{i_0,j})(1+B_{i_0+1,j})}{r_{i_0}+1}\right\} \prod_{m=1}^{k} \frac{(v_m \log \rho_{k-m+1})^{r_m}}{r_m!} \\
&=: \sum_{j=1}^{k+1} T_j.
\end{aligned} \tag{5.9}$$

We fix $j \in \{1, \ldots, k+1\}$ and bound T_j. We have that $\boldsymbol{r} \in \mathcal{R}_j$ if, and only if,

$$\sum_{m=i}^{j-1} \log(k-m+2)(r_m - v_m) \geq 0 \quad (1 \leq i \leq j-1) \tag{5.10}$$

and

$$\sum_{m=j}^{i}\log(k-m+2)(r_m-v_m)\le 0\quad (j\le i\le k). \tag{5.11}$$

Let $\mathcal{R}_{1,j}$ be the set of vectors $\boldsymbol{r}_1=(r_1,\dots,r_{j-1})\in(\mathbb{N}\cup\{0\})^{j-1}$ such that (5.10) holds and let $\mathcal{R}_{2,j}$ be the set of vectors $\boldsymbol{r}_2=(r_j,\dots,r_k)\in(\mathbb{N}\cup\{0\})^{k-j+1}$ such that (5.11) holds. Note that if $\boldsymbol{r}_1\in\mathcal{R}_{1,j}$, then

$$1+B_{i_0,j}=1+B_{i_0,1}+B_{1,j}\le(1+\max\{0,B_{i_0,1}\})(1+B_{1,j})\ll_k(1+\ell_1+\cdots+\ell_{i_0-1})(1+B_{1,j}),$$

since (5.10) implies that $B_{1,j}\ge 0$. Similarly, if $\boldsymbol{r}_2\in\mathcal{R}_{2,j}$, then

$$1+B_{i_0+1,j}=1+B_{i_0+1,k+1}+B_{k+1,j}\ll_k(1+r_{i_0+1}+\cdots+r_k)(1+B_{k+1,j}),$$

since (5.11) implies that $B_{k+1,j}\ge 0$. So, if we set

$$\beta(\boldsymbol{r})=\min\left\{1,\frac{(1+\ell_1+\cdots+\ell_{i_0-1})(1+r_{i_0+1}+\cdots+r_k)}{r_{i_0}+1}\right\},$$

then we have that

$$T_j\ll_k\sum_{\substack{\boldsymbol{r}_i\in\mathcal{R}_{i,j}\\ i\in\{1,2\}}}\left(\prod_{m=1}^{j-1}(k-m+2)^{v_m}\right)\left(\prod_{m=j}^{k}(k-m+2)^{r_m}\right)$$
$$\times\beta(\boldsymbol{r})(1+B_{1,j})(1+B_{k+1,j})\prod_{m=1}^{k}\frac{(v_m\log\rho_{k-m+1})^{r_m}}{r_m!}.$$

For $s\in\{0,1,\dots,k\}$ set

$$T_{j,s}=\sum_{\substack{\boldsymbol{r}_i\in\mathcal{R}_{i,j}\\ i\in\{1,2\}}}\left(\prod_{m=1}^{j-1}(k-m+2)^{v_m}\right)\left(\prod_{m=j}^{k}(k-m+2)^{r_m}\right)$$
$$\times(1+B_{1,j})(1+B_{k+1,j})\frac{r_s+1}{r_{i_0}+1}\prod_{m=1}^{k}\frac{(v_m\log\rho_{k-m+1})^{r_m}}{r_m!},$$

where $r_0=0$. Then

$$T_j\ll_k\min\left\{T_{j,i_0},(1+\ell_1+\cdots+\ell_{i_0-1})(T_{j,0}+T_{j,i_0+1}+T_{j,i_0+2}+\cdots+T_{j,k})\right\}. \tag{5.12}$$

Observe that $T_{j,s}$ may be written as a product of two sums, with the first one ranging over $\boldsymbol{r}_1\in\mathcal{R}_{1,j}$ and the second one over $\boldsymbol{r}_2\in\mathcal{R}_{2,j}$. Lemma 4.3(a) can be applied to the first of these sums (with $j-1$ in place of k, $\{v_i\log(\rho_{k-i+1})\}_{i=1}^{j-1}$ in place of $\{z_i\}_{i=1}^{k}$, $\{\log(k-i+2)\}_{i=1}^{j-1}$ in place of $\{\lambda_i\}_{i=1}^{k}$ and $\{k-i+1\}_{i=1}^{j-1}$ in place of $\{\mu_i\}_{i=1}^{k}$). Similarly, Lemma 4.3(b) can be applied to the second sum. As a result, we deduce that

$$T_{j,s}\ll_k\frac{1+\ell_s}{1+\ell_{i_0}}\left(\prod_{m=1}^{k}(k-m+2)^{v_m}\right)\sum_{\substack{\boldsymbol{r}_i\in\mathcal{R}'_{i,j}\\ i\in\{1,2\}}}\prod_{m=1}^{k}\frac{(v_m\log\rho_{k-m+1})^{r_m}}{r_m!}, \tag{5.13}$$

where $\ell_0 = 0$,

$$\mathcal{R}'_{1,j} = \left\{ \boldsymbol{r}_1 \in (\mathbb{N}\cup\{0\})^{j-1} : -\log(k+1) \le \sum_{m=1}^{j-1} \log(k-m+2)(r_m - v_m) \le 0 \right\}$$

and

$$\mathcal{R}'_{2,j} = \left\{ \boldsymbol{r}_2 \in (\mathbb{N}\cup\{0\})^{k-j+1} : -\log(k+1) \le \sum_{m=j}^{k} \log(k-m+2)(r_m - v_m) \le 0 \right\}.$$

Clearly, we have that

$$\mathcal{R}'_{1,j} \times \mathcal{R}'_{2,j} \subset \left\{ \boldsymbol{r} \in (\mathbb{N}\cup\{0\})^{k} : -2\log(k+1) \le \sum_{m=1}^{k} \log(k-m+2)(r_m - v_m) \le 0 \right\},$$

which, in combination with relation (5.13) and Lemmas 5.1 and 4.2(b), implies that

$$T_{j,s} \ll_k \frac{\ell_s + 1}{\ell_{i_0}+1} \frac{1}{\sqrt{\log\log y_k}} \prod_{i=1}^{k} \left(\frac{\log y_i}{\log y_{i-1}} \right)^{k-i+2-Q((k-i+2)^{\alpha})}.$$

By the above estimate and (5.12) we deduce that

$$T_j \ll_k \frac{\min\left\{1, \dfrac{(1+\ell_1+\cdots+\ell_{i_0-1})(1+\ell_{i_0+1}+\cdots+\ell_k)}{\ell_{i_0}}\right\}}{\sqrt{\log\log y_k}} \prod_{i=1}^{k} \left(\frac{\log y_i}{\log y_{i-1}} \right)^{k-i+2-Q((k-i+2)^{\alpha})}.$$

Finally, inserting this inequality and (2.10) into (5.9) proves the lemma. □

6. The lower bound in Theorem 1.5: outline of the proof

As in the proof of the upper bound in Theorem 1.5, our starting point in order to prove the corresponding lower bound is Theorem 1.7. Also, we may assume that the numbers $\ell_1, \dots, \ell_k$ are large enough, by Corollary 1.8. However, the arguments deviate significantly from those in Section 5. As in [Fo08a, Fo08b, K10a], our strategy is to construct a subset of $\mathcal{P}^k_*(\boldsymbol{y})$ which contributes a positive proportion to $S^{(k+1)}(\boldsymbol{y})$ and on which we have good control of the size of $L^{(k+1)}(\boldsymbol{a})$ via Hölder's inequality. First, for $P \in (1, +\infty)$ and $\boldsymbol{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$ set

$$W^P_{k+1}(\boldsymbol{a}) = \sum_{\substack{d_1\cdots d_i | a_1\cdots a_i \\ 1\le i\le k}} \left(\sum_{\substack{d'_1\cdots d'_i | a_1\cdots a_i \\ |\log(d'_i/d_i)|<\log 2 \\ 1\le i\le k}} 1 \right)^{P-1}.$$

We have the following inequality.

Lemma 6.1. *Let $P \in (1, +\infty)$ and consider a finite set $\mathcal{A} \subset \mathbb{N}^k$. Then*

$$\left(\sum_{\boldsymbol{a}\in\mathcal{A}} \frac{W^P_{k+1}(\boldsymbol{a})}{a_1\cdots a_k} \right)^{1/P} \left(\frac{1}{(\log 2)^k} \sum_{\boldsymbol{a}\in\mathcal{A}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1\cdots a_k} \right)^{1-1/P} \ge \sum_{\boldsymbol{a}\in\mathcal{A}} \frac{\tau_{k+1}(\boldsymbol{a})}{a_1\cdots a_k}.$$

Proof. The proof is similar to the proof of Lemma 3.3 in [K10a] □

Our next goal is to bound

$$\sum_{\boldsymbol{a}\in\mathcal{A}} \frac{W_{k+1}^P(\boldsymbol{a})}{a_1\cdots a_k}$$

from above for suitably chosen sets $\mathcal{A} \subset \mathbb{N}^k$. In order to construct these sets, recall the definition of the numbers $\lambda_{i,j}$ and v_i and of the sets $D_{i,j}$ from the beginning of Subsection 5.1. Then for $\boldsymbol{g} = (\boldsymbol{g}_1, \dots, \boldsymbol{g}_k) \in (\mathbb{N}\cup\{0\})^{v_1} \times \cdots \times (\mathbb{N}\cup\{0\})^{v_k}$ with $\boldsymbol{g}_i = (g_{i,1}, \dots, g_{i,v_i})$ let

$$\mathcal{A}(\boldsymbol{g}) = \mathcal{A}_1(\boldsymbol{g}_1) \times \cdots \times \mathcal{A}_k(\boldsymbol{g}_k),$$

where for each $i \in \{1, \dots, k\}$ $\mathcal{A}_i(\boldsymbol{g}_i)$ is defined to be the set of square-free integers composed of exactly $g_{i,j}$ prime factors from $D_{i,j}$ for each $j \in \{1, \dots, v_i\}$. Set $G_{i,0} = 0$ and $G_{i,j} = g_{i,1} + \cdots + g_{i,j}$, $j = 1, \dots, v_i$. We shall estimate

$$\sum_{\boldsymbol{a}\in\mathcal{A}(\boldsymbol{g})} \frac{W_{k+1}^P(\boldsymbol{a})}{a_1\cdots a_k},$$

but first we need to introduce some additional notation. Fix $P \in (1, 2]$ and set

$$t_{i,j} = \frac{j + (k-i+2-j)^P}{k-i+2} \quad (1 \le i \le k,\ 0 \le j \le k-i+1).$$

Also, for integers $1 \le i \le k$, $\nu \ge 0$ and $n \ge 0$ with $\nu + n \le k - i + 1$ and for $\boldsymbol{g}_i \in (\mathbb{N}\cup\{0\})^{v_i}$, set

$$T_i(\boldsymbol{g}_i; \nu, n) = \sum_{0=s_0\le s_1\le\cdots\le s_n\le s_{n+1}=v_i} (\rho_{k-i+1}^{P-1})^{-(s_1+\cdots+s_n)} \prod_{j=0}^{n} (t_{i,\nu+j})^{G_{i,s_{j+1}}-G_{i,s_j}}.$$

Lastly, we define

$$T(\boldsymbol{g}) = \sum_{\substack{0=J_0\le J_1\le\cdots\le J_k\le k\\ J_i\ge i\ (1\le i\le k)}} \prod_{i=1}^{k} (\rho_{k-i+1}^{P-1})^{-(k-J_i)v_i} T_i(\boldsymbol{g}_i; J_{i-1} - i + 1, J_i - J_{i-1}).$$

Lemma 6.2. *Let $\boldsymbol{r} \in \mathbb{N}^k$ and $\boldsymbol{g} = (\boldsymbol{g}_1, \dots, \boldsymbol{g}_k) \in (\mathbb{N}\cup\{0\})^{v_1} \times \cdots \times (\mathbb{N}\cup\{0\})^{v_k}$ such that $G_{i,v_i} = r_i$ for all $i \in \{1, \dots, k\}$. Then*

$$\sum_{\boldsymbol{a}\in\mathcal{A}(\boldsymbol{g})} \frac{W_{k+1}^P(\boldsymbol{a})}{a_1\cdots a_k} \ll_k T(\boldsymbol{g}) \prod_{i=1}^{k} \frac{((k-i+2)\log\rho_{i-1+1})^{r_i}}{g_{i,1}!\cdots g_{i,v_i}!}.$$

The proof of Lemma 6.2 will be given in Section 7. Next, we use the above result to show that $W_{k+1}^P(\boldsymbol{a})$ is bounded on average over a union of suitably chosen sets $\mathcal{A}(\boldsymbol{g})$, which we construct below. Define

$$\mathcal{R}^* = \Bigg\{(r_1, \dots, r_k) \in (\mathbb{N}\cup\{0\})^k : -\log(k+1) \le \sum_{i=1}^{k} \log(k-i+2)(r_i - v_i) \le 0,$$
$$|r_i - (k-i+2)^\alpha \ell_i| \le \sqrt{\ell_i} \quad (1 \le i \le k)\Bigg\}.$$

Fix $\boldsymbol{r} \in \mathcal{R}^*$ and $i \in \{1, \dots, k\}$ and set

$$u_i' = 1 + \frac{1}{\log(k-i+2)} \sum_{j=1}^{i-1} \log(k-j+2)(v_j - r_j)$$

and

$$w_i' = u_i' + v_i - r_i = 1 + \frac{1}{\log(k-i+2)} \sum_{j=1}^{i} \log(k-j+2)(v_j - r_j).$$

By Lemma 2.2 and the definition of i_0 (see also the derivation of (2.6)), we have that

$$u_i' \asymp_k \begin{cases} 1 + \ell_1 + \cdots + \ell_{i-1} & \text{if } 1 \le i \le i_0, \\ 1 + \ell_i + \cdots + \ell_k & \text{if } i_0 + 1 \le i \le k, \end{cases} \tag{6.1}$$

and

$$w_i' \asymp_k \begin{cases} 1 + \ell_1 + \cdots + \ell_i & \text{if } 1 \le i \le i_0 - 1, \\ 1 + \ell_{i+1} + \cdots + \ell_k & \text{if } i_0 \le i \le k. \end{cases} \tag{6.2}$$

Define

$$u_i = \min\left\{ u_i', \frac{r_i - v_i + \sqrt{(r_i - v_i)^2 + 4r_i}}{2} \right\}$$

and

$$w_i = u_i + v_i - r_i = \min\left\{ w_i', \frac{v_i - r_i + \sqrt{(r_i - v_i)^2 + 4r_i}}{2} \right\}.$$

Note that $u_i \gg_k 1$ and $w_i \gg_k 1$, since $r_i \asymp_k v_i$ for $\boldsymbol{r} \in \mathcal{R}^*$. Also, since

$$u_i'w_i' = (u_i')^2 + (v_i - r_i)u_i',$$

we have that $u_i'w_i' \le r_i$ exactly when

$$u_i' \le \frac{r_i - v_i + \sqrt{(r_i - v_i)^2 + 4r_i}}{2},$$

in which case $u_i = u_i'$ and $w_i = w_i'$. On the other hand, if $u_i'w_i' > r_i$, then we find similarly that

$$u_i = \frac{r_i - v_i + \sqrt{(r_i - v_i)^2 + 4r_i}}{2} \quad \text{and} \quad w_i = \frac{v_i - r_i + \sqrt{(r_i - v_i)^2 + 4r_i}}{2}.$$

In any case, we have that

$$\beta_i := \frac{u_i w_i}{r_i} = \min\left\{ 1, \frac{u_i'w_i'}{r_i} \right\}. \tag{6.3}$$

Lastly, observe that

$$\beta_i \asymp_k \begin{cases} \beta & \text{if } i = i_0, \\ 1 & \text{otherwise}, \end{cases} \tag{6.4}$$

by relations (6.1), (6.2) and (2.10). For every $i \in \{2, \dots, k\}$ let $\mathcal{G}_i(r_i)$ be the set of vectors $\boldsymbol{g}_i \in (\mathbb{N} \cup \{0\})^{v_i}$ such that

$$G_{i,v_i} = r_i \quad \text{and} \quad G_{i,j} \leq j + u_i \quad (1 \leq j \leq v_i). \tag{6.5}$$

Also, let $\mathcal{G}_1(r_1)$ be the set of vectors $\boldsymbol{g}_1 = (g_{1,1}, \dots, g_{1,v_1}) \in (\mathbb{N} \cup \{0\})^{v_1}$ that satisfy (6.5) with $i = 1$ and have the additional property that $g_{1,j} = 0$ for $1 \leq j \leq N - 1$, where $N = N(k)$ is a sufficiently large constant to be chosen later. Finally, let $\mathcal{G}(\boldsymbol{r}) = \mathcal{G}_1(r_1) \times \dots \times \mathcal{G}_k(r_k)$. Then the following estimates hold.

Lemma 6.3. *For every $\boldsymbol{r} \in \mathcal{R}^*$ we have that*

$$\sum_{\boldsymbol{g} \in \mathcal{G}(\boldsymbol{r})} \sum_{\boldsymbol{a} \in \mathcal{A}(\boldsymbol{g})} \frac{1}{a_1 \cdots a_k} \gg_k \beta \prod_{i=1}^{k} \frac{\ell_i^{r_i}}{r_i!},$$

provided that N is large enough.

Lemma 6.4. *Assume that α satisfies* (1.1) *for some fixed $\epsilon > 0$. If $P = P(k, \epsilon)$ is close enough to 1, then for $\boldsymbol{r} \in \mathcal{R}^*$ we have that*

$$\sum_{\boldsymbol{g} \in \mathcal{G}(\boldsymbol{r})} \sum_{\boldsymbol{a} \in \mathcal{A}(\boldsymbol{g})} \frac{W_{k+1}^P(\boldsymbol{a})}{a_1 \cdots a_k} \ll_{k,\epsilon} \beta \prod_{i=1}^{k} \frac{((k-i+2)\ell_i)^{r_i}}{r_i!}.$$

Lemmas 6.3 and 6.4 will be proven in Section 8. Using these results, we complete the proof of Theorem 1.5.

Proof of Theorem 1.5 (lower bound). Assume that α satisfies (1.1) for some fixed $\epsilon > 0$. Fix $\boldsymbol{r} \in \mathcal{R}^*$. For every $\boldsymbol{a} \in \bigcup_{\boldsymbol{g} \in \mathcal{G}(\boldsymbol{r})} \mathcal{A}(\boldsymbol{g})$ we have that

$$\tau_{k+1}(\boldsymbol{a}) = \prod_{i=1}^{k} (k-i+2)^{r_i} \asymp_k \prod_{i=1}^{k} (k-i+2)^{v_i} \asymp_k \prod_{i=1}^{k} \left(\frac{\log y_i}{\log y_{i-1}} \right)^{k-i+1},$$

by Lemma 5.1 and the definition of $\mathcal{R}^*$. Therefore

$$\sum_{\boldsymbol{g} \in \mathcal{G}(\boldsymbol{r})} \sum_{\boldsymbol{a} \in \mathcal{A}(\boldsymbol{g})} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1 \cdots a_k} \gg_{k,\epsilon} \beta \prod_{i=1}^{k} \left(\frac{\log y_i}{\log y_{i-1}} \right)^{k-i+1} \prod_{i=1}^{k} \frac{\ell_i^{r_i}}{r_i!}, \tag{6.6}$$

by Lemmas 6.1, 6.3 and 6.4. Also, relation (5.2) implies that

$$\bigcup_{\boldsymbol{r} \in \mathcal{R}^*} \bigcup_{\boldsymbol{g} \in \mathcal{G}(\boldsymbol{r})} \mathcal{A}(\boldsymbol{g}) \subset \mathcal{P}_*^k(\boldsymbol{y}).$$

Hence, combining (6.6) with Theorem 1.7, we deduce that

$$\frac{H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y})}{x} \gg_{k,\epsilon} \beta e^{-(\ell_1 + \dots + \ell_k)} \sum_{\boldsymbol{r} \in \mathcal{R}^*} \prod_{i=1}^{k} \frac{\ell_i^{r_i}}{r_i!}.$$

Finally, we have that

$$e^{-(\ell_1 + \dots + \ell_k)} \sum_{\boldsymbol{r} \in \mathcal{R}^*} \prod_{i=1}^{k} \frac{\ell_i^{r_i}}{r_i!} \gg_k \frac{1}{\sqrt{\log \log y_k}} \prod_{i=1}^{k} \left(\frac{\log y_i}{\log y_{i-1}} \right)^{-Q((k-i+2)^\alpha)},$$

by Lemma 4.2(a), which completes the proof. □

7. The method of low moments

This section is devoted to establishing Lemma 6.2. This will be done in three steps. Throughout this entire section we fix a vector $\boldsymbol{r} \in \mathbb{N}^k$ and a vector $\boldsymbol{g} = (\boldsymbol{g}_1, \dots, \boldsymbol{g}_k) \in (\mathbb{N} \cup \{0\})^{v_1} \times \cdots \times (\mathbb{N} \cup \{0\})^{v_k}$ with $G_{i,v_i} = r_i$ for all $i \in \{1, \dots, k\}$. We set $R_i = \sum_{j=1}^{i} r_j$ and define

$$\mathcal{P}_{\boldsymbol{r}} = \{(Y_1, \dots, Y_k) : Y_i \subset \{1, \dots, R_i\},\ Y_i \cap Y_j = \emptyset \text{ if } i \neq j\}.$$

Also, we set

$$\mathcal{R}_i = \begin{cases} \{0, 1, \dots, R_1\} & \text{if } i = 1 \\ \{R_{i-1} + 1, \dots, R_i\} & \text{if } 2 \leq i \leq k. \end{cases}$$

For $I \in \{0, 1, \dots, R_k\}$, we define $E_{\boldsymbol{g}}(I) \in \bigcup_{i=1}^{k} \{0, 1, \dots, v_i\}$ as follows: if $I = 0$, we set $E_{\boldsymbol{g}}(I) = 0$; else, we let i be the unique number in $\{1, \dots, k\}$ such that $R_{i-1} < I \leq R_i$ and we define $E_{\boldsymbol{g}}(I)$ by

$$G_{i, E_{\boldsymbol{g}}(I)-1} < I - R_{i-1} \leq G_{i, E_{\boldsymbol{g}}(I)}.$$

For $\boldsymbol{Y} = (Y_1, \dots, Y_k) \in \mathcal{P}_{\boldsymbol{r}}$, $\boldsymbol{m} = \{m_1, \dots, m_k\}$ a permutation of $\{1, \dots, k\}$ and $I_1, \dots, I_k \in \{0, 1, \dots, R_k\}$ we put

$$\begin{aligned} &M_{\boldsymbol{r}}(\boldsymbol{Y}; \boldsymbol{I}; \boldsymbol{m}) \\ &\quad = \left| \left\{ (Z_1, \dots, Z_k) \in \mathcal{P}_{\boldsymbol{r}} : \bigcup_{i=j}^{k} (Z_{m_i} \cap (I_j, R_k]) = \bigcup_{i=j}^{k} (Y_{m_i} \cap (I_j, R_k])\ (1 \leq j \leq k) \right\} \right|. \end{aligned}$$

In addition, we let

$$\mathcal{J} = \left\{ (\mathcal{J}_1, \dots, \mathcal{J}_k) : \mathcal{J}_i \subset \{1, \dots, k\},\ \sum_{m=1}^{i} |\mathcal{J}_m| \geq i\ (1 \leq i \leq k),\ \mathcal{J}_i \cap \mathcal{J}_j = \emptyset \text{ if } i \neq j \right\}$$

and, for $(\mathcal{J}_1, \dots, \mathcal{J}_k) \in \mathcal{J}$, we set $J_i = |\mathcal{J}_1| + \cdots + |\mathcal{J}_i| \geq i$ for all $i \in \{0, \dots, k\}$. Lastly, for a family of sets $\{X_i\}_{i \in I}$ we define

$$\mathcal{U}(\{X_i : i \in I\}) := \left\{ x \in \bigcup_{i \in I} X_i : |\{j \in I : x \in X_j\}| = 1 \right\}.$$

In particular, $\mathcal{U}(Y, Z) = Y \triangle Z$, the symmetric difference of Y and Z.

Remark 7.1. Assume that $Y_1, \dots, Y_n$ and $Z_1, \dots, Z_n$ satisfy $Y_i \cap Y_j = Z_i \cap Z_j = \emptyset$ for $i \neq j$. Then

$$\mathcal{U}(\{Y_j \triangle Z_j : 1 \leq j \leq n\}) = \left(\bigcup_{j=1}^{n} Y_j \right) \triangle \left(\bigcup_{j=1}^{n} Z_j \right).$$

7.1. **Interpolating between L^1 and L^2 estimates.** The main difficulty in bounding $W^P_{k+1}(\boldsymbol{a})$ when $P \in (1,2)$ is that it is hard to use combinatorial arguments directly due to the presence of the fractional exponent $P-1$ in the definition of $W^P_{k+1}(\boldsymbol{a})$. To overcome this difficulty, we perform a special type of interpolation between L^1 and L^2 estimates. This is accomplished in Lemma 7.2 below, which is a generalization of Lemma 3.5 in [K10a].

Lemma 7.2. *Let $P \in (1,2]$, $\boldsymbol{r} \in (\mathbb{N}\cup\{0\})^k$ and $\boldsymbol{g} = (\boldsymbol{g}_1, \dots, \boldsymbol{g}_k) \in (\mathbb{N}\cup\{0\})^{v_1} \times \cdots \times (\mathbb{N}\cup\{0\})^{v_k}$ such that $G_{i,v_i} = r_i$ for $i = 1, \dots, k$. Then*

$$\sum_{\boldsymbol{a}\in\mathcal{A}(\boldsymbol{g})} \frac{W^P_{k+1}(\boldsymbol{a})}{a_1\cdots a_k} \ll_k \sum_{(\mathcal{J}_1,\dots,\mathcal{J}_k)\in\mathcal{J}} \sum_{\boldsymbol{m}} \sum_{\substack{I_j\in\mathcal{R}_i\\ 1\le i\le k, j\in\mathcal{J}_i}} \sum_{\boldsymbol{Y}\in\mathcal{P}_{\boldsymbol{r}}} \left(M_{\boldsymbol{r}}(\boldsymbol{Y};\boldsymbol{I};\boldsymbol{m})\right)^{P-1}$$
$$\times \prod_{i=1}^k \frac{(\log(\rho_{k-i+1}))^{r_i} (\rho^{P-1}_{k-i+1})^{-(k-J_i)v_i}}{g_{i,1}!\cdots g_{i,v_i}!} \prod_{j\in\mathcal{J}_i} (\rho^{P-1}_{k-i+1})^{-E_{\boldsymbol{g}}(I_j)}.$$

Proof. Consider

$$\boldsymbol{a} = (a_1,\dots,a_k) = (p_1\cdots p_{R_1}, p_{R_1+1}\cdots p_{R_2}, \dots, p_{R_{k-1}+1}\cdots p_{R_k}) \in \mathcal{A}(\boldsymbol{g}) \tag{7.1}$$

such that

$$p_{R_{i-1}+G_{i,j-1}+1}, \dots, p_{R_{i-1}+G_{i,j}} \in D_{i,j} \quad (1\le i\le k,\ 1\le j\le v_i) \tag{7.2}$$

and the primes in each interval $D_{i,j}$ for $i = 1, \dots, k$ and $j = 1, \dots, v_i$ are unordered. Since the number $\prod_{i=1}^k a_i$ is square-free and $\omega(a_i) = r_i$ for all $i \in \{1, \dots, k\}$, the k-tuples $(d_1, \dots, d_k)$ with $d_1\cdots d_i | a_1\cdots a_i$ for $1 \le i \le k$ are in one to one correspondence with the k-tuples $(Y_1, \dots, Y_k) \in \mathcal{P}_{\boldsymbol{r}}$ via the relation

$$d_j = \prod_{i\in Y_j} p_i \quad (1\le j\le k).$$

Using this observation twice, we find that

$$W^P_{k+1}(\boldsymbol{a}) = \sum_{(Y_1,\dots,Y_k)\in\mathcal{P}_{\boldsymbol{r}}} \left(\sum_{\substack{(Z_1,\dots,Z_k)\in\mathcal{P}_{\boldsymbol{r}}\\ (7.3)}} 1 \right)^{P-1},$$

where for two k-tuples $(Y_1, \dots, Y_k) \in \mathcal{P}_{\boldsymbol{r}}$ and $(Z_1, \dots, Z_k) \in \mathcal{P}_{\boldsymbol{r}}$ condition (7.3) is defined by

$$-\log 2 < \sum_{i\in Y_j} \log p_i - \sum_{i\in Z_j} \log p_i < \log 2 \quad (1\le j\le k). \tag{7.3}$$

Moreover, each k-tuple $(a_1, \dots, a_k) \in \mathcal{A}(\boldsymbol{g})$ has exactly $\prod_{i,j} g_{i,j}!$ representations of the form given in (7.1), corresponding to all the possible permutations of the prime numbers

$p_1,\dots,p_{R_k}$ under condition (7.2). Hence

$$\sum_{\boldsymbol{a}\in\mathcal{A}(\boldsymbol{g})}\frac{W_{k+1}^P(\boldsymbol{a})}{a_1\cdots a_k}=\left(\prod_{\substack{1\le i\le k\\1\le j\le v_i}}\frac{1}{g_{i,j}!}\right)\sum_{\substack{p_1,\dots,p_{R_k}\\(7.2)}}\frac{1}{p_1\cdots p_{R_k}}\sum_{(Y_1,\dots,Y_k)\in\mathcal{P}_{\boldsymbol{r}}}\left(\sum_{\substack{(Z_1,\dots,Z_k)\in\mathcal{P}_{\boldsymbol{r}}\\(7.3)}}1\right)^{P-1}$$
$$=\left(\prod_{\substack{1\le i\le k\\1\le j\le v_i}}\frac{1}{g_{i,j}!}\right)\sum_{(Y_1,\dots,Y_k)\in\mathcal{P}_{\boldsymbol{r}}}\sum_{\substack{p_1,\dots,p_{R_k}\\(7.2)}}\frac{1}{p_1\cdots p_{R_k}}\left(\sum_{\substack{(Z_1,\dots,Z_k)\in\mathcal{P}_{\boldsymbol{r}}\\(7.3)}}1\right)^{P-1}.$$

So Hölder's inequality yields that

$$\sum_{\boldsymbol{a}\in\mathcal{A}(\boldsymbol{g})}\frac{W_{k+1}^P(\boldsymbol{a})}{a_1\cdots a_k}\le\left(\prod_{\substack{1\le i\le k\\1\le j\le v_i}}\frac{1}{g_{i,j}!}\right)\sum_{(Y_1,\dots,Y_k)\in\mathcal{P}_{\boldsymbol{r}}}\left(\sum_{\substack{p_1,\dots,p_{R_k}\\(7.2)}}\frac{1}{p_1\cdots p_{R_k}}\sum_{\substack{(Z_1,\dots,Z_k)\in\mathcal{P}_{\boldsymbol{r}}\\(7.3)}}1\right)^{P-1}$$
$$\times\left(\sum_{\substack{p_1,\dots,p_{R_k}\\(7.2)}}\frac{1}{p_1\cdots p_{R_k}}\right)^{2-P}.$$

Note that

$$\sum_{\substack{p_1,\dots,p_{R_k}\\(7.2)}}\frac{1}{p_1\cdots p_{R_k}}\le\prod_{i=1}^k\prod_{j=1}^{v_i}\left(\sum_{p\in D_{i,j}}\frac{1}{p}\right)^{g_{i,j}}\le\prod_{i=1}^k(\log(\rho_{k-i+1}))^{r_i}$$

by (5.1) and, consequently,

$$\sum_{\boldsymbol{a}\in\mathcal{A}(\boldsymbol{g})}\frac{W_{k+1}^P(\boldsymbol{a})}{a_1\cdots a_k}\le\left(\prod_{i=1}^k\frac{(\log(\rho_{k-i+1}))^{(2-P)r_i}}{g_{i,1}!\cdots g_{i,v_i}!}\right)$$
$$\times\sum_{(Y_1,\dots,Y_k)\in\mathcal{P}_{\boldsymbol{r}}}\left(\sum_{(Z_1,\dots,Z_k)\in\mathcal{P}_{\boldsymbol{r}}}\sum_{\substack{p_1,\dots,p_{R_k}\\(7.2),(7.3)}}\frac{1}{p_1\cdots p_{R_k}}\right)^{P-1}. \tag{7.4}$$

Next, we estimate the sum over the primes above. In order to do so, we need to understand condition (7.3). Note that (7.3) is equivalent to

$$-\log 2<\sum_{i\in Y_j\setminus Z_j}\log p_i-\sum_{i\in Z_j\setminus Y_j}\log p_i<\log 2\quad(1\le j\le k). \tag{7.5}$$

Fix two k-tuples $(Y_1,\dots,Y_k)\in\mathcal{P}_{\boldsymbol{r}}$ and $(Z_1,\dots,Z_k)\in\mathcal{P}_{\boldsymbol{r}}$ and define the numbers $I_1,\dots,I_k$ and $m_1,\dots,m_k$ with $I_i\in(Y_{m_i}\triangle Z_{m_i})\cup\{0\}$ for all $i\in\{1,\dots,k\}$ inductively, as follows (see the proof of [K10a, Lemma 3.5] for the motivation behind these definitions). Let

$$I_1 = \max\left\{\mathcal{U}(Y_1\triangle Z_1,\dots,Y_k\triangle Z_k)\cup\{0\}\right\}.$$

If $I_1 = 0$, set $m_1 = k$. Else, define m_1 to be the unique element of $\{1,\dots,k\}$ such that $I_1 \in Y_{m_1}\triangle Z_{m_1}$. Assume we have defined $I_1,\dots,I_i$ for some $i\in\{1,\dots,k-1\}$ with $I_r \in (Y_{m_r}\triangle Z_{m_r})\cup\{0\}$ for $r=1,\dots,i$. Then set

$$I_{i+1} = \max\left\{\mathcal{U}\left(\{Y_j\triangle Z_j : j\in\{1,\dots,k\}\setminus\{m_1,\dots,m_i\}\}\right)\cup\{0\}\right\}.$$

If $I_{i+1}=0$, set $m_{i+1}=\max\{\{1,\dots,k\}\setminus\{m_1,\dots,m_i\}\}$. Otherwise, define m_{i+1} to be the unique element of $\{1,\dots,k\}\setminus\{m_1,\dots,m_i\}$ such that $I_{i+1}\in Y_{m_{i+1}}\triangle Z_{m_{i+1}}$. This completes the inductive step.

Note that we must have $\{m_1,\dots,m_k\}=\{1,\dots,k\}$. Also, if we set

$$\mathcal{J}_i = \{1\le j\le k : I_j \in \mathcal{R}_i\}\quad (1\le i\le k),$$

then observe that $(\mathcal{J}_1,\dots,\mathcal{J}_k)\in\mathcal{J}$, since

$$J_i = \sum_{m=1}^{i}|\mathcal{J}_m| = |\{1\le j\le k : I_j\le R_i\}| \ge |\{1\le j\le k : m_j\le i\}| = i$$

for all $i\in\{1,\dots,k\}$. Set $\mathcal{I}=\{I_j : 1\le j\le k, I_j>0\}$ and fix for the moment the primes p_i for $i\in\{1,\dots,R_k\}\setminus\mathcal{I}$. Then (7.5) becomes a system of linear inequalities with respect to the set of variables $\{\log p_I : I\in\mathcal{I}\}$ that corresponds to a triangular matrix, up to a permutation of its rows. So a straightforward manipulation of the inequalities which constitute (7.5) implies that $p_I\in[X_I,4^kX_I]$ for $I\in\mathcal{I}$, where the numbers X_I depend only on the primes p_i for $i\in\{1,\dots,R_k\}\setminus\mathcal{I}$ and the k-tuples $(Y_1,\dots,Y_k)$ and $(Z_1,\dots,Z_k)$, which we have fixed. Consequently,

$$\sum_{\substack{p_I,\ I\in\mathcal{I}\\ (7.2),(7.5)}}\prod_{I\in\mathcal{I}}\frac{1}{p_I} \ll_k \prod_{i=1}^{k}\prod_{\substack{j\in\mathcal{J}_i\\ I_j>0}}\frac{1}{\log(\max\{\lambda_{i,E_{\boldsymbol{g}}(I)-1},X_{I_j}\})} \ll_k \prod_{i=1}^{k}\prod_{j\in\mathcal{J}_i}\frac{(\rho_{k-i+1})^{-E_{\boldsymbol{g}}(I_j)}}{\log y_{i-1}},$$

by Lemma 5.1. So we find that

$$\sum_{\substack{p_1,\dots,p_{R_k}\\ (7.2),(7.5)}}\frac{1}{p_1\cdots p_{R_k}} \ll_k \prod_{i=1}^{k}(\log(\rho_{k-i+1}))^{r_i}\prod_{j\in\mathcal{J}_i}\frac{(\rho_{k-i+1})^{-E_{\boldsymbol{g}}(I_j)}}{\log y_{i-1}}$$

which, together with (7.4), implies that

$$\begin{aligned}\sum_{\boldsymbol{a}\in\mathcal{A}(\boldsymbol{g})}\frac{W_{k+1}^{P}(\boldsymbol{a})}{a_1\cdots a_k} &\ll_k \left(\prod_{i=1}^{k}\frac{(\log(\rho_{k-i+1}))^{r_i}}{g_{i,1}!\cdots g_{i,v_i}!}\right)\\ &\qquad \sum_{(Y_1,\dots,Y_k)\in\mathcal{P}_{\boldsymbol{r}}}\left(\sum_{(Z_1,\dots,Z_k)\in\mathcal{P}_{\boldsymbol{r}}}\prod_{i=1}^{k}\prod_{j\in\mathcal{J}_i}\frac{(\rho_{k-i+1})^{-E_{\boldsymbol{g}}(I_j)}}{\log y_{i-1}}\right)^{P-1}.\end{aligned}\tag{7.6}$$

Note that

$$\prod_{i=1}^{k}(\log y_{i-1})^{|\mathcal{J}_i|} \asymp_k \prod_{i=1}^{k} e^{(k-J_i)\ell_i} \asymp_k \prod_{i=1}^{k}(\rho_{k-i+1})^{(k-J_i)v_i}, \tag{7.7}$$

by Lemma 5.1. Moreover, the definition of the numbers $I_1,\dots,I_k$ and $m_1,\dots,m_k$ implies that

$$(I_j,R_k]\cap\mathcal{U}(\{Y_{m_r}\triangle Z_{m_r}: j\le r\le k\})=\emptyset \quad (1\le j\le k),$$

which is equivalent to

$$\bigcup_{r=j}^{k}(Z_{m_r}\cap(I_j,R_k])=\bigcup_{r=j}^{k}(Y_{m_r}\cap(I_j,R_k]) \quad (1\le j\le k),$$

by Remark 7.1. Hence for fixed $(Y_1,\dots,Y_k)\in\mathcal{P}_{\boldsymbol{r}}$, $0\le I_1,\dots,I_k\le R_k$ and $\boldsymbol{m}=\{m_1,\dots,m_k\}$, a permutation of $\{1,\dots,k\}$, the number of admissible k-tuples $(Z_1,\dots,Z_k)\in\mathcal{P}_{\boldsymbol{r}}$ is at most $M_{\boldsymbol{r}}(\boldsymbol{Y};\boldsymbol{I};\boldsymbol{m})$. Combining this observation with (7.6) and (7.7) we deduce that

$$\begin{aligned}\sum_{\boldsymbol{a}\in\mathcal{A}(\boldsymbol{g})}\frac{W_{k+1}^P(\boldsymbol{a})}{a_1\cdots a_k} \ll_k &\left(\prod_{i=1}^{k}\frac{(\log(\rho_{k-i+1}))^{r_i}}{g_{i,1}!\cdots g_{i,v_i}!}\right)\\ &\times\sum_{(Y_1,\dots,Y_k)\in\mathcal{P}_{\boldsymbol{r}}}\left(\sum_{I_1,\dots,I_k}\sum_{\boldsymbol{m}}M_{\boldsymbol{r}}(\boldsymbol{Y};\boldsymbol{I};\boldsymbol{m})\prod_{i=1}^{k}(\rho_{k-i+1})^{-(k-J_i)v_i}\prod_{j\in\mathcal{J}_i}(\rho_{k-i+1})^{-E_{\boldsymbol{g}}(I_j)}\right)^{P-1}.\end{aligned}$$

Finally, the inequality $(a+b)^{P-1}\le a^{P-1}+b^{P-1}$ for $a\ge0$ and $b\ge0$, which holds precisely when $1<P\le2$, completes the proof of the lemma. □

7.2. **Combinatorial arguments.** In this subsection we use combinatorial arguments to calculate $M_{\boldsymbol{r}}(\boldsymbol{Y};\boldsymbol{I};\boldsymbol{m})$ and, as a result, simplify the estimate given by Lemma 7.2. Note that the following lemma is similar to Lemma 3.6 in [K10a].

Lemma 7.3. *Let $P\in(1,+\infty)$, $\boldsymbol{r}\in(\mathbb{N}\cup\{0\})^k$, $\boldsymbol{m}=\{m_1,\dots,m_k\}$ a permutation of $\{1,\dots,k\}$, $(\mathcal{J}_1,\dots,\mathcal{J}_k)\in\mathcal{J}$ and $0\le I_1,\dots,I_k\le R_k$ such that $I_s\in\mathcal{R}_i$ for $s\in\mathcal{J}_i$ and $1\le i\le k$. Assume that $\sigma\in S_k$ is a permutation such that $I_{\sigma(1)}\le\dots\le I_{\sigma(k)}$. Then*

$$\sum_{\boldsymbol{Y}\in\mathcal{P}_{\boldsymbol{r}}}(M_{\boldsymbol{r}}(\boldsymbol{Y};\boldsymbol{I};\boldsymbol{m}))^{P-1}\le\prod_{i=1}^{k}\left((k-i+2)^{r_i}\frac{(t_{i,J_i-i+1})^{R_i}}{(t_{i,J_{i-1}-i+1})^{R_{i-1}}}\prod_{J_{i-1}<j\le J_i}\left(\frac{t_{i,j-i}}{t_{i,j-i+1}}\right)^{I_{\sigma(j)}}\right).$$

Proof. Set $\sigma(0)=0$, $\sigma(k+1)=k+1$, $I_0=0$ and $I_{k+1}=R_k$. First, we calculate $M_{\boldsymbol{r}}(\boldsymbol{Y};\boldsymbol{I};\boldsymbol{m})$ for fixed $\boldsymbol{Y}\in\mathcal{P}_{\boldsymbol{r}}$. Let

$$\mathcal{N}_{i,j}=\mathcal{R}_i\cap(I_{\sigma(j)},I_{\sigma(j+1)}] \quad (1\le i\le k,\ J_{i-1}\le j\le J_i)$$

and

$$Y_{s,i,j}=Y_s\cap\mathcal{N}_{i,j},\quad y_{s,i,j}=|Y_{s,i,j}| \quad (0\le s\le k,\ 1\le i\le k,\ J_{i-1}\le j\le J_i),$$

where

$$Y_0=\{1,\dots,R_k\}\setminus\bigcup_{i=1}^{k}Y_i.$$

The k-tuple $(Z_1,\dots,Z_k)\in\mathcal{P}_{\boldsymbol{r}}$ is counted by $M_{\boldsymbol{r}}(\boldsymbol{Y};\boldsymbol{I};\boldsymbol{m})$ when

$$\bigcup_{s=j}^{k}(Z_{m_s}\cap(I_j,R_k])=\bigcup_{s=j}^{k}(Y_{m_s}\cap(I_j,R_k])\quad(1\le j\le k). \tag{7.8}$$

So if we set

$$Z_{s,i,j}=Z_s\cap\mathcal{N}_{i,j}\quad(0\le s\le k,\ 1\le i\le k,\ J_{i-1}\le j\le J_i),$$

where

$$Z_0=\{1,\dots,R_k\}\setminus\bigcup_{i=1}^{k}Z_i,$$

then (7.8) can be written as

$$\bigcup_{s=\sigma(t)}^{k}Z_{m_s,i,j}=\bigcup_{s=\sigma(t)}^{k}Y_{m_s,i,j}\quad(1\le i\le k,\ J_{i-1}\le j\le J_i,\ 0\le t\le j). \tag{7.9}$$

For $j\ge 0$ let

$$\chi_j:\{0,1,\dots,j,j+1\}\to\{\sigma(0),\sigma(1),\dots,\sigma(j),\sigma(k+1)\}$$

be the bijection uniquely determined by the property that $\chi_j(0)<\dots<\chi_j(j+1)$. So the sequence $\chi_j(0),\dots,\chi_j(j+1)$ is the sequence $\sigma(0),\dots,\sigma(j),\sigma(k+1)$ ordered increasingly. In particular, $\chi_j(0)=\sigma(0)=0$ and $\chi_j(j+1)=\sigma(k+1)=k+1$. Note that $Z_{m_s,i,j}=Y_{m_s,i,j}=\emptyset$ if $1\le m_s<i$, by the definition of $\mathcal{P}_{\boldsymbol{r}}$. So if we set $m_0=0$ and

$$A_{t,i,j}=\{\chi_j(t)\le s<\chi_j(t+1):m_s\ge i\text{ or }s=0\}\quad(1\le i\le k,\ J_{i-1}\le j\le J_i,\ 0\le t\le j),$$

then (7.9) is equivalent to

$$\bigcup_{s\in A_{t,i,j}}Z_{m_s,i,j}=\bigcup_{s\in A_{t,i,j}}Y_{m_s,i,j}\quad(0\le t\le j), \tag{7.10}$$

for all $1\le i\le k$ and $J_{i-1}\le j\le J_i$. For such a pair (i,j), let $M_{i,j}$ be the set of mutually disjoint $(k-i+2)$-tuples $(Z_{0,i,j},Z_{i,i,j},Z_{i+1,i,j},\dots,Z_{k,i,j})$ that satisfy (7.10). Then

$$M_{\boldsymbol{r}}(\boldsymbol{Y};\boldsymbol{I};\boldsymbol{m})=\prod_{\substack{1\le i\le k\\ J_{i-1}\le j\le J_i}}M_{i,j}. \tag{7.11}$$

Moreover, it is immediate from the definition of $M_{i,j}$ that

$$M_{i,j}=\prod_{t=0}^{j}|A_{t,i,j}|^{\sum_{s\in A_{t,i,j}}y_{m_s,i,j}}$$

(with the standard notational convention that $0^0=1$). Let

$$W_{t,i,j}=\bigcup_{s\in A_{t,i,j}}Y_{m_s,i,j}\quad\text{and}\quad w_{t,i,j}=|W_{t,i,j}|\quad(1\le i\le k,\ J_{i-1}\le j\le J_i,\ 0\le t\le j). \tag{7.12}$$

With this notation, we have that

$$M_{i,j}=\prod_{t=0}^{j}|A_{t,i,j}|^{w_{t,i,j}}.$$

Inserting the above relation into (7.11), we deduce that

$$M_{\boldsymbol{r}}(\boldsymbol{Y};\boldsymbol{I};\boldsymbol{m}) = \prod_{i=1}^{k}\prod_{j=J_{i-1}}^{J_i}\prod_{t=0}^{j}|A_{t,i,j}|^{w_{t,i,j}}.$$

Therefore

$$S := \sum_{\boldsymbol{Y}\in\mathcal{P}_{\boldsymbol{r}}}(M_{\boldsymbol{r}}(\boldsymbol{Y};\boldsymbol{I};\boldsymbol{m}))^{P-1} = \prod_{i=1}^{k}\prod_{j=J_{i-1}}^{J_i}\sum_{Y_{0,i,j},Y_{i,i,j},\dots,Y_{k,i,j}}\prod_{t=0}^{j}|A_{t,i,j}|^{(P-1)w_{t,i,j}}.$$

Next, for fixed $i\in\{1,\dots,k\}$, $j\in\{J_{i-1},\dots,J_i\}$ and $W_{0,i,j},\dots,W_{j,i,j}$, a partition of $\mathcal{N}_{i,j}$, the number of $Y_{0,i,j},Y_{i,i,j},\dots,Y_{k,i,j}$ that satisfy (7.12) is equal to

$$\prod_{t=0}^{j}|A_{t,i,j}|^{w_{t,i,j}}.$$

Consequently,

$$S = \prod_{i=1}^{k}\prod_{j=J_{i-1}}^{J_i}\sum_{W_{0,i,j},\dots,W_{j,i,j}}\prod_{t=0}^{j}|A_{t,i,j}|^{Pw_{t,i,j}} = \prod_{i=1}^{k}\prod_{j=J_{i-1}}^{J_i}\left(|A_{0,i,j}|^P+\cdots+|A_{j,i,j}|^P\right)^{|\mathcal{N}_{i,j}|}, \tag{7.13}$$

by the multinomial theorem. Fix $1\le i\le k$ and $J_{i-1}\le j\le J_I$ and set

$$K_{i,j} = \{0\le t\le j : |A_{t,i,j}|\ge 1\}.$$

We claim that

$$j-i+2\le |K_{i,j}|\le k-i+2. \tag{7.14}$$

Indeed, we have that

$$\{0\}\cup\{1\le s\le k : m_s\ge i\} = \bigcup_{t\in K_{i,j}} A_{t,i,j}\subset\bigcup_{t\in K_{i,j}}\{s\in\mathbb{Z}:\chi_j(t)\le s<\chi_j(t+1)\}.$$

The above relation implies that

$$k-i+2 = |\{0\}\cup\{1\le s\le k : m_s\ge i\}| = \sum_{t\in K_{i,j}}|A_{t,i,j}|\ge |K_{i,j}|$$

and

$$\begin{aligned} k-i+2 &\le \left|\bigcup_{t\in K_{i,j}}\{s\in\mathbb{Z}:\chi_j(t)\le s<\chi_j(t+1)\}\right| \\ &= k+1-\left|\bigcup_{t\in\{0,1,\dots,j\}\setminus K_{i,j}}\{s\in\mathbb{Z}:\chi_j(t)\le s<\chi_j(t+1)\}\right|\le k-j+|K_{i,j}|, \end{aligned}$$

which together prove (7.14). Lastly, note that for $n\le X$ we have that

$$\max\left\{\sum_{j=1}^{n}x_j^P : \sum_{j=1}^{n}x_j = X,\ x_j\ge 1\ (1\le j\le n)\right\} = n-1+(X-n+1)^P,$$

since the maximum of a convex function in a simplex occurs at its vertices. Therefore

$$\begin{aligned}|A_{0,i,j}|^P+\cdots+|A_{j,i,j}|^P\le|K_{i,j}|-1+(k-i+3-|K_{i,j}|)^P&\le j-i+1+(k-j+1)^P\\&=(k-i+2)t_{i,j-i+1},\end{aligned}$$

by (7.14). Finally, inserting the above inequality into (7.13) yields

$$\begin{aligned}S&\le\prod_{i=1}^{k}(k-i+2)^{r_i}\prod_{j=J_{i-1}}^{J_i}(t_{i,j-i+1})^{|\mathcal{N}_{i,j}|}\\&=\prod_{i=1}^{k}(k-i+2)^{r_i}(t_{i,J_{i-1}-i+1})^{I_{\sigma(J_{i-1}+1)}-R_{i-1}}\\&\qquad\times\left(\prod_{j=J_{i-1}+1}^{J_i-1}(t_{i,j-i+1})^{I_{\sigma(j+1)}-I_{\sigma(j)}}\right)(t_{i,J_i-i+1})^{R_i-I_{\sigma(J_i)}},\end{aligned}$$

which completes the proof of the lemma. □

7.3. **Proof of Lemma 6.2.** In this last subsection we combine the results of Subsections 7.1 and 7.2 to show Lemma 6.2.

Proof of Lemma 6.2. By Lemmas 7.2 and 7.3 we have that

$$\begin{aligned}\sum_{\boldsymbol{a}\in\mathcal{A}(\boldsymbol{g})}\frac{W_{k+1}^P(\boldsymbol{a})}{a_1\cdots a_k}\ll_k&\sum_{\substack{0=J_0\le J_1\le\cdots\le J_k\le k\\ J_i\ge i\ (1\le i\le k)}}\ \sum_{\substack{0\le I_1\le\cdots\le I_k\le R_k\\ I_j\in\mathcal{R}_i,\ J_{i-1}<j\le J_i\\ 1\le i\le k}}\prod_{i=1}^{k}\Bigg(\frac{((k-i+2)\log(\rho_{k-i+1}))^{r_i}}{g_{i,1}!\cdots g_{i,v_i}!}\\&\times\frac{(t_{i,J_i-i+1})^{R_i}}{(\rho_{k-i+1}^{P-1})^{(k-J_i)v_i}(t_{i,J_{i-1}-i+1})^{R_{i-1}}}\prod_{J_{i-1}<j\le J_i}(\rho_{k-i+1}^{P-1})^{-E_{\boldsymbol{g}}(I_j)}\left(\frac{t_{i,j-i}}{t_{i,j-i+1}}\right)^{I_j}\Bigg).\end{aligned}\tag{7.15}$$

Write $e_j=E_{\boldsymbol{g}}(I_j)$ for $i\in\{1,\dots,k\}$ and note that

$$0\le e_{J_{i-1}+1}\le\cdots\le e_{J_i}\le v_i\quad(1\le i\le k).$$

Moreover, for $1\le i\le k$ and $J_{i-1}<j\le J_i$ we have that

$$\sum_{I_j\in\mathcal{R}_i,\ E_{\boldsymbol{g}}(I_j)=e_j}\left(\frac{t_{i,j-i}}{t_{i,j-i+1}}\right)^{I_j}\le\sum_{G_{i,e_j-1}+R_{i-1}\le I_j\le G_{i,e_j}+R_{i-1}}\left(\frac{t_{i,j-i}}{t_{i,j-i+1}}\right)^{I_j}\ll_{k,P}\left(\frac{t_{i,j-i}}{t_{i,j-i+1}}\right)^{G_{i,e_j}+R_{i-1}},$$

since $t_{i,j-i}>t_{i,j-i+1}$. Inserting the above inequality into (7.15) completes the proof. □

8. The lower bound in Theorem 1.5: completion of the proof

In this section we complete the proof of Theorem 1.5 by showing Lemmas 6.3 and 6.4.

8.1. **Preliminaries.** We state here some inequalities we will need later. For $0 < h \le x$ set

$$F(x,h) = \frac{(x+1)\log(x+1) - (x-h+1)\log(x-h+1)}{h}.$$

Lemma 8.1. *The function F has the following properties:*

(a) *For $0 < h \le x$ we have*

$$\frac{\partial F(x,h)}{\partial h} < 0, \quad \frac{\partial F(x,h)}{\partial x} > 0 \quad \textit{and} \quad \frac{\partial (F(x,x))}{\partial x} > 0.$$

(b) *For $0 < h \le x-1$ we have*

$$F(x,h) > F(x-h,1).$$

Proof. (a) We have that

$$\frac{\partial F(x,h)}{\partial h} = \frac{1}{h^2}\left[h + (x+1)\log\left(1 - \frac{h}{x+1}\right)\right] < 0 \quad (0 < h \le x).$$

Also,

$$\frac{\partial F(x,h)}{\partial x} = \frac{1}{h}\log\left(\frac{x+1}{x-h+1}\right) > 0 \quad (0 < h \le x).$$

Finally,

$$\frac{\partial (F(x,x))}{\partial x} = \frac{x - \log(x+1)}{x^2} > 0 \quad (x > 0).$$

(b) Fix $x > 1$ and note that it suffices to show that

$$g(h) = (x+1)\log(x+1) - (h+1)(x-h+1)\log(x-h+1) + h(x-h)\log(x-h) > 0$$

for $0 < h \le x-1$. Since $g(0) = 0$, it is enough to show that $g'(h) > 0$. We have that

$$g'(h) = 1 + (2h - x)\log\left(\frac{x-h+1}{x-h}\right).$$

If $h \ge x/2$, then $g'(h) \ge 1$. If $0 < h < x/2$, then

$$g'(h) > 1 - \frac{x-2h}{x-h} = \frac{h}{x-h} > 0.$$

In any case, we have that $g'(h) > 0$, which completes the proof of the lemma. □

Finally, we have the following lemma.

Lemma 8.2. *The sequence*

$$\left\{1 - \frac{1}{\log(n+2)}\log\left(\frac{(n+2)\log(n+2) - \log 4}{n}\right)\right\}_{n\in\mathbb{N}}$$

is strictly increasing.

Proof. For $x > 0$ set

$$g(x) = \frac{(x+2)\log(x+2) - \log 4}{x} \quad \text{and} \quad G(x) = 1 - \frac{\log(g(x))}{\log(x+2)}.$$

First, we check numerically that $G(1) < G(2) < \cdots < G(14)$. Next, we handle the larger terms of the sequence. We have

$$G'(x) = \frac{h(x)}{x(x+2)[(x+2)\log(x+2) - \log 4]\log^2(x+2)},$$

where

$$h(x) = x[(x+2)\log(x+2) - \log 4]\log(g(x)) - (x+2)\log(x+2)(x - 2\log(x+2) + \log 4).$$

Observe that for $x \geq 14$ we have $\log(g(x)) \geq \log\log(x+2) \geq 1$. Consequently,

$$\begin{aligned} h(x) &\geq -x\log 4 + 2(x+2)\log^2(x+2) - (\log 4)(x+2)\log(x+2) \\ &\geq (-x + 3(x+2)\log(x+2))\log 4 > 0 \end{aligned}$$

for all $x \geq 14$, that is $G'(x) > 0$ for $x \geq 14$ and the desired result follows. □

8.2. **Estimates from order statistics.** Throughout this subsection we fix a vector $\boldsymbol{r} \in \mathcal{R}^*$. Our goal is to bound on average the quantities $T_i(\boldsymbol{g}_i; \nu, n)$, which were defined in Section 6, on average. To achieve this, we appeal to certain estimates from probability theory proven by Ford in [Fo08c]. Recall that

$$\Delta_r = \{\boldsymbol{\xi} = (\xi_1, \ldots, \xi_r) \in \mathbb{R}^r : 0 \leq \xi_1 \leq \cdots \leq \xi_r \leq 1\}.$$

For $r \in \mathbb{N}$, $u > 0$ and $v \geq 1$ set

$$\begin{aligned} Q_r(u,v) &= r!\,\mathrm{Vol}\left(\left\{\boldsymbol{\xi} \in \Delta_r : \xi_i \geq \frac{i-u}{v} \quad (1 \leq i \leq r)\right\}\right) \\ &= \mathbf{Prob}\left(\xi_i \geq \frac{i-u}{v}\ (1 \leq i \leq r) \,\middle|\, \boldsymbol{\xi} \in \Delta_r\right). \end{aligned}$$

Then we have the following estimate, which essentially follows from Theorem 1 in [Fo08c]. This estimate is stated in [Fo07] too without proof. For the sake of completeness, we supply the details of its proof.

Lemma 8.3. *Let $r \in \mathbb{N}$, $u \geq 1$ and $v \geq 1$. If $w = u + v - r \geq 1$, then*

$$Q_r(u,v) \asymp \min\left\{1, \frac{uw}{r}\right\}.$$

Proof. The desired upper bound follows immediately by Theorem 1 in [Fo08c] and the trivial bound $Q_r(u,v) \leq 1$. For the lower bound we distinguish several cases. First, assume that $v \geq 2r$. Then

$$Q_r(u,v) \geq Q_r(1,v) = \frac{1+v-r}{v}\left(1 + \frac{1}{v}\right)^{r-1} \asymp 1 = \min\left\{1, \frac{uw}{r}\right\},$$

by [Fo08c, Lemma 2.1(i)]. Next, consider the case $v \le 2r$. Set

$$u' = \min\left\{u, \frac{r-v+\sqrt{(r-v)^2+4r}}{2}\right\} \ge \frac{1}{2}$$

and

$$w' = u'+v-r = \min\left\{w, \frac{v-r+\sqrt{(r-v)^2+4r}}{2}\right\} \ge \frac{1}{2}$$

By a similar argument with the one leading to (6.3), we have that

$$\min\left\{1, \frac{uw}{r}\right\} = \frac{u'w'}{r}.$$

Fix some constant C. If $u' \ge C$ and $w' \ge C$, then the lower bound follows by Theorem 1 in [Fo08c] applied to $Q_r(u',v) \le Q_r(u,v)$, provided that C is large enough. If $1/2 \le u' \le w'$ and $u' \le C$, then $r \le v \le 2r$ and thus

$$Q_r(u,v) \ge Q_r(1,v) = \frac{1+v-r}{v}\left(1+\frac{1}{v}\right)^{r-1} \asymp_C \frac{u'+v-r}{r} \asymp_C \frac{u'w'}{r},$$

by [Fo08c, Lemma 2.1(i)]. Finally, if $1/2 \le w' \le u'$ and $w' \le C$, then $v \le r$ and thus

$$Q_r(u,v) \ge Q_r(1+r-v,v) \gg \frac{1+r-v}{r} \asymp_C \frac{w'+r-v}{r} \asymp_C \frac{u'w'}{r},$$

by [Fo08b, Lemma 11.1]. In any case, we obtain the desired result. □

For $r, v \in \mathbb{N}$ and $u \ge 0$ set

$$\mathcal{G}_r(u,v) = \{(g_1,\dots,g_v) \in (\mathbb{N}\cup\{0\})^v : g_1+\dots+g_v = r,\ g_1+\dots+g_i \le i+u\ (1 \le i \le v)\}.$$

Then an equivalent formulation of Lemma 8.3 is the following result.

Lemma 8.4. *Let $r \in \mathbb{N}$, $v \in \mathbb{N}$ and $u \ge 0$. If $w = u+v-r \ge 0$, then*

$$\sum_{\boldsymbol{g}\in\mathcal{G}_r(u,v)} \frac{1}{g_1!\cdots g_v!} \asymp \frac{v^r}{r!}\min\left\{1, \frac{(u+1)(w+1)}{r}\right\}.$$

Proof. For every $\boldsymbol{g} \in \mathcal{G}_r(u,v)$, let $R(\boldsymbol{g})$ be the set of $\boldsymbol{\xi} \in \Delta_r$ such that, for any $i \in \{1,\dots,v\}$, exactly g_i of the numbers ξ_j lie in $[(i-1)/v, i/v)$. Then

$$\mathrm{Vol}(R(\boldsymbol{g})) = \frac{1}{v^r}\frac{1}{g_1!\cdots g_v!}. \tag{8.1}$$

Also, we have that $g_1+\dots+g_i \le i+u$ if, and only if, $\xi_{i+\lfloor u+1\rfloor} \ge \frac{i}{v}$. Hence summing (8.1) over $\boldsymbol{g} \in \mathcal{G}_r(u,v)$ and applying Lemma 8.3 completes the proof. □

Lemma 8.5. *Let $\boldsymbol{r} \in \mathcal{R}^*$. Consider integers $1 \le i \le k$, $\nu \ge 0$ and $n \ge 1$ with $\nu+n \le k-i+1$ and $\eta \in (0,1]$. There exists a constant $c_k' > 0$ such that the following hold:*

(a) *If*

$$\left|\frac{F(k-i+1-\nu,n)}{(k-i+2)^{1-\alpha}}-1\right|\ge\eta,$$

$P\le 1+\eta/c_k'$ and $v_i\ge(c_k'/\eta)^2$, then

$$\sum_{\boldsymbol{g}_i\in\mathcal{G}_{r_i}(u_i,v_i)}\frac{T_i(\boldsymbol{g}_i;\nu,n)}{g_{i,1}!\cdots g_{i,v_i}!}\ll_{k,P,\eta}\beta_i\frac{v_i^{r_i}}{r_i!}\max_{j\in\{0,n\}}(\rho_{k-i+1}^{P-1})^{-(n-j)v_i}(t_{i,\nu+j})^{r_i}.$$

(b) *If $P\le 1+1/c_k'$ and $v_i\ge c_k'$, then*

$$\sum_{\boldsymbol{g}_i\in\mathcal{G}_{r_i}(u_i,v_i)}\frac{T_i(\boldsymbol{g}_i;\nu,n)}{g_{i,1}!\cdots g_{i,v_i}!}\ll_{k,P}\beta_i\frac{v_i^{r_i}e^{O_k((P-1)^2v_i)}}{r_i!}\max_{j\in\{0,n\}}(\rho_{k-i+1}^{P-1})^{-(n-j)v_i}(t_{i,\nu+j})^{r_i}.$$

Proof. For now, we treat parts (a) and (b) together. Their proofs will deviate only towards the end. Set

$$\begin{aligned}S&=\sum_{\boldsymbol{g}_i\in\mathcal{G}_{r_i}(u_i,v_i)}\frac{T_i(\boldsymbol{g}_i;\nu,n)}{g_{i,1}!\cdots g_{i,v_i}!}\\&=\sum_{0=s_0\le s_1\le\cdots\le s_n\le s_{n+1}=v_i}\ \sum_{\boldsymbol{g}_i\in\mathcal{G}_{r_i}(u_i,v_i)}\frac{1}{g_{i,1}!\cdots g_{i,v_i}!}\prod_{j=0}^{n}(t_{i,\nu+j})^{G_{i,s_{j+1}}-G_{i,s_j}}.\end{aligned}$$

Fix $0=s_0\le s_1\le\cdots\le s_n\le s_{n+1}=v_i$ and let $m_1,\dots,m_{n+1}$ be non-negative integers with

$$M_j=m_1+\cdots+m_j\le s_j+u_i\quad(1\le j\le n),\quad M_{n+1}=m_1+\cdots+m_{n+1}=r_i.$$

Also, put $M_0=0$. Then we have that

$$\begin{aligned}\sum_{\substack{\boldsymbol{g}_i\in\mathcal{G}_{r_i}(u_i,v_i)\\G_{i,s_j}=M_j\\1\le j\le n}}\frac{1}{g_{i,1}!\cdots g_{i,v_i}!}&=\prod_{j=0}^{n}\ \sum_{\substack{(g_{i,s_j+1},\dots,g_{i,s_{j+1}})\\\in\mathcal{G}_{m_{j+1}}(u_i+s_j-M_j,s_{j+1}-s_j)}}\frac{1}{g_{i,s_j+1}!\cdots g_{i,s_{j+1}}!}\\&\ll\min\left\{\frac{u_i(u_i+s_1-m_1+1)}{m_1+1},\frac{w_i(u_i+s_n-M_n+1)}{m_{n+1}+1}\right\}\prod_{j=0}^{n}\frac{(s_{j+1}-s_j)^{m_{j+1}}}{m_{j+1}!},\end{aligned}$$

by Lemma 8.4 applied for $j=0$ and $j=n$. Also, note that

$$\begin{aligned}\frac{u_i(u_i+s_1-m_1+1)}{m_1+1}\le\frac{u_i(w_i+r_i-m_1+1)}{m_1+1}&\le\frac{u_i(w_i+1)(r_i-m_1+1)}{m_1+1}\\&\ll\beta_i\frac{r_i(r_i-m_1+1)}{m_1+1}\end{aligned}$$

and, similarly,

$$\frac{w_i(u_i+s_n-M_n+1)}{m_{n+1}+1}\le\frac{w_i(u_i+1)(s_n+1)}{m_{n+1}+1}\ll\beta_i\frac{r_i(s_n+1)}{m_{n+1}+1}.$$

So

$$S \ll \beta_i r_i \sum_{0=s_0\le s_1\le\cdots\le s_n\le s_{n+1}=v_i} (\rho_{k-i+1}^{P-1})^{-(s_1+\cdots+s_n)} \tag{8.2}$$
$$\times \sum_{m_1+\cdots+m_{n+1}=r_i} \min\left\{\frac{r_i-m_1+1}{m_1+1}, \frac{s_n+1}{m_{n+1}+1}\right\} \prod_{j=0}^{n} \frac{(t_{i,\nu+j}(s_{j+1}-s_j))^{m_{j+1}}}{m_{j+1}!}$$

The inner sum in the right hand side of (8.2) satisfies the following two upper bounds: it is at most

$$\sum_{m_1=0}^{r_i} \frac{r_i-m_1+1}{m_1+1} \frac{(t_{i,\nu}s_1)^{m_1}}{m_1!} \frac{\left(\sum_{j=1}^{n} t_{i,\nu+j}(s_{j+1}-s_j)\right)^{r_i-m_1}}{(r_i-m_1)!}$$
$$\ll_k \frac{1}{r_i!}\frac{v_i-s_1+1}{s_1+1}\left(\sum_{j=0}^{n} t_{i,\nu+j}(s_{j+1}-s_j)\right)^{r_i}$$

and, also, it is at most

$$(s_n+1)\sum_{m_{n+1}=0}^{r_i} \frac{(t_{i,\nu+n}(s_{n+1}-s_n))^{m_{n+1}}}{(m_{n+1}+1)!} \frac{\left(\sum_{j=0}^{n-1} t_{i,\nu+j}(s_{j+1}-s_j)\right)^{r_i-m_{n+1}}}{(r_i-m_{n+1})!}$$
$$\ll_k \frac{1}{r_i!}\frac{s_n+1}{v_i-s_n+1}\left(\sum_{j=0}^{n} t_{i,\nu+j}(s_{j+1}-s_j)\right)^{r_i}.$$

Consequently,

$$S \ll_k \beta_i \frac{v_i^{r_i+1}}{r_i!} \sum_{0\le s_1\le\cdots\le s_n\le v_i} (\rho_{k-i+1}^{P-1})^{-(s_1+\cdots+s_n)} \min\left\{\frac{v_i-s_1+1}{s_1+1}, \frac{s_n+1}{v_i-s_n+1}\right\}$$
$$\times\left(\sum_{j=1}^{n}(t_{i,\nu+j-1}-t_{i,\nu+j})\frac{s_j}{v_i}+t_{i,\nu+n}\right)^{r_i} \tag{8.3}$$
$$= \beta_i\frac{v_i^{r_i+1}}{r_i!}\sum_{0\le s_1\le\cdots\le s_n\le v_i} g(s_1,s_n)\exp\{G(s_1,\dots,s_n)\},$$

where for $\boldsymbol{x}=(x_1,\dots,x_n)\in[0,+\infty)^n$ we have set

$$G(\boldsymbol{x}) = \log\left((\rho_{k-i+1}^{P-1})^{-(x_1+\cdots+x_n)}\left(\sum_{j=1}^{n}(t_{i,\nu+j-1}-t_{i,\nu+j})\frac{x_j}{v_i}+t_{i,\nu+n}\right)^{r_i}\right)$$

and for $(x,y)\in[0,+\infty)^2$ we have set

$$g(x,y)=\min\left\{\frac{v_i-x+1}{x+1},\frac{y+1}{v_i-y+1}\right\}.$$

We claim that

$$S \ll_{k,P} \beta_i \frac{v_i^{r_i+1}}{r_i!} \sum_{0\le s\le v_i} g(s,s)\exp\{G(s,\dots,s)\}. \tag{8.4}$$

To show (8.4) we will make extensive use of the following simple fact: if $b:[m,m+1]\to\mathbb{R}$ is a differentiable function satisfying $b'(x)\ge\delta>0$ for all $x\in(m,m+1)$, where δ is a fixed positive number, then

$$\frac{e^{b(m+1)}}{e^{b(m)}} \ge e^{\delta}, \tag{8.5}$$

by the Mean Value Theorem. Fix a small positive constant $\eta_0=\eta_0(k)$ to be chosen later and define $J\in\{0,1,\dots,n-1\}$ as follows. If

$$\frac{F(k-i+1-\nu,1)}{(k-i+2)^{1-\alpha}} < 1+\eta_0,$$

then set $J=0$; else, put

$$J=\max\left\{1\le j\le n-1 : \frac{F(k-i+1-\nu,j)}{(k-i+2)^{1-\alpha}} \ge 1+\eta_0\right\}.$$

Observe that

$$t_{i,j}=1+\frac{(k-i+2-j)\log(k-i+2-j)}{k-i+2}(P-1)+O_k((P-1)^2) \quad (0\le j\le k-i+1).$$

Therefore if $1\le j\le J$, then Lemma 8.1(a) yields that

(8.6)

$$\begin{aligned}
&\frac{\partial}{\partial x_j}(G(\underbrace{x_j,\dots,x_j}_{j\text{ times}},x_{j+1},x_{j+2},\dots,x_n))\\
&\quad= -j(P-1)\log(\rho_{k-i+1}) + \frac{r_i(t_{i,\nu}-t_{i,\nu+j})}{(t_{i,\nu}-t_{i,\nu+j})x_j+\sum_{m=j+1}^{n}(t_{i,\nu+m-1}-t_{i,\nu+m})x_m+t_{i,\nu+n}v_i}\\
&\quad= j(P-1)\log(\rho_{k-i+1})\left(-1+\frac{F(k-i+1-\nu,j)}{(k-i+2)^{1-\alpha}}+O_k\left(P-1+v_i^{-1/2}\right)\right)\\
&\quad\ge \frac{\eta_0(P-1)j\log(\rho_{k-i+1})}{2} > 0
\end{aligned}$$

uniformly in $0\le x_j\le\cdots\le x_n\le v_i$, provided that c_k' is large enough. So if $J\ge1$ and we fix $0\le s_2\le\cdots\le s_n$, then we have that

$$\sum_{0\le s_1\le s_2} g(s_1,s_n)\exp\{G(s_1,\dots,s_n)\} \ll_{k,P} g(s_2,s_n)\exp\{G(s_2,s_2,s_3,\dots,s_n)\},$$

by (8.6) with $j=1$, and (8.5). Similarly, if $J\ge2$ and we fix $0\le s_3\le\cdots\le s_n\le v_i$, then

$$\sum_{0\le s_2\le s_3} g(s_2,s_n)\exp\{G(s_2,s_2,s_3,\dots,s_n)\} \ll_{k,P} g(s_3,s_n)\exp\{G(s_3,s_3,s_3,s_4,\dots,s_n)\}.$$

Continuing in the above fashion, we deduce that

$$\begin{aligned}(8.7)\qquad &\sum_{0\le s_1\le\cdots\le s_n\le v_i} g(s_1,s_n)\exp\{G(s_1,\dots,s_n)\}\\ &\ll_{k,P} \sum_{0\le s_{J+1}\le\cdots\le s_n\le v_i} g(s_{J+1},s_n)\exp\{G(\underbrace{s_{J+1},\dots,s_{J+1}}_{J+1\text{ times}},s_{J+2},\dots,s_n)\},\end{aligned}$$

which also holds trivially if $J=0$. If, now, $J=n-1$, then (8.4) follows immediately by (8.7). So assume that $J<n-1$. Then Lemma 8.1(b) implies that

$$F(k-i-\nu-J,1)<F(k-i+1-\nu,J+1)\le 1+\eta_0$$

and hence

$$F(k-i+2-\nu-j,1)\le F(k-i-\nu-J,1)\le 1-\eta_0\quad (J+2\le j\le n),$$

provided that $2\eta_0\le F(k-i+1-\nu,J+1)-F(k-i-\nu-J,1)$. Consequently,

$$\begin{aligned}(8.8)\qquad \frac{\partial G}{\partial x_j}(\boldsymbol{x}) &= (P-1)\log(\rho_{k-i+1})\left(-1+\frac{F(k-i+2-\nu-j,1)}{(k-i+2)^{1-\alpha}}+O_k\left(P-1+v_i^{-1/2}\right)\right)\\ &\le -\frac{\eta_0(P-1)\log(\rho_{k-i+1})}{2}<0\quad (J+2\le j\le n)\end{aligned}$$

uniformly in $0\le x_1\le\cdots\le x_n\le v_i$, provided that c_k' is large enough. Thus, if we fix $s_{J+1}\ge 0$ and $v_i\ge s_n\ge s_{n-1}\ge\cdots\ge s_{J+3}\ge s_{J+1}$, then we find that

$$\sum_{s_{J+1}\le s_{J+2}\le s_{J+3}}\exp\{G(\underbrace{s_{J+1},\dots,s_{J+1}}_{J+1\text{ times}},s_{J+2},\dots,s_n)\}\ll_{k,P}\exp\{G(\underbrace{s_{J+1},\dots,s_{J+1}}_{J+2\text{ times}},s_{J+3},\dots,s_n)\},$$

by (8.8) with $j=J+2$, and (8.5). Similarly, if we fix $s_{J+1}\ge 0$ and $v_i\ge s_n\ge s_{n-1}\ge\cdots\ge s_{J+4}\ge s_{J+1}$, then we have

$$\sum_{s_{J+1}\le s_{J+3}\le s_{J+4}}\exp\{G(\underbrace{s_{J+1},\dots,s_{J+1}}_{J+2\text{ times}},s_{J+3},\dots,s_n)\}\ll_{k,P}\exp\{G(\underbrace{s_{J+1},\dots,s_{J+1}}_{J+3\text{ times}},s_{J+4},\dots,s_n)\}.$$

Continuing in this fashion, we deduce that

$$\begin{aligned}&\sum_{0\le s_{J+1}\le\cdots\le s_n\le v_i} g(s_{J+1},s_n)\exp\{G(\underbrace{s_{J+1},\dots,s_{J+1}}_{J+1\text{ times}},s_{J+2},\dots,s_n)\}\\ &\qquad\ll_{k,P}\sum_{0\le s_{J+1}\le v_i} g(s_{J+1},s_{J+1})\exp\{G(s_{J+1},\dots,s_{J+1})\}\end{aligned}$$

which, together with (8.7) and (8.3), proves (8.4) in this case too. Finally, we use (8.4) to prove parts (a) and (b).

(a) First, assume that

$$\frac{F(k-i+1-\nu,n)}{(k-i+2)^{1-\alpha}}\ge 1+\eta.$$

Note that

$$\frac{\partial}{\partial x}(G(x,\dots,x)) = n(P-1)\log(\rho_{k-i+1})\left(-1+\frac{F(k-i+1-\nu,n)}{(k-i+2)^{1-\alpha}}+O_k(P-1+v_i^{-1/2})\right)$$
$$\geq \frac{\eta(P-1)n\log(\rho_{k-i+1})}{2} > 0$$

uniformly in $0 \leq x \leq v_i$, provided that c'_k is large enough. Hence

$$\sum_{0\leq s\leq v_i} g(s,s)\exp\{G(s,\dots,s)\} \ll_{k,\eta,P} g(v_i,v_i)\exp\{G(v_i,\dots,v_i)\} = \frac{\exp\{G(v_i,\dots,v_i)\}}{v_i+1},$$

by (8.5), which together with (8.4) yields the desired result. Similarly, if

$$\frac{F(k-i+1-\nu,n)}{(k-i+2)^{1-\alpha}} \leq 1-\eta,$$

then we find that

$$\frac{\partial}{\partial x}(G(x,\dots,x)) \leq -\frac{\eta(P-1)n\log(\rho_{k-i+1})}{2} < 0,$$

uniformly in $0 \leq x \leq v_i$, and therefore

$$\sum_{0\leq s\leq v_i} g(s,s)\exp\{G(s,\dots,s)\} \ll_{k,\eta,P} \frac{\exp\{G(0,\dots,0)\}}{v_i}.$$

Inserting this estimate into (8.4) gives us the desired result in this case as well.

(b) By (8.4), we have that

$$S \ll_{k,P} \beta_i \frac{v_i^{r_i+2}}{r_i!}\max_{0\leq s\leq v_i} e^{G(s,\dots,s)} \ll_{k,P} \beta_i \frac{e^{O_k((P-1)^2v_i)}v_i^{r_i}}{r_i!}\max_{0\leq s\leq v_i} e^{G(s,\dots,s)}. \tag{8.9}$$

Since $t_{i,j} = 1+O_k(P-1)$ for all j, we find that, for any $0 \leq s \leq v_i$, we have that

$$\log\left((t_{i,\nu}-t_{i,\nu+n})\frac{s}{v_i}+t_{i,\nu+n}\right) = (t_{i,\nu}-t_{i,\nu+n})\frac{s}{v_i}+t_{i,\nu+n}-1+O_k((P-1)^2)$$
$$= \frac{s}{v_i}\log\left(\frac{t_{i,\nu}}{t_{i,\nu+n}}\right)+\log(t_{i,\nu+n})+O_k((P-1)^2)$$

and, consequently,

$$\max_{0\leq s\leq v_i} G(s,\dots,s) = \max\{r_i\log(t_{i,\nu+n}), -(P-1)nv_i\log(\rho_{k-i+1})+r_i\log(t_{i,\nu})\}$$
$$+O_k((P-1)^2v_i).$$

Inserting the above estimate into (8.9) completes the proof of the lemma. □

The proof of the next lemma uses some ideas from the proof of [Fo08b, Lemmas 4.8 and 11.1] and [K10a, Lemma 3.8].

Lemma 8.6. *Let $\boldsymbol{r} \in \mathcal{R}^*$ and $i \in \{1, \dots, k\}$. There is a constant $c_k'' > 0$ such that*

$$\sum_{\boldsymbol{g}_i \in \mathcal{G}_{r_i}(u_i, v_i)} \frac{T_i(\boldsymbol{g}_i; 0, k-i+1)}{g_{i,1}! \cdots g_{i,v_i}!} \ll_{k,P} \beta_i \frac{v_i^{r_i}}{r_i!} \prod_{j=1}^{i-1} (k-j+2)^{(P-1)(v_j - r_j)},$$

provided that $P \le 1 + 1/c_k''$.

Proof. Since $t_{i,0} > \cdots > t_{i,k-i} > t_{i,k-i+1} = 1$, we have that

$$T_i(\boldsymbol{g}_i; 0, k-i+1) = \sum_{0 \le s_1 \le \cdots \le s_{k-i+1} \le v_i} \prod_{j=1}^{k-i+1} (\rho_{k-i+1}^{P-1})^{-s_j} \left(\frac{t_{i,j-1}}{t_{i,j}} \right)^{G_{i,s_j}}.$$

Also,

$$\prod_{j=1}^{k-i+1} \left(\frac{t_{i,j-1}}{t_{i,j}} \right)^{G_{i,s_j}} \le \left(\frac{t_{i,0}}{t_{i,1}} \right)^{G_{i,s_1}} \prod_{j=2}^{k-i+1} \left(\frac{t_{i,j-1}}{t_{i,j}} \right)^{s_j + u_i} = \left(\frac{t_{i,0}}{t_{i,1}} \right)^{G_{i,s_1}} (t_{i,1})^{s_1 + u_i} \prod_{j=2}^{k-i+1} (t_{i,j-1})^{s_j - s_{j-1}}.$$

Thus, by setting

$$\lambda = \frac{t_{i,0}}{t_{i,1}} = \frac{(\rho_{k-i+1}^{P-1})^{k-i+1}}{t_{i,1}},$$

$m_1 = s_1$ and $m_j = s_j - s_{j-1}$ for $j = 2, \dots, k-i+1$, we deduce that

$$T_i(\boldsymbol{g}_i; 0, k-i+1) \le (t_{i,1})^{u_i} \sum_{\substack{m_1 + \cdots + m_{k-i+1} \le v_i \\ m_j \ge 0 \ (1 \le j \le k-i+1)}} \lambda^{G_{i,m_1} - m_1} \prod_{j=2}^{k-i+1} \left(\frac{t_{i,j-1}}{(\rho_{k-i+1}^{P-1})^{k-i+2-j}} \right)^{m_j}. \tag{8.10}$$

Note that

$$\begin{aligned} \log(t_{i,j-1}) &= (P-1) \frac{(k-i-j+3)\log(k-i-j+3)}{k-i+2} + O_k((P-1)^2) \\ &< (P-1)(k-i-j+2)\log(\rho_{k-i+1}) \quad (2 \le j \le k-i+1), \end{aligned}$$

provided that $P - 1$ is small enough, by Lemma 8.1(a). Combining the above relation with (8.10) and summing the resulting inequality over $\boldsymbol{g}_i \in \mathcal{G}_{r_i}(u_i, v_i)$, we find that

$$\sum_{\boldsymbol{g}_i \in \mathcal{G}_{r_i}(u, v_i)} \frac{T_i(\boldsymbol{g}_i; 0, k-i+1)}{g_{i,1}! \cdots g_{i,v_i}!} \ll_{k,P} (t_{i,1})^{u_i} \sum_{\boldsymbol{g}_i \in \mathcal{G}_{r_i}(u, v_i)} \frac{1}{g_{i,1}! \cdots g_{i,v_i}!} \sum_{m=0}^{v_i} \lambda^{G_{i,m} - m} =: (t_{i,1})^{u_i} T. \tag{8.11}$$

Next, we claim that

$$T \le \frac{v_i^{r_i}}{1 - 1/\lambda} \int_{\mathcal{D}} \left(1 + \sum_{j=1}^{r_i} \lambda^{j - v_i \xi_j} \right) d\boldsymbol{\xi}, \tag{8.12}$$

where

$$\mathcal{D} = \left\{ \boldsymbol{\xi} \in \Delta_{r_i} : \xi_j \ge \frac{j - \lfloor u_i + 1 \rfloor}{v_i} \quad (1 \le j \le r_i) \right\}.$$

To see this, fix $\boldsymbol{g}_i \in \mathcal{G}_{r_i}(u_i, v_i)$ and consider the set $I(\boldsymbol{g}_i)$ of vectors $\boldsymbol{\xi} \in \Delta_{r_i}$ such that

$$|\{1 \le j \le r_i : s - 1 \le v_i \xi_j < s\}| = g_{i,s} \quad (1 \le s \le v_i).$$

Notice that if $\boldsymbol{\xi} \in I(\boldsymbol{g}_i)$, then $v_i\xi_{j+\lfloor u_i+1\rfloor} \ge j$ for all j because $\boldsymbol{g}_i \in \mathcal{G}_{r_i}(u_i, v_i)$, that is to say, $\boldsymbol{\xi} \in \mathcal{D}$. Moreover,

$$\begin{aligned}
\frac{1}{1-1/\lambda}\sum_{j=1}^{r_i}\lambda^{j-v_i\xi_j} &\ge \frac{1}{1-1/\lambda}\sum_{s=1}^{v_i}\lambda^{-s}\sum_{j:\ v_i\xi_j\in[s-1,s)}\lambda^j \\
&\ge \sum_{s=1}^{v_i}\sum_{m=s}^{v_i}\lambda^{-m}\sum_{j:\ v_i\xi_j\in[s-1,s)}\lambda^j \\
&= \sum_{m=1}^{v_i}\lambda^{-m}\sum_{j:\ v_i\xi_j<m}\lambda^j \\
&\ge \sum_{\substack{1\le m\le v_i\\ G_{i,m}>0}}\lambda^{-m+G_{i,m}} \\
&\ge -\frac{1}{1-1/\lambda}+\sum_{m=0}^{v_i}\lambda^{-m+G_{i,m}}.
\end{aligned}$$

Lastly, we have that

$$\mathrm{Vol}(I(\boldsymbol{g}_i)) = \frac{1}{v_i^{r_i}}\frac{1}{g_{i,1}\cdots g_{i,v_i}}.$$

Combining the above remarks, (8.12) follows. To bound the integral in the right hand side of (8.12), we proceed as in the proof of Lemma 4.9 in [Fo08b]. The only difference is that we use Lemma 8.3 from this paper in place of [Fo08b, Lemma 11.1]. This method gives us

$$T \ll_{k,P} \beta_i\frac{v_i^{r_i}}{r_i!}\lambda^{u_i}.$$

By the above estimate, (8.11) and (8.12), we deduce that

$$\sum_{\boldsymbol{g}_i\in\mathcal{G}_{r_i}(u_i,v_i)}\frac{T_i(\boldsymbol{g}_i;0,k-i+1)}{g_{i,1}!\cdots g_{i,v_i}!} \ll_k \beta_i\frac{v_i^{r_i}}{r_i!}(k-i+2)^{(P-1)u_i}.$$

To complete the proof of the lemma, recall that

$$u_i \le 1+\frac{1}{\log(k-i+2)}\sum_{j=1}^{i-1}\log(k-j+2)(v_j-r_j).$$

□

8.3. **Proof of Lemmas 6.3 and 6.4.** In this subsection we establish Lemmas 6.3 and 6.4 thus completing all the steps in the proof of the lower bound implicit in Theorem 1.5.

Proof of Lemma 6.3. Since $g_{1,j} \le G_{1,j} \le j + u_1 \le j+1$ for all $j \in \{1,\dots,v_1\}$, we have that

$$\begin{aligned}\sum_{a_1\in\mathcal{A}_1(\boldsymbol{g}_1)} \frac{1}{a_1} &= \prod_{j=N}^{v_1} \frac{1}{g_{1,j}!} \left(\sum_{p_1\in D_{1,j}} \frac{1}{p_1} \sum_{\substack{p_2\in D_{1,j} \\ p_2\neq p_1}} \frac{1}{p_2} \cdots \sum_{\substack{p_{g_{1,j}}\in D_{1,j} \\ p_{g_{1,j}}\notin\{p_1,\dots,p_{g_{1,j}-1}\}}} \frac{1}{p_{g_{1,j}}} \right) \\ &\ge \frac{1}{g_{1,1}!\cdots g_{1,v_1}!} \prod_{j=N}^{v_1} \left(\log\rho_k - \frac{g_{1,j}}{\lambda_{1,j-1}} \right)^{g_{1,j}} \\ &\ge \prod_{i=1}^{k} \frac{(\log\rho_k)^{r_1}}{g_{1,1}!\cdots g_{1,v_1}!} \prod_{j=N}^{v_1} \left(1 - \frac{j+1}{(\log\rho_k)\exp\left\{\rho_k^{j-L_k-1}\right\}} \right)^{j+1} \\ &\ge \frac{1}{2} \frac{(\log\rho_k)^{r_1}}{g_{1,N}!\cdots g_{1,v_1}!},\end{aligned} \tag{8.13}$$

by Lemma 5.1, provided that N is large. Similarly, if $i \in \{2,\dots,k\}$, then we have that $g_{i,j} \le G_{i,j} \le j + u_i \le j + c\log\log y_{i-1}$ for some $c = c(k)$. Therefore

$$\begin{aligned}\sum_{a_i\in\mathcal{A}_i(\boldsymbol{g}_i)} \frac{1}{a_i} &\ge \frac{(\log(\rho_{k-i+1}))^{r_i}}{g_{i,1}!\cdots g_{i,v_i}!} \prod_{j=1}^{v_i} \left(1 - \frac{j + c\log\log y_{i-1}}{(\log(\rho_{k-i+1}))(\log y_{i-1})\exp\left\{\rho_{k-i+1}^{j-L_k-1}\right\}} \right)^{j+c\log\log y_{i-1}} \\ &\ge \frac{1}{2} \prod_{i=1}^{k} \frac{(\log(\rho_{k-i+1}))^{r_i}}{g_{i,N}!\cdots g_{i,v_i}!},\end{aligned} \tag{8.14}$$

provided that $y_{i-1} \ge y_1 \ge C'_k$ is large enough. Combine (8.13) and (8.14) with Lemmas 5.1 and 8.4 and relation (6.4) to complete the proof. $\square$

Proof of Lemma 6.4. Fix $\boldsymbol{r} \in \mathcal{R}^*$. In view of Lemmas 5.1 and 6.2, it suffices to show that

$$\prod_{i=1}^{k} (\rho_{k-i+1}^{P-1})^{-(k-J_i)v_i} \sum_{\boldsymbol{g}_i\in\mathcal{G}_{r_i}(u_i,v_i)} \frac{T_i(\boldsymbol{g}_i; J_{i-1}-i+1, J_i-J_{i-1})}{g_{i,1}!\cdots g_{i,v_i}!} \ll_k \beta \prod_{i=1}^{k} \frac{v_i^{r_i}}{r_i!} \tag{8.15}$$

for every choice of integers $0 = J_0 \le J_1 \le \cdots \le J_k \le k$ with $J_i \ge i$ for all $i \in \{1,\dots,k\}$. So fix such a $(k+1)$-tuple $(J_0, J_1, \dots, J_k)$ and set

$$T_i = (\rho_{k-i+1}^{P-1})^{-(k-J_i)v_i} \sum_{\boldsymbol{g}_i\in\mathcal{G}_{r_i}(u_i,v_i)} \frac{T_i(\boldsymbol{g}_i; J_{i-1}-i+1, J_i-J_{i-1})}{g_{i,1}!\cdots g_{i,v_i}!} \quad (1 \le i \le k).$$

Also, let

$$I = \min\{1 \le i \le k : J_i = k\}$$

(note that $J_k = k$, so I is well-defined). We claim that

(8.16)
$$T_i \ll_{k,\epsilon} \beta_i \frac{v_i^{r_i}}{r_i!} \times \begin{cases} (k-i+2)^{(P-1)(r_i-v_i)} & \text{if } 1 \le i < I, \\ \max\left\{1, (k-I+2)^{(P-1)(r_I-v_I)}, \prod_{j=1}^{I-1}(k-j+2)^{(P-1)(v_j-r_j)}\right\} & \text{if } i = I, \\ 1 & \text{if } I < i \le k. \end{cases}$$

Note that if inequality (8.16) is indeed true, then

$$\prod_{i=1}^{k} T_i \ll_{k,\epsilon} \left(\prod_{i=1}^{k} \beta_i \frac{v_i^{r_i}}{r_i!}\right) \max_{m\in\{1,I,I+1\}} \prod_{j=1}^{m-1}(k-j+2)^{(P-1)(r_j-v_j)} \ll_k \beta \prod_{i=1}^{k} \frac{v_i^{r_i}}{r_i!},$$

by relations (6.1), (6.2) and (6.4), that is (8.15) holds. So establishing (8.16) will complete the proof of the lemma.

Before embarking on the proof of (8.16), we introduce some notation and prove an intermediate result. For $i \in \{1, \dots, k\}$ define $J_i' \in \{J_{i-1}, J_i\}$ by

$$(\rho_{k-i+1}^{P-1})^{-(J_i-J_i')v_i}(t_{i,J_i'-i+1})^{r_i} = \max_{j\in\{J_{i-1},J_i\}} (\rho_{k-i+1}^{P-1})^{-(J_i-j)v_i}(t_{i,j-i+1})^{r_i}.$$

We claim that if $i \le J_i' \le k-1$, then

$$T_i \ll_{k,\epsilon} \beta_i \frac{v_i^{r_i}}{r_i!}(k-i+2)^{(P-1)(r_i-v_i)}, \tag{8.17}$$

provided that $P-1$ is small enough. Indeed, Lemmas 5.1 and 8.5(b) give us that

$$\begin{aligned} &T_i \cdot (k-i+2)^{(P-1)(v_i-r_i)} \\ &\quad \ll_k \frac{e^{O_k((P-1)^2 v_i)} v_i^{r_i}}{r_i!} \frac{(\rho_{k-i+1}^{P-1})^{-(k-J_i)v_i}(\rho_{k-i+1}^{P-1})^{-(J_i-J_i')v_i}(t_{i,J_i'-i+1})^{r_i}}{(k-i+2)^{(P-1)(r_i-v_i)}} \\ &\quad = \frac{e^{O_k((P-1)^2 v_i)} v_i^{r_i}}{r_i!}(\rho_{k-i+1}^{P-1})^{(J_i'-i+1)v_i}\left(\frac{t_{i,J_i'-i+1}}{(k-i+2)^{P-1}}\right)^{r_i} \end{aligned}$$

So

$$\begin{aligned} &\log\left(\frac{T_i \cdot (k-i+2)^{(P-1)(v_i-r_i)}}{v_i^{r_i}/r_i!}\right) \\ &\quad = (P-1)(J_i'-i+1)\left(\ell_i - \frac{F(k-i+1, J_i'-i+1)}{k-i+2} r_i + O_k((P-1)\ell_i + 1)\right) \\ &\quad = (P-1)(J_i'-i+1)\ell_i\left(1 - \frac{F(k-i+1, J_i'-i+1)}{(k-i+2)^{1-\alpha}} + O_k\left(P-1+\ell_i^{-1/2}\right)\right). \end{aligned} \tag{8.18}$$

For every $i \in \{1, \dots, k-1\}$, condition (1.1) and Lemma 8.2 imply that

$$\alpha \ge 1 + \epsilon - \frac{1}{\log(k-i+2)} \log\left(\frac{(k-i+2)\log(k-i+2) - 2\log 2}{k-i}\right)$$

or, equivalently, that

$$(k-i+2)^{\alpha-1} F(k-i+1, k-i) \ge (k-i+2)^{\epsilon}.$$

So if $i \le J_i' \le k-1$, then

$$(k-i+2)^{\alpha-1}F(k-i+1, J_i'-i+1) \ge (k-i+2)^{\alpha-1}F(k-i+1,k-i) \ge (k-i+2)^{\epsilon}, \tag{8.19}$$

by Lemma 8.1(a). Inserting the above inequality into (8.18) proves (8.17).

We are now in position to show (8.16). First, if $I < i \le k$, then $J_i = J_{i-1} = k$. So

$$T_i(\boldsymbol{g}_i; J_{i-1}-i+1, J_i - J_{i-1}) = T_i(\boldsymbol{g}_i; k-i+1, 0) = 1$$

for every $\boldsymbol{g}_i \in \mathcal{G}_i(r_i)$ and (8.16) follows immediately by Lemma 8.4. Next, let $1 \le i < I$. If $J_i' \ge i$, then (8.16) follows by (8.17), since we also have that $J_i' \le J_i \le J_{I-1} \le k-1$. Assume now that $J_i' = i-1$, in which case $J_{i-1} = i-1$. Then

$$\begin{aligned}\frac{F(k-i+1-(J_{i-1}-i+1), J_i-J_{i-1})}{(k-i+2)^{1-\alpha}} = \frac{F(k-i+1, J_i-i+1)}{(k-i+2)^{1-\alpha}} &\ge \frac{F(k-i+1,k-i)}{(k-i+2)^{1-\alpha}} \\ &\ge (k-i+2)^{\epsilon},\end{aligned}$$

by Lemma 8.1(a) and relation (8.19). The above inequality allows us to apply Lemma 8.5(a) with $\eta = (k-i+2)^{\epsilon} - 1 > 0$. Therefore we deduce that

$$T_i \ll_{k,\epsilon} \beta_i \frac{v_i^{r_i}}{r_i!} (\rho_{k-i+1}^{P-1})^{-(k-J_i)v_i} (\rho_{k-i+1}^{P-1})^{-(J_i-J_i')v_i} (t_{i,J_i'-i+1})^{r_i} = \beta_i \frac{v_i^{r_i}}{r_i!}(k-i+2)^{(P-1)(r_i-v_i)},$$

that is (8.16) holds in this case too. Finally, we bound from above T_I. If $I \le J_I' \le k-1$ or $J_{I-1} = I-1$, then (8.16) follows immediately by relation (8.17) or Lemma 8.6, respectively. So suppose that $J_I' \in \{I-1, k\}$ and $J_{I-1} \ge I$, in which case we must have $J_I' = J_I = k$. We separate two cases. Set

$$\eta_1 = \frac{F(k-I+1,k-I+1) - F(k-I,k-I)}{2(k-I+2)^{1-\alpha}} > 0$$

and assume first that

$$\frac{F(k-I+1, J_I'-I+1)}{(k-I+2)^{1-\alpha}} = \frac{F(k-I+1,k-I+1)}{(k-I+2)^{1-\alpha}} \ge 1 + \eta_1.$$

Inserting the above inequality into (8.18) implies that

$$T_I \ll_k \beta_I \frac{v_I^{r_I}}{r_I!}(k-I+2)^{(P-1)(r_I-v_I)},$$

provided that $P-1$ is small enough, thus proving (8.16) in this case. Finally, assume that

$$\frac{F(k-I+1,k-I+1)}{(k-I+2)^{1-\alpha}} \le 1 + \eta_1.$$

Then

$$\begin{aligned}\frac{F(k-I+1-(J_{I-1}-I+1), J_I-J_{I-1})}{(k-I+2)^{1-\alpha}} = \frac{F(k-J_{I-1}, k-J_{I-1})}{(k-I+2)^{1-\alpha}} &\le \frac{F(k-I,k-I)}{(k-I+2)^{1-\alpha}} \\ &\le 1 - \eta_1\end{aligned}$$

which, together with Lemma 8.5(a), shows that

$$T_I \ll_{k,\epsilon} \beta_I \frac{v_I^{r_I}}{r_I!} (\rho_{k-I+1}^{P-1})^{-(k-J_I)v_I} (\rho_{k-I+1}^{P-1})^{-(J_I-J_I')v_I} (t_{I,J_I'-I+1})^{r_I} = \beta_I \frac{v_I^{r_I}}{r_I!},$$

thus proving (8.16) in this last case too. This completes the proof of the lemma. □

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