ON THE DUFFIN-SCHAEFFER CONJECTURE

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ABSTRACT. Let $\psi : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ be an arbitrary function from the positive integers to the non-negative reals. Consider the set $\mathcal{A}$ of real numbers α for which there are infinitely many reduced fractions a/q such that $|\alpha - a/q| \leqslant \psi(q)/q$. If $\sum_{q=1}^{\infty} \psi(q)\varphi(q)/q = \infty$, we show that $\mathcal{A}$ has full Lebesgue measure. This answers a question of Duffin and Schaeffer. As a corollary, we also establish a conjecture due to Catlin regarding non-reduced solutions to the inequality $|\alpha - a/q| \leqslant \psi(q)/q$, giving a refinement of Khinchin's Theorem.

1. INTRODUCTION

Let $\psi : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ be an arbitrary function from the positive integers to the non-negative reals. Given $\alpha \in \mathbb{R}$, we wish to understand when we can find infinitely many integers a and q such that

$$\left|\alpha - \frac{a}{q}\right| \leqslant \frac{\psi(q)}{q}. \tag{1.1}$$

Clearly, it suffices to restrict our attention to numbers $\alpha \in [0, 1]$.

When $\psi(q) = 1/q$ for all q, Dirichlet's approximation theorem implies that, given *any* $\alpha \in [0, 1]$, there are infinitely many coprime integers a and q satisfying (1.1). On the other hand, the situation can become significantly more complicated if ψ behaves more irregularly. Even small irregularities can cause (1.1) to have no solutions for certain numbers α. However, there are several results in the literature that show that, under rather general conditions on ψ, (1.1) has infinitely many solutions for *almost all* $\alpha \in [0, 1]$, in the sense that the residual set has null Lebesgue measure.

The prototypical such 'metric' result was proven by Khinchin in 1924 [12] (see also [13, Theorem 32]). To state his result, we let λ denote the Lebesgue measure on $\mathbb{R}$.

Khinchin's theorem. *Consider a function $\psi : \mathbb{N} \to [0, +\infty)$ such that the sequence $(q\psi(q))_{q=1}^{\infty}$ is decreasing, and let $\mathcal{K}$ denote the set of real numbers $\alpha \in [0, 1]$ for which (1.1) has infinitely many solutions $(a, q) \in \mathbb{Z}^2$ with $0 \leqslant a \leqslant q$.*

(a) If $\sum_{q\geqslant 1} \psi(q) < \infty$, then $\lambda(\mathcal{K}) = 0$.
(b) If $\sum_{q\geqslant 1} \psi(q) = \infty$, then $\lambda(\mathcal{K}) = 1$.

There is an intuitive way to explain why Khinchin's result ought to be true. Consider the sets

$$\mathcal{K}_q = [0, 1] \cap \bigcup_{a=0}^{q} \Big[\frac{a - \psi(q)}{q}, \frac{a + \psi(q)}{q}\Big], \tag{1.2}$$

Date: July 10, 2019.
2010 *Mathematics Subject Classification.* Primary: 11J83. Secondary: 05C40.
Key words and phrases. Diophantine approximation, Metric Number Theory, Duffin-Schaeffer conjecture, graph theory, density increment, compression arguments.

so that[1]

$$\mathcal{K} = \limsup_{q\to\infty} \mathcal{K}_q.$$

In addition,

$$\min\{\psi(q), 1/2\} \leqslant \lambda(\mathcal{K}_q) \leqslant 2\min\{\psi(q), 1/2\}.$$

Thus, part (a) of Khinchin's theorem is an immediate corollary of the 'easy' direction of the Borel-Cantelli lemma from Probability Theory [10, Lemma 1.2] applied to the probability space $[0,1]$ equipped with the measure λ. If we knew, in addition, that the sets $\mathcal{K}_q$ were mutually independent, then we could apply the 'hard' direction of the Borel-Cantelli lemma [10, Lemma 1.3] to deduce part (b) of Khinchin's theorem. Of course, the sets $\mathcal{K}_q$ are not mutually independent, so the difficult part in Khinchin's proof is to show that there is enough 'approximately independence', so that $\mathcal{K}$ still has full measure.

In 1941, Duffin and Schaeffer [7] undertook a study of the limitations to the validity of Khinchin's theorem, since the condition that $q\psi(q)$ is decreasing is not a necessary condition. They discovered that it is more natural to focus on reduced solutions a/q to (1.1) that avoid overcounting issues arising when working with arbitrary fractions a/q. To this end, let

$$\mathcal{A}_q := [0,1] \cap \bigcup_{\substack{1\leqslant a\leqslant q\\ \gcd(a,q)=1}} \Big[\frac{a-\psi(q)}{q}, \frac{a+\psi(q)}{q}\Big]. \tag{1.3}$$

and

$$\mathcal{A} := \limsup_{q\to\infty} \mathcal{A}_q. \tag{1.4}$$

Just like before, using the 'easy' direction of the Borel-Cantelli lemma, we immediately find that

$$\sum_{q=1}^{\infty} \frac{\varphi(q)\psi(q)}{q} < \infty \qquad \Longrightarrow \qquad \lambda(\mathcal{A}) = 0. \tag{1.5}$$

In analogy to Khinchin's result, Duffin and Schaeffer conjectured that we also have the implication

$$\sum_{q=1}^{\infty} \frac{\varphi(q)\psi(q)}{q} = \infty \qquad \Longrightarrow \qquad \lambda(\mathcal{A}) = 1. \tag{1.6}$$

The main result of the present paper is a proof of the Duffin-Schaeffer conjecture:

Theorem 1. *Let $\psi:\mathbb{N}\to\mathbb{R}_{\geqslant 0}$ be a function such that*

$$\sum_{q=1}^{\infty} \frac{\psi(q)\varphi(q)}{q} = \infty.$$

Let $\mathcal{A}$ be the set of $\alpha\in[0,1]$ for which the inequality

$$\Big|\alpha - \frac{a}{q}\Big| \leqslant \frac{\psi(q)}{q} \tag{1.7}$$

has infinitely many coprime solutions a and q. Then $\mathcal{A}$ has Lebesgue measure 1.

As a direct corollary, we obtain Catlin's conjecture [6] that deals with solutions to (1.7) where the approximations are not necessarily reduced fractions, giving an extension of Khinchin's Theorem.

[1] Recall that if $X_1, X_2, \dots$ is a sequence of sets of real numbers, then $\limsup_{n\to\infty} X_n$ denotes the set of real numbers lying in infinitely many X_n's.

Theorem 2. *Let $\psi : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ and let $\mathcal{K}$ denote the set of $\alpha \in [0,1]$ for which the inequality (1.7) has infinitely many solutions $(a,q) \in \mathbb{Z}$ with $0 \leqslant a \leqslant q$. Then the following hold:*

(a) If $\sum_{q=1}^{\infty} \varphi(q) \sup_{n\in\mathbb{N}:\, q|n}(\psi(n)/n) < \infty$, then $\lambda(\mathcal{K}) = 0$.

(b) If $\sum_{q=1}^{\infty} \varphi(q) \sup_{n\in\mathbb{N}:\, q|n}(\psi(n)/n) = \infty$, then $\lambda(\mathcal{K}) = 1$.

There has been much partial progress on the Duffin-Schaeffer conjecture in previous work. The assumption that the sequence $(q\psi(q))_{q=1}^{\infty}$ is decreasing implies that $(\psi(q)/q)_{q=1}^{\infty}$ is also decreasing. In particular, if a/q is a fraction satisfying (1.1), then so is its reduction a_1/q_1. Thus, as observed by Walfisz [17] (in work predating Duffin and Schaeffer's conjecture), Khinchin's Theorem implies the Duffin-Schaeffer conjecture when $q\psi(q)$ is decreasing. In the same paper, he strengthened part (b) of Khinchin's theorem as follows: if $\sum_{q\geqslant 1} \psi(q) = \infty$ and $\psi(q) \ll \psi(2q)$ for all $q \in \mathbb{N}$, then the set of $\alpha \in [0,1]$ for which (1.1) has infinitely many coprime solutions a and q has Lebesgue measure 1.

Duffin and Schaeffer [7] had already established their conjecture (1.6) when ψ is sufficiently 'regular', in the sense that the function $\varphi(q)/q$ behaves like the constant function 1 when weighted with ψ. More precisely, they proved (1.6) under the assumption that

$$\limsup_{Q\to\infty} \frac{\sum_{q\leqslant Q} \psi(q)\varphi(q)/q}{\sum_{q\leqslant Q} \psi(q)} > 0.$$

Since then, a variety of results towards the Duffin-Schaeffer conjecture have been proven. The first significant step was achieved by Erdős [8] and then improved by Vaaler [16], who demonstrated (1.6) when $\psi(q) = O(1/q)$. In addition, Pollington and Vaughan [14] proved that the d-dimensional analogue of the Duffin-Schaeffer conjecture holds for any $d \geqslant 2$.

The proof of all three aforementioned results can be found in Harman's book [10] (see Theorems 2.5, 2.6 and 3.6, respectively), along with various other cases of the Duffin-Schaeffer conjecture (see Theorems 2.9, 2.10, 3.7 and 3.8).

More recently, the focus shifted towards establishing variations of (1.6), where the assumption that the series $\sum_{q\geqslant 1} \psi(q)\varphi(q)/q$ diverges is replaced by a slightly stronger assumption. The first result of this kind was proven in 2006 by Haynes, Pollington and Velani [11], and was improved in 2013 by Beresnevich, Harman, Haynes and Velani [4]. The strongest such result is the recent result of Aistleitner, Lachmann, Munsch, Technau and Zafeiropoulos [2] who showed that

$$\sum_{q=1}^{\infty} \frac{\varphi(q)\psi(q)}{q(\log q)^{\varepsilon}} = \infty \qquad \Longrightarrow \qquad \lambda(\mathcal{A}) = 1,$$

for any fixed $\varepsilon > 0$. In 2014, Aistleitner [1] established a sort of a 'companion result' to the above one. He showed that if $\sum_{q=1}^{\infty} \psi(q)\varphi(q)/q$ diverges in such a way that

$$\sum_{2^{2^j} < q \leqslant 2^{2^{j+1}}} \frac{\psi(q)\varphi(q)}{q} = O(1/j)$$

for all $j \geqslant 1$, then $\lambda(\mathcal{A}) = 1$.

Finally, Beresnevich and Velani [5] have proven that the Duffin-Schaeffer conjecture implies a Hausdorff measure version of itself. An immediate corollary of their results when combined with Theorem 1 is the following.

Corollary 3. *Let $\psi : \mathbb{N} \to [0, 1/2]$. Let $\mathcal{A}$ be the set of $\alpha \in [0, 1]$ such that (1.7) has infinitely many coprime solutions a and q. Let*

$$s = \inf\Big\{\beta \in \mathbb{R}_{\geqslant 0} : \sum_{q=1}^{\infty} \varphi(q)(\psi(q)/q)^{\beta} < \infty\Big\}.$$

Then the Hausdorff dimension $\dim_{\mathcal{H}}(\mathcal{A})$ of $\mathcal{A}$ satisfies

$$\dim_{\mathcal{H}}(\mathcal{A}) = \min(s, 1).$$

The proof of Theorem 2, assuming Theorem 1, is explained in Section 2. For an outline of the proof of Theorem 1, we refer the readers to Section 3. Finally, the structure of the rest of the paper is presented in Section 4.

Notation. The letter μ will always denote a generic measure on $\mathbb{N}$. We reserve the letter λ for the Lebesgue measure on $\mathbb{R}$.

Sets will be typically denoted by capital calligraphic letters such as $\mathcal{A}, \mathcal{V}$ and $\mathcal{E}$. A triple $G = (\mathcal{V}, \mathcal{W}, \mathcal{E})$ denotes a bipartite graph with vertex sets $\mathcal{V}$ and $\mathcal{W}$ and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{W}$.

Given a set or an event $\mathcal{E}$, we let $\mathbb{1}_{\mathcal{E}}$ denote its indicator function.

The letter p will always denote a prime number. We also write $p^k \| n$ to mean that p^k is the exact power of p dividing the integer n.

When we write (a, b), we mean the pair of a and b. In contrast, we write $\gcd(a, b)$ for the greatest common divisor of the integers a and b, whereas $\mathrm{lcm}[a, b]$ denotes their least common multiple.

Finally, we adopt the usual asymptotic notation of Vinogradov: given two functions $f, g : X \to \mathbb{R}$ and a set $Y \subseteq X$, we write "$f(x) \ll g(x)$ for all $x \in Y$" if there is a constant $c = c(f, g, Y) > 0$ such that $|f(x)| \leqslant cg(x)$ for all $x \in Y$. The constant is absolute unless otherwise noted by the presence of a subscript. If $h : X \to \mathbb{R}$ is a third function, we use Landau's notation $f = g + O(h)$ to mean that $|f - g| \ll h$.

We introduce several new quantities and associated notation in Section 6 which are tailored to our application. In the interests of concreteness we have decided to use explicit constants in several parts of the argument, but we encourage the reader not to concern themselves with numerics on a first reading.

Acknowledgements. First and foremost, we would like to thank Sam Chow, Leo Goldmakher and Andrew Pollington for their valuable insights to this project: we have had extended discussions with them on various aspects of the Duffin-Schaeffer conjecture and are indebted to them for their contributions. In addition, we would like to thank Sam Chow for pointing out the connection of our paper to Catlin's conjecture, and Sanju Velani for introducing J.M. to this problem. Finally, we are grateful to Christopher Aistleitner and Sam Chow for sending us various comments and corrections on an earlier version of our paper.

Our project began in the Spring of 2017 during our visit to the Mathematical Sciences Research Institute in Berkeley, California (supported by the National Science Foundation under Grant No. DMS-1440140). In addition, a significant part of our work took place during two visits of J.M. to the Centre de recherche mathématiques in Montréal in November 2017 and May 2018, and during the visit of D.K. to the University of Oxford in the Spring of 2019 (supported by Ben Green's Simons Investigator Grant 376201). We would like to thank our hosts for their support and hospitality.

D.K. was also supported by Natural Sciences and Engineering Research Council of Canada (Discovery Grant 2018-05699) and by the Fonds de recherche du Québec - Nature et technologies (projet de recherche en équipe - 256442). J.M. was also supported by a Clay Research Fellowship during the first half of this project.

2. DEDUCTION OF THEOREM 2 FROM THEOREM 1

Most of the details of this deduction can be found in Catlin's original paper [6]. We give them here as well for the sake of completeness. For easy reference, let

$$S = \sum_{q=1}^{\infty} \varphi(q) \sup_{\substack{n\in\mathbb{N}\\ q|n}} \frac{\psi(n)}{n}.$$

Firstly, we deal with a rather trivial case.

Case 1: There is a sequence of integers $q_1 < q_2 < \cdots$ such that $\psi(q_i) \geqslant 1/2$ for all i.

By passing to a subsequence if necessary, we may assume that $q_{i+1} \geqslant (2q_i)^2$ for all i. Recall the definition of the set $\mathcal{K}_q$ from (1.2). Since $\psi(q_i) \geqslant 1/2$, we infer that $K_{q_i} = [0,1]$ for each i. As a consequence, $\mathcal{K} = [0,1]$. We claim that we also have $S = \infty$. Indeed, for each $d|q_i$, we have

$$\sup_{\substack{n\in\mathbb{N}\\ d|n}} \frac{\psi(n)}{n} \geqslant \frac{\psi(q_i)}{q_i} \geqslant \frac{1}{2q_i}.$$

Consequently,

$$\sum_{q_{i-1}<q\leqslant q_i} \varphi(q) \sup_{\substack{n\in\mathbb{N}\\ q|n}} \frac{\psi(n)}{n} \geqslant \sum_{\substack{q_{i-1}<q\leqslant q_i\\ q|q_i}} \frac{\varphi(q)}{2q_i} \geqslant \frac{1}{2q_i}\sum_{q|q_i}\varphi(q) - \frac{1}{2q_i}\sum_{q\leqslant q_{i-1}}\varphi(q) \geqslant \frac{1}{4}, \tag{2.1}$$

since $\sum_{q|q_i}\varphi(q) = q_i$ and $\sum_{q|q_i,\ d\leqslant q_{i-1}}\varphi(q) \leqslant q_{i-1}^2 \leqslant q_i/2$. Summing (2.1) over all $i \geqslant 2$ proves our claim that $S = \infty$.

Hence, if we are in Case 1, we see that $S = \infty$ and $\mathcal{K} = [0,1]$, so that Theorem 2 holds.

Case 2: There are finitely many $q \in \mathbb{N}$ with $\psi(q) \geqslant 1/2$.

Note that in this case replacing ψ by $\min\{\psi, 1/2\}$ does not affect neither the convergence of S, nor which numbers lie in the set $\mathcal{K} = \limsup_{q\to\infty}\mathcal{K}_q$. Hence, we may assume without loss of generality that $\psi \leqslant 1/2$. In particular, we have that $\lim_{n\to\infty}\psi(n)/n = 0$, so that we may replace $\sup$ by $\max$ in the definition of S. We now follow an argument due to Catlin.

Consider the function ξ defined by

$$\frac{\xi(q)}{q} = \max_{\substack{n\in\mathbb{N}\\ q|n}} \frac{\psi(n)}{n}$$

and the sets

$$\mathcal{C}_q = [0,1] \cap \bigcup_{\substack{1\leqslant a\leqslant q\\ \gcd(a,q)=1}} \Big[\frac{a-\xi(q)}{q}, \frac{a+\xi(q)}{q}\Big] \quad \text{and} \quad \mathcal{C} := \limsup_{q\to\infty}\mathcal{C}_q$$

These are the analogues of the sets $\mathcal{A}_q$ and $\mathcal{A}$ that appear in Theorem 1, but with ξ in place of ψ. We claim that

$$\mathcal{C}\setminus\mathbb{Q} = \mathcal{K}\setminus\mathbb{Q}. \tag{2.2}$$

This will immediately complete the proof of Theorem 2(b) by applying Theorem 1. In addition, Theorem 2(a) will follow from (1.5).

Indeed, if $\alpha \in \mathcal{C} \setminus \mathbb{Q}$, then there are infinitely many reduced fractions a_j/q_j such $|\alpha - a_j/q_j| \leqslant \xi(q_j)/q_j$. By the definition of ξ, there is some n_j that is a multiple of q_j such that $\xi(q_j)/q_j = \psi(n_j)/n_j$. If we let $m_j = a_j n_j/q_j$, then $|\alpha - m_j/n_j| \leqslant \psi(n_j)/n_j$ for all j, whence $\alpha \in \mathcal{K}$.

Conversely, let $\alpha \in \mathcal{K} \setminus \mathbb{Q}$. Then there are infinitely many pairs $(m_j, n_j) \in \mathbb{N}^2$ such that $|\alpha - m_j/n_j| \leqslant \psi(n_j)/n_j$. If we let a_j/q_j be the fraction m_j/n_j in reduced form, we also have that $|\alpha - a_j/q_j| \leqslant \psi(n_j)/n_j \leqslant \xi(q_j)/q_j$, where the last inequality follows by noticing that $q_j | n_j$. This shows that $\alpha \in \mathcal{C}$, as long as we can show that infinitely many of the fractions a_j/q_j are distinct. But if this were not the case, there would exist a fraction a/q such that $a_j/q_j = a/q$ for infinitely many j, so that $|\alpha - a/q| \leqslant \psi(n_j)/n_j \leqslant 1/(2n_j)$ for all such j. Letting $j \to \infty$, we find that $\alpha = a/q \in \mathbb{Q}$, a contradiction.

This completes the proof of (2.2), and hence of Theorem 2 in all cases.

3. Outline of the proof of Theorem 1

The purpose of this section is to explain in rough terms the main ideas that go into the proof of our main result. To simplify various technicalities, let us consider the special case where the function ψ satisfies the following conditions:

(a) $\psi(q) = 0$ or $\psi(q) = q^{-c}$ for every $q \in \mathbb{N}$;
(b) ψ is non-zero only on square-free integers q;
(c) There exists an infinite sequence $2 < x_1 < x_2 < \dots$ such that:
  (i) $x_j > x_{j-1}^2$;
  (ii) ψ is supported on $\cup_{i=1}^{\infty}[x_i, 2x_i]$;
  (iii) For each i we have
$$\sum_{q \in [x_i, 2x_i]} \frac{\varphi(q)}{q}\psi(q) \in [1, 2] .$$

In this set-up, it follows from a well-known second moment argument that to establish the Duffin-Schaeffer conjecture it is sufficient to show that for any $x \in \{x_1, x_2, \dots\}$ we have
$$\sum_{q, r \in \mathcal{S}} \frac{\varphi(q)}{q} \cdot \frac{\varphi(r)}{r} \cdot P(q, r) \ll x^{2c},$$
where
$$\mathcal{S} := \{q \in \mathbb{Z} \cap [x, 2x] : \psi(q) \neq 0\},$$
$$P(q, r) := \prod_{\substack{p | qr/\gcd(q,r)^2 \\ p > M(q,r)/\gcd(q,r)}} \Big(1 + \frac{1}{p}\Big),$$
$$M(q, r) := \max\{q\psi(r), r\psi(q)\} \asymp x^{1-c}.$$

Note that we have the estimate
$$\sum_{q \in \mathcal{S}} \frac{\varphi(q)}{q} \asymp x^c \sum_{q \in \mathcal{S}} \frac{\varphi(q)}{q}\psi(q) \asymp x^c,$$
so the key to the proof is to show that $P(q, r) \ll 1$ on average over $q, r \in \mathcal{S}$. This would then show suitable 'approximate independence' of the sets $\mathcal{A}_q$ mentioned in the introduction. The size

of $P(q,r)$ is controlled by small primes dividing exactly one of q,r. With this in mind, let us consider separately the contribution from q,r with

$$\sum_{\substack{p|qr/\gcd(q,r)^2\\ p\geqslant t}}\frac{1}{p}\approx 1 \tag{3.1}$$

for different thresholds t (which we think of as small compared with x). A calculation then shows that it is sufficient to show that for each t

$$\sum_{\substack{q,r\in\mathcal{S}\\ \gcd(q,r)\geqslant x^{1-c}/t}}\frac{\varphi(q)}{q}\cdot\frac{\varphi(r)}{r}\ll\frac{x^{2c}}{t}. \tag{3.2}$$

In particular, we need to understand the structure of a set $\mathcal{S}$ where many of the pairs $(q,r)\in\mathcal{S}^2$ have a large common factor. Given $q\in\mathcal{S}$, there are $x^{o(1)}$ divisors of q that are at least x^{1-c}/t. In turn, given such a divisor d, there are $O(x^c t)$ integers $r\in[x,2x]$ which are a multiple of d (forgetting the constraint $r\in\mathcal{S}$). This gives a bound $tx^{2c+o(1)}$ for the sum in (3.2), and so the key problem is to win back a little bit more than the $x^{o(1)}$ factor from the divisor bound. We wish to do this by gaining a structural understanding of sets $\mathcal{S}$ where many pairs have a large GCD. One way that many pairs in $\mathcal{S}$ can have a large GCD is if a positive proportion of elements of $\mathcal{S}$ are a multiple of some fixed divisor d. It is natural to ask if this is the only such construction. If we ignore the $\varphi(q)/q$ weights, this leads to the following model question.

Question. *Let $\mathcal{S}\subseteq[x,2x]$ satisfy $\#\mathcal{S}\asymp x^c$ and be such that there are $\#\mathcal{S}^2/100$ pairs $(a_1,a_2)\in\mathcal{S}^2$ with $\gcd(a_1,a_2)>x^{1-c}$. Must it be the case that there is an integer $d\gg x^{1-c}$ which divides $\gg\#\mathcal{S}$ elements of $\mathcal{S}$?*

To attack this problem, we use a 'compression' argument. We will repeatedly pass to subsets of $\mathcal{S}$ where we have increasing control over whether given primes occur in the GCDs or not, whilst at the same time showing that the size of the original set is controlled in terms of the size of the new set. At the end of the iteration procedure we will then have arrived at a subset which controls the size of $\mathcal{S}$, and where we know that all large GCDs are caused by a fixed divisor.

Since the final set then has a very simple GCD structure, we will have enough information to establish (3.2).

To enable the iterations, we pass to a bipartite setup. We will start out with $\mathcal{V}_0=\mathcal{W}_0=\mathcal{S}$ and repeatedly pass to subsets. Given two sets $\mathcal{V},\mathcal{W}\subseteq\mathcal{S}$ and a prime p, we wish to pass to subsets $\mathcal{V}'\subseteq\mathcal{V}$ and $\mathcal{W}'\subseteq\mathcal{W}$ where either $\mathcal{V}'$ is all elements of $\mathcal{V}$ that are divisible by p, or $\mathcal{V}'$ is all elements of $\mathcal{V}$ coprime to p (and similarly with $\mathcal{W}'$). Since we're assuming that $\mathcal{S}$ contains only square-free integers, we then will completely know the p-divisibility of all elements of $\mathcal{V}'$ and $\mathcal{W}'$, so in particular all GCDs between an element of $\mathcal{V}'$ and $\mathcal{W}'$ will either be multiple of p, or all will be coprime to p. After repeating this procedure for each prime occurring in any large GCD between an element of $\mathcal{V}$ and $\mathcal{W}$, we end up with sets $\mathcal{V}''\subseteq\mathcal{S}$ and $\mathcal{W}''\subseteq\mathcal{S}$ and integers a,b such that all elements of $\mathcal{V}''$ are a multiple of a, all elements of $\mathcal{W}''$ are a multiple of b, and all large GCDs between an element of $\mathcal{V}''$ and $\mathcal{W}''$ are exactly equal to $\gcd(a,b)$.

We choose whether to pass to all elements of $\mathcal{V}$ which are a multiple of p or all which are coprime to p (and similarly for $\mathcal{W}$) in such a way that we increase the amount of structure at each stage. This will enable us to control a quantity like the left hand side of (3.2) in terms of a related quantity for $\mathcal{V}'$ and $\mathcal{W}'$. An initially appealing choice to measure the 'structure' might be

$$\delta(\mathcal{V},\mathcal{W})=\frac{\#\{(v,w)\in\mathcal{V}\times\mathcal{W}:\ \gcd(v,w)>x^{1-c}/t\}}{\#\mathcal{V}\cdot\#\mathcal{W}},$$

namely the density of pairs (v,w) with large GCD. Iteratively increasing this quantity would try to mimic a 'density increment' strategy such as that used in the proof of Roth's Theorem on arithmetic progressions. Unfortunately, such an argument loses all control over the size of the vertex sets, and so we lose control over the sum in (3.2).

An alternative suggestion might be to consider a different quantity which focuses on the size of the vertex sets. If all elements of $\mathcal{V}$ are a multiple of a, all elements of $\mathcal{W}$ are a multiple of b, and all edges come from pairs (v,w) with $\gcd(v,w)=\gcd(a,b)$, then $\gcd(a,b)>x^{1-c}/t$. Thus

$$\#\mathcal{V}\cdot\#\mathcal{W}\leqslant\#\{(v,w)\in(\mathbb{Z}\cap[x,2x])^2: a|v,\ b|w\}\ll\frac{x^2}{ab}\leqslant t^2x^{2c}\cdot\frac{\gcd(a,b)^2}{ab}, \tag{3.3}$$

where we used that $\mathcal{V}$ and $\mathcal{W}$ are subsets of $[x,2x]$ in (3.3). Thus, one might try to iteratively increase the quantity

$$\#\mathcal{V}\cdot\#\mathcal{W}\cdot\frac{ab}{\gcd(a,b)^2}.$$

(Here a is the fixed factor of all elements of $\mathcal{V}$ and b the fixed factor of elements of $\mathcal{W}$ which come from when we restrict to all elements being a multiple of p.) This would adequately control (3.2), but unfortunately it is not possible to guarantee that this quantity increases at each stage, and so this proposal also fails.

However, the variant

$$\delta(\mathcal{V},\mathcal{W})^{10}\cdot\#\mathcal{V}\cdot\#\mathcal{W}\cdot\frac{ab}{\gcd(a,b)^2} \tag{3.4}$$

turns out to (more-or-less) work well. Indeed, if the quantity (3.4) increases at each iteration, and at the final iteration all elements of $\mathcal{V}''$ are a multiple of a, all elements of $\mathcal{W}''$ are a multiple of b, and all edges come from pairs (v,w) with $\gcd(v,w)=\gcd(a,b)>x^{1-c}/t$, then we find that

$$\delta(\mathcal{S},\mathcal{S})^{10}\#\mathcal{S}^2\leqslant\delta(\mathcal{V}'',\mathcal{W}'')^{10}\#\mathcal{V}''\#\mathcal{W}''\frac{ab}{\gcd(a,b)^2}\leqslant\#\mathcal{V}''\mathcal{W}''\frac{ab}{\gcd(a,b)^2}\ll t^2x^{2c}. \tag{3.5}$$

We note that in our setup $\#\mathcal{S}\asymp x^c$, and that

$$\sum_{\substack{q,r\in\mathcal{S}\\ \gcd(q,r)>x^{1-c}/t}}1=\delta(\mathcal{S},\mathcal{S})\#\mathcal{S}^2. \tag{3.6}$$

If it so happens that $\delta(\mathcal{S},\mathcal{S})\leqslant 1/t$, then we trivially obtain (3.2) (ignoring the $\varphi(q)/q$ weighting) from (3.6). On the other hand, if $\delta(\mathcal{S},\mathcal{S})\gg 1/t$, then (3.5) falls short of (3.2) only by a factor t^{12}.

Finally, to win the additional factor of t^{12} we make use of the fact that any edge (q,r) in our graph satisfies (3.1). The crucial estimate is that

$$\#\Big\{n<x:\sum_{\substack{p|n\\ p\geqslant t}}\frac{1}{p}\geqslant 1\Big\}\ll e^{-t}x. \tag{3.7}$$

This was the crucial idea in the earlier work of Erdős [8] and Vaaler [16] on the Duffin-Schaeffer conjecture. In our situation, our iteration procedure has essentially reduced the proof to a similar situation to their work.

Indeed, in (3.3), we may restrict our attention to pairs (v, w) such that $a|v$, $b|w$ and

$$\sum_{\substack{p|vw/\gcd(v,w)^2 \\ p\geqslant t}} \frac{1}{p} \approx 1. \tag{3.8}$$

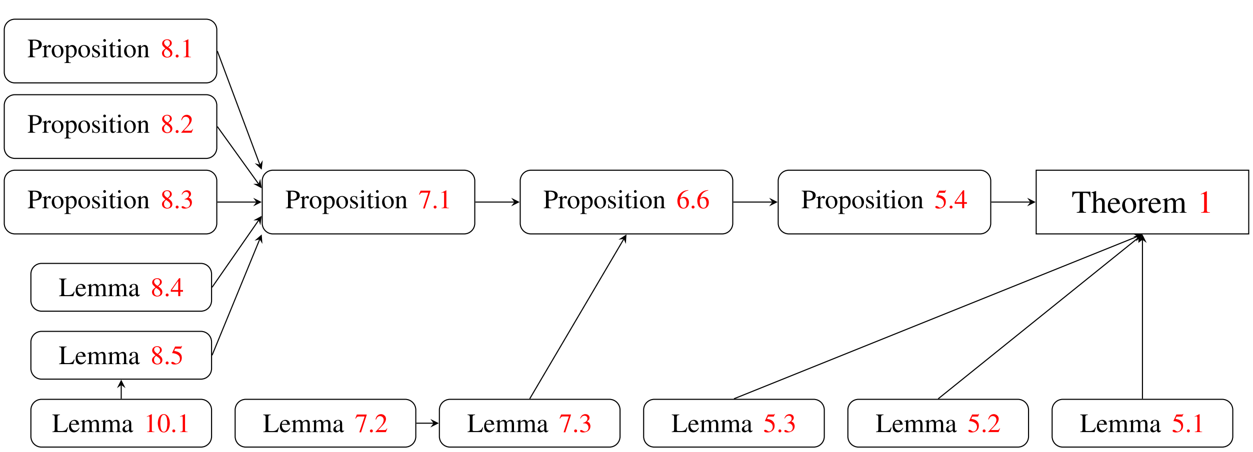
Unless most of the contribution to the above sum of comes from primes in a and b, we can apply (3.7) to win a factor of size $e^{-t} = o(t^{-12})$ in (3.3). Finally, if the small primes in a and b do cause a problem, then a more careful analysis of our iteration procedure shows that we actually are able to increase the quantity (3.4) by more than t^{12}, which also suffices for establishing (3.2) in this case.

This description has ignored several important technicalities; it turns out that the $\varphi(q)/q$ weights are vital for our argument to work (see the discussion in Section 15). In addition, we do not quite work with (3.4) but with a closely related (but more complicated) expression to enable this quantity to increase at each iteration. The iteration procedure of our argument is broken up into different stages. In between two of the principal iterative stages, we perform a certain 'clean-up' step at which we allow a small loss in the quantity (3.4). This step is essential in order to keep track track of the condition (3.8) (which could otherwise become meaningless after too many iterations).

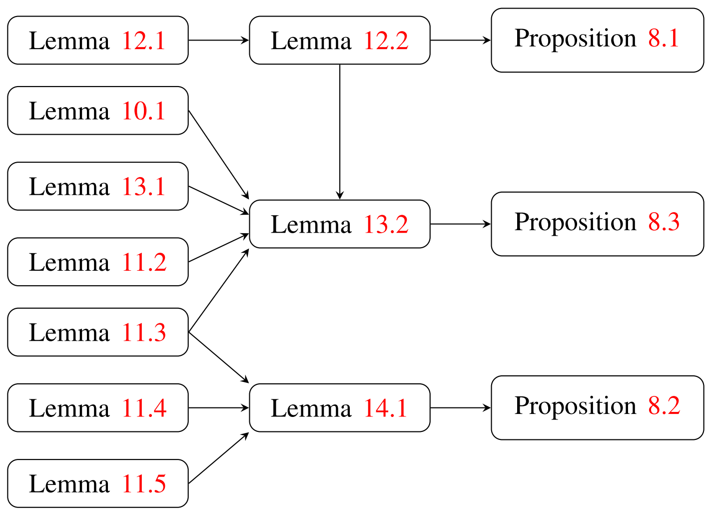
4. Structure of the paper

In the first half of the paper that consists of Sections 5-10, we reduce the proof of Theorem 1 to three technical iterative statements about particular graphs, which we call 'GCD graphs' (see Definition 6.1). Specifically, in Section 5 we use a second moment argument to reduce the proof to Proposition 5.4, which claims a suitable bound for sums of the form (3.2). Here, we make use of Lemmas 5.1-5.3 which are standard results from the literature. In Section 6 we introduce the key terminology of the paper and translate Proposition 5.4 into Proposition 6.6, a statement about edges in a particular 'GCD graph'. In Section 7 we use results about the anatomy of integers (Lemmas 7.2 and 7.3) to reduce the situation to establishing Proposition 7.1, a technical statement claiming the existence of a 'good' GCD subgraph (where 'good' means that there are integers a and b such that all vertices in $\mathcal{V}$ are divisible by a, those in $\mathcal{W}$ are divisible by b, and if (v, w) is an edge, then $\gcd(v, w) = \gcd(a, b)$). Then in Section 8, we reduce the proof of Proposition 7.1 to five iterative claims: Propositions 8.1-8.3 and Lemmas 8.4-8.5. In Sections 9 and 10 we then directly establish Lemmas 8.4 and 8.5, respectively, leaving the second half of the paper to demonstrate the key statements of Propositions 8.1-8.3.

The dependency diagram for the first half of the paper is as follows:



The second half of the paper consists of Sections 11-14, and it is devoted to proving each of Proposition 8.1, 8.2 and 8.3. Before we embark on the proofs directly, we first establish several preparatory lemmas in Section 11. In particular we prove Lemmas 11.2-11.5 which are minor results on GCD graphs we will use later on. Section 12 is dedicated to the proof of Proposition 8.1, which is the easier iteration step, and relies on two auxiliary results: Lemmas 12.1 and 12.2. Section 13 is dedicated to the proof of Proposition 8.3, the iteration procedure for small primes. This proposition follows from Lemma 13.2, in turn relying on Lemmas 11.2, 11.3 and 13.1. Finally, in Section 14 we prove Proposition 8.2, which is the most delicate part of the iteration procedure. This follows quickly from Lemma 14.1, which in turn relies on Lemmas 11.3-11.5. The dependency diagram for the second half of the paper is as follows:



(We have not included the essentially trivial statement of Lemma 11.1 or Lemma 6.5 which is used frequently in the later sections.) All lemmas are proven in the section where they appear with the exception of Lemma 8.4 and Lemma 8.5, which are proven in Sections 9 and 10 respectively. All propositions are proven in sections later than they appear.

5. PRELIMINARIES

We first reduce the proof of Theorem 1 to a second moment bound given by Proposition 5.4 below. This reduction is standard and appears in several previous works on the Duffin-Schaeffer conjecture. In particular, a vital component is an ergodic 0-1 law due to Gallagher (Lemma 5.1).

Lemma 5.1 (Gallagher's 0-1 law)**.** *Consider a function* $\psi : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ *and let* $\mathcal{A}$ *be as in* (1.4)*. Then either* $\lambda(\mathcal{A}) = 0$ *or* $\lambda(\mathcal{A}) = 1$.

Proof. This is Theorem 1 of [9]. □

Lemma 5.2 (The Duffin-Schaeffer Conjecture when ψ only takes large values)**.** *Let* $\psi : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ *be a function, and let* $\mathcal{A}$ *be as in* (1.4)*. Assume, further, that:*

(a) For every $q \in \mathbb{Z}$*, either* $\psi(q) = 0$ *or* $\psi(q) \geqslant 1/2$*;*
(b) $\sum_{q=1}^{\infty} \psi(q)\varphi(q)/q = \infty$.

Then $\lambda(\mathcal{A}) = 1$.

Proof. This follows from [14, Theorem 2]. □

Lemma 5.3 (Bound for $\lambda(\mathcal{A}_q \cap \mathcal{A}_r)$)**.** *Consider a function $\psi : \mathbb{N} \to [0, 1/2]$ and let $\mathcal{A}_q$ be as in (1.3). In addition, given $q, r \in \mathbb{N}$, set*

$$M(q, r) := \max\{r\psi(q), q\psi(r)\}.$$

If $q \neq r$, then we have

$$\frac{\lambda(\mathcal{A}_q \cap \mathcal{A}_r)}{\lambda(\mathcal{A}_q)\lambda(\mathcal{A}_r)} \ll 1 + \mathbb{1}_{M(q,r) \geqslant \gcd(q,r)} \prod_{\substack{p \mid qr/\gcd(q,r)^2 \\ p > M(q,r)/\gcd(q,r)}} \left(1 + \frac{1}{p}\right).$$

Proof. This bound is given in [14, p. 195-196]. □

Given the above lemma, we introduce the notation

$$L_t(a, b) := \sum_{\substack{p \mid ab/\gcd(a,b)^2 \\ p \geqslant t}} \frac{1}{p} \tag{5.1}$$

for $a, b \in \mathbb{N}$ and $t \geqslant 1$. The key result to proving Theorem 1 is:

Proposition 5.4 (Second moment bound)**.** *Let ψ, $\mathcal{A}_q$ and $M(q, r)$ be as in as in Lemma 5.3, and consider $Y \geqslant X \geqslant 1$ such that*

$$1 \leqslant \sum_{X \leqslant q \leqslant Y} \frac{\psi(q)\varphi(q)}{q} \leqslant 2.$$

For each $t \geqslant 1$, set

$$\mathcal{E}_t = \left\{(v, w) \in (\mathbb{Z} \cap [X, Y])^2 : \gcd(v, w) \geqslant t^{-1} \cdot M(v, w),\ L_t(v, w) \geqslant 10\right\}. \tag{5.2}$$

Then

$$\sum_{(v,w) \in \mathcal{E}_t} \frac{\varphi(v)\psi(v)}{v} \cdot \frac{\varphi(w)\psi(w)}{w} \ll \frac{1}{t}.$$

Proof of Theorem 1 assuming Proposition 5.4. We wish to prove that

$$\lambda(\mathcal{A}) = 1, \tag{5.3}$$

where $\mathcal{A} = \limsup_{q \to \infty} \mathcal{A}_q$ with $\mathcal{A}_q$ defined by (1.4). We first write

$$\psi(q) = \psi_1(q) + \psi_2(q), \qquad \text{where} \qquad \psi_1(q) = \begin{cases} \psi(q) & \text{if } \psi(q) > 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\psi_2(q) = \psi(q)$ if $\psi(q) \leqslant 1/2$ and 0 otherwise.

If it so happens that $\sum_{q=1}^{\infty} \psi_1(q)\varphi(q)/q = \infty$, then we apply Lemma 5.2 to ψ_1 to find that $\lambda(\limsup_{q \to \infty} \mathcal{B}_q) = 1$, where $\mathcal{B}_q$ is defined as $\mathcal{A}_q$ but with ψ replaced by ψ_1. This proves (5.3), since $\psi_1(q) \leqslant \psi(q)$, and so $\mathcal{B}_q \subseteq \mathcal{A}_q$.

Therefore we may assume without loss of generality that $\sum_{q=1}^{\infty} \psi_1(q)\varphi(q)/q < \infty$, and so $\sum_{q=1}^{\infty} \psi_1 2(q)\varphi(q)/q = \infty$. Thus, we have reduced Theorem 1 to the case when

$$\psi(q) \leqslant 1/2 \quad \text{for all } q \geqslant 1.$$

By Lemma 5.1, the Duffin-Schaeffer conjecture will follow if we prove that $\lambda(\mathcal{A})>0$, since this means $\mathcal{A}$ cannot have measure 0. Note that

$$\mathcal{A}=\limsup_{q\to\infty}\mathcal{A}_q=\bigcap_{j=1}^{\infty}\bigcup_{q\geqslant j}\mathcal{A}_q. \tag{5.4}$$

Now, let X be a large parameter and fix $Y=Y(X)$ to be minimal such that

$$\sum_{X\leqslant q\leqslant Y}\frac{\varphi(q)\psi(q)}{q}\in[1,2].$$

(Such a Y exists since $\psi(q)\leqslant 1/2$ for all q.) Hence, we see that it suffices to prove that

$$\lambda\Big(\bigcup_{X\leqslant q\leqslant Y}\mathcal{A}_q\Big)\gg 1 \tag{5.5}$$

uniformly for all large enough X, since this implies that $\lambda(\mathcal{A})>0$ by virtue of (5.4), and hence Theorem 1 follows.

For each $\alpha\in\mathbb{R}$, consider the counting function

$$Q(\alpha)=\#\{q\in\mathbb{Z}\cap[X,Y]:\alpha\in\mathcal{A}_q\}.$$

We then have

$$\mathrm{supp}(Q)=\bigcup_{X\leqslant q\leqslant Y}\mathcal{A}_q,$$

$$\int_0^1 Q(\alpha)\mathrm{d}\alpha=\sum_{X\leqslant q\leqslant Y}\lambda(\mathcal{A}_q)\geqslant\sum_{X\leqslant q\leqslant Y}\frac{\varphi(q)\psi(q)}{q}\geqslant 1,$$

$$\int_0^1 Q(\alpha)^2\mathrm{d}\alpha=\sum_{X\leqslant q,r\leqslant Y}\lambda(\mathcal{A}_q\cap\mathcal{A}_r).$$

Hence, the Cauchy-Schwarz inequality implies that

$$\lambda\Big(\bigcup_{X\leqslant q\leqslant Y}\mathcal{A}_q\Big)\int_0^1 Q(\alpha)^2\mathrm{d}\alpha\geqslant\Big(\int_0^1 Q(\alpha)\mathrm{d}\alpha\Big)^2\geqslant 1.$$

Thus, to establish (5.5), it is enough to prove that

$$\sum_{X\leqslant q,r\leqslant Y}\lambda(\mathcal{A}_q\cap\mathcal{A}_r)\ll 1.$$

The terms with $q=r$ contribute a total

$$\sum_{X\leqslant q\leqslant Y}\lambda(\mathcal{A}_q)\leqslant\sum_{X\leqslant q\leqslant Y}\frac{2\varphi(q)\psi(q)}{q}\leqslant 4,$$

and so we only need to consider the contribution of those terms with $q\neq r$. Applying Lemma 5.3, we see that

$$\lambda(\mathcal{A}_q\cap\mathcal{A}_r)\ll\frac{\varphi(q)\psi(q)}{q}\cdot\frac{\varphi(r)\psi(r)}{r}\cdot\Bigg[1+\mathbb{1}_{M(q,r)\geqslant\gcd(q,r)}\prod_{\substack{p|qr/\gcd(q,r)^2\\ p>M(q,r)/\gcd(q,r)}}\Big(1+\frac{1}{p}\Big)\Bigg],$$

where we recall that

$$M(q,r)=\max\{r\psi(q),q\psi(r)\}.$$

Since

$$\sum_{X\leqslant q,r\leqslant Y}\frac{\varphi(q)\psi(q)}{q}\cdot\frac{\varphi(r)\psi(r)}{r}=\Big(\sum_{X\leqslant q\leqslant Y}\frac{\varphi(q)\psi(q)}{q}\Big)^2\leqslant 4,$$

it suffices to show that

$$\sum_{\substack{X\leqslant q,r\leqslant Y\\ M(q,r)\geqslant\gcd(q,r)}}\frac{\varphi(q)\psi(q)}{q}\cdot\frac{\varphi(r)\psi(r)}{r}\prod_{\substack{p\mid qr/\gcd(q,r)^2\\ p>M(q,r)/\gcd(q,r)}}\Big(1+\frac{1}{p}\Big)\ll 1. \tag{5.6}$$

To prove this inequality, we divide the range of q and r into convenient subsets.

The pairs $(q,r)\in(\mathbb{Z}\cap[X,Y])^2$ with

$$\prod_{\substack{p\mid qr/\gcd(q,r)^2\\ p>M(q,r)/\gcd(q,r)}}\Big(1+\frac{1}{p}\Big)<e^{100}$$

contribute a total of at most

$$e^{100}\Big(\sum_{q\in[X,Y]}\frac{\varphi(q)\psi(q)}{q}\Big)^2\leqslant 4e^{100}$$

to the right hand side of (5.6), and so can be ignored.

For any other pair (q,r), we see that

$$e^{100}\leqslant\prod_{\substack{p\mid qr/\gcd(q,r)^2\\ p>M(q,r)/\gcd(q,r)}}\Big(1+\frac{1}{p}\Big)\leqslant\exp\Big(\sum_{\substack{p\mid qr/\gcd(q,r)^2\\ p>M(q,r)/\gcd(q,r)}}\frac{1}{p}\Big),$$

so certainly we have

$$\sum_{p\mid qr/\gcd(q,r)^2}\frac{1}{p}\geqslant 100.$$

For any such pair, we let $j=j(q,r)$ to be the smallest integer such that

$$\sum_{\substack{p\mid qr/\gcd(q,r)^2\\ p\geqslant\exp\exp(j)}}\frac{1}{p}\geqslant 10.$$

Since j is chosen minimally, we have

$$\sum_{\substack{p\mid qr/\gcd(q,r)^2\\ p\geqslant\exp\exp(j+1)}}\frac{1}{p}<10.$$

Mertens' theorem then implies that

$$\sum_{\substack{p\mid qr/\gcd(q,r)^2\\ p\geqslant\exp\exp(j)}}\frac{1}{p}=\sum_{\substack{p\mid qr/\gcd(q,r)^2\\ \exp\exp(j)\leqslant p<\exp\exp(j+1)}}\frac{1}{p}+\sum_{\substack{p\mid qr/\gcd(q,r)^2\\ p\geqslant\exp\exp(j+1)}}\frac{1}{p}\ll 1.$$

Therefore

$$\prod_{\substack{p\mid qr/\gcd(q,r)^2\\ p>M(q,r)/\gcd(q,r)}}\Big(1+\frac{1}{p}\Big)\ll\prod_{M(q,r)/\gcd(q,r)<p\leqslant\exp\exp(j)}\Big(1+\frac{1}{p}\Big)$$
$$\ll\begin{cases}1 & \text{if } M(q,r)/\gcd(q,r)\geqslant\exp\exp(j),\\ e^j & \text{otherwise.}\end{cases}$$

As above, those pairs with

$$\prod_{\substack{p\mid qr/\gcd(q,r)^2\\ p>M(q,r)/\gcd(q,r)}}\Big(1+\frac{1}{p}\Big)\ll 1$$

make an acceptable contribution to (5.6). Therefore we only need to consider pairs (q,r) with $M(q,r)/\gcd(q,r)<\exp\exp(j)$.

We have thus reduced (5.6) to showing that

$$\sum_{j\geqslant 0}e^j\sum_{(q,r)\in\mathcal{E}_{\exp\exp(j)}}\frac{\varphi(q)\psi(q)}{q}\cdot\frac{\varphi(r)\psi(r)}{r}\ll 1, \tag{5.7}$$

where $\mathcal{E}_t$ is defined by (5.2). To prove (5.7), we apply Proposition 5.4. This completes the proof of Theorem 1. □

Thus we are left to establish Proposition 5.4.

6. Bipartite GCD graphs

In this section we introduce the key notation that will underlie the rest of the paper. In particular, we show that Proposition 5.4 follows from a statement given by Proposition 6.6 about a weighted graph with additional information about divisibility of the integers making up its vertices. The rest of the paper is then dedicated to establishing suitable properties of such graphs, which we call 'GCD graphs'.

If we let

$$\mathcal{V}=\{q\in\mathbb{Z}\cap[X,Y]:\psi(q)\neq 0\}$$

and we weight the elements of $\mathcal{V}$ with the measure

$$\mu(v)=\frac{\varphi(q)\psi(q)}{q},$$

then Proposition 5.4 can be interpreted as an estimate for the weighted edge density of the graph with set of vertices $\mathcal{V}$ and set of edges $\mathcal{E}_t$ defined by (5.2).

Our strategy for proving Proposition 5.4 is to use a 'compression' argument. More precisely, if G_1 denotes the graph described in the above parapraph, we will construct a finite sequence of graphs $G_1,\dots,G_J$ where we make a small local change to pass from G_j to G_{j+1} that increases the amount of structure in the graph. The final graph G_J will then be highly structured and easy to analyze. To keep control over the procedure, we keep track of how certain statistics of the graph change at each step. This enables us to show that the relevant properties of G_j are suitably controlled by G_{j+1}, and so G_1 is controlled by G_J, where everything is explicit.

To perform the above construction, we introduce some new notation to take into account the extra information about prime power divisibility which we need to carry at each stage.

Definition 6.1 (GCD graph)**.** Let G be a septuple $(\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ such that:

(a) μ is a measure on $\mathbb{N}$ that we extend to $\mathbb{N}^2$ by letting

$$\mu(\mathcal{N}) := \sum_{(n_1,n_2)\in\mathcal{N}} \mu(n_1)\mu(n_2) \quad \text{for} \quad \mathcal{N} \subseteq \mathbb{N}^2;$$

(b) $\mathcal{V}$ and $\mathcal{W}$ are finite sets of integers such that $0 < \mu(\mathcal{V}), \mu(\mathcal{W}) < \infty$;

(c) $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{W}$, that is to say the triplet $(\mathcal{V}, \mathcal{W}, \mathcal{E})$ is a bipartite graph.

(d) $\mathcal{P}$ is a set of primes;

(e) f and g are functions from $\mathcal{P}$ to $\mathbb{Z}_{\geqslant 0}$ such that for all $p \in \mathcal{P}$ we have:

  (i) $|f(p) - g(p)| \leqslant 1$;

  (ii) $p^{f(p)}|v$ for all $v \in \mathcal{V}$, and $p^{g(p)}|w$ for all $w \in \mathcal{W}$;

  (iii) if $(v, w) \in \mathcal{E}$, then $p^{\min\{f(p),g(p)\}}\| \gcd(v, w)$;

  (iv) if $f(p) \neq g(p)$, then $p^{f(p)}\|v$ for all $v \in \mathcal{V}$, and $p^{g(p)}\|w$ for all $w \in \mathcal{W}$.

We then call G a (bipartite) *GCD graph* with *sets of vertices* $(\mathcal{V}, \mathcal{W})$, *set of edges* $\mathcal{E}$ and *multiplicative data* $(\mathcal{P}, f, g)$. We will also refer to $\mathcal{P}$ as the *set of primes* of G. If $\mathcal{P} = \emptyset$, we say that G has *trivial* set of primes and we view $f = f_\emptyset$ and $g = g_\emptyset$ as two copies of the empty function from $\emptyset$ to $\mathbb{Z}_{\geqslant 0}$.

Definition 6.2 (GCD subgraph)**.** Let $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ and $G' = (\mu', \mathcal{V}', \mathcal{W}', \mathcal{E}', \mathcal{P}', f', g')$ be two GCD graphs. We say that G' is a *GCD subgraph* of G if:

$$\mu' = \mu, \quad \mathcal{V}' \subseteq \mathcal{V}, \quad \mathcal{W}' \subseteq \mathcal{W}, \quad \mathcal{E}' \subseteq \mathcal{E}, \quad \mathcal{P}' \supseteq \mathcal{P}, \quad f'\big|_{\mathcal{P}} = f, \quad g'\big|_{\mathcal{P}} = g.$$

We write $G' \preceq G$ if G' is a GCD subgraph of G.

We thus see from the above definition that we only accept G' as a subgraph of G if we have at least as much information about the divisibility of the vertices of G' compared to those of G. In particular, we have that $p^{\min(f(p),g(p))}\| \gcd(v', w')$ for all $(v', w') \in \mathcal{E}'$ and all $p \in \mathcal{P}$.

We will devise an iterative argument that adds one prime at a time to $\mathcal{P}$, so that we will eventually control very well the multiplicative structure of GCDs of connected vertices in the graph we end up with at the end of this process.

The main way we will produce a GCD subgraph of a GCD graph G is by restricting to vertex sets with certain divisibility properties. Since we will use this several times, we introduce a specific notation for these GCD subgraphs:

Definition 6.3 (Special GCD subgraphs from prime power divisibility)**.** Let p be a prime number, and let $k, \ell \in \mathbb{Z}_{\geqslant 0}$.

(a) If $\mathcal{V}$ is a set of integers and $k \in \mathbb{Z}_{\geqslant 0}$, we set

$$\mathcal{V}_{p^k} = \{v \in \mathcal{V} : p^k\|v\},$$

that is to say $\mathcal{V}_{p^k}$ is the set of integers in $\mathcal{V}$ whose p-adic valuation is exactly k. Here we have the understanding that $\mathcal{V}_{p^0}$ denotes the set of $v \in \mathcal{V}$ that are coprime to p. In particular, $\mathcal{V}_{2^0}$ and $\mathcal{V}_{3^0}$ denote different sets of integers.

(b) Let $G = (\mathcal{V}, \mathcal{W}, \mathcal{E})$ be a bipartite graph. If $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{W}' \subseteq \mathcal{W}$, we define

$$\mathcal{E}(\mathcal{V}', \mathcal{W}') := \mathcal{E} \cap (\mathcal{V}' \times \mathcal{W}').$$

We also write for brevity

$$\mathcal{E}_{p^k,p^\ell} := \mathcal{E}(\mathcal{V}_{p^k}, \mathcal{W}_{p^\ell}).$$

(c) Let $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ be a GCD graph such that $p \in \mathcal{R}(G)$. We then define the septuple

$$G_{p^k,p^\ell} = (\mu, \mathcal{V}_{p^k}, \mathcal{W}_{p^\ell}, \mathcal{E}_{p^k,p^\ell}, \mathcal{P} \cup \{p\}, f_{p^k}, g_{p^\ell})$$

where the functions f_{p^k}, g_{p^ℓ} are defined on $\mathcal{P} \cup \{p\}$ by the relations $f_{p^k}|_{\mathcal{P}} = f$, $g_{p^\ell}|_{\mathcal{P}} = g$,

$$f_{p^k}(p) = k \quad \text{and} \quad g_{p^\ell}(p) = \ell.$$

It is easy to check that G_{p^k,p^ℓ} is a GCD subgraph of G.

The aim of our iterative procedure is to obtain a simple GCD subgraph G' of our initial graph G where the key quantitative aspects of G are controlled by the corresponding quantities of G'. Here 'simple' graphs have many primes occurring in $\gcd(v, w)$ for $(v, w) \in \mathcal{E}$ to a fixed exponent, whilst for subgraphs to maintain control over the original graph we need to maintain sufficiently many edges relative to the number of vertices. This leads us to our last four definitions:

Definition 6.4 (Quantities associated to GCD graphs)**.** Let $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ be a GCD graph.

(a) The *edge density* of G is defined by

$$\delta = \delta(G) := \frac{\mu(\mathcal{E})}{\mu(\mathcal{V})\mu(\mathcal{W})}.$$

(b) The *neighbourhood sets* are defined by

$$\Gamma_G(v) := \{w \in \mathcal{W} :\ (v, w) \in \mathcal{E}\} \quad \text{for any } v \in \mathcal{V},$$

and similarly

$$\Gamma_G(w) := \{v \in \mathcal{V} :\ (v, w) \in \mathcal{E}\} \quad \text{for any } w \in \mathcal{W}.$$

(c) We let $\mathcal{R}(G)$ be given by

$$\mathcal{R}(G) := \{p \notin \mathcal{P} : \exists (v, w) \in \mathcal{E} \text{ such that } p | \gcd(v, w)\}.$$

That is to say $\mathcal{R}(G)$ is the set of primes occurring in a GCD which we haven't yet accounted for. We split this into two further subsets:

$$\mathcal{R}^\sharp(G) := \left\{ p \in \mathcal{R}(G) : \exists k \in \mathbb{Z}_{\geqslant 0} \text{ such that } \frac{\mu(\mathcal{V}_{p^k})}{\mu(\mathcal{V})}, \frac{\mu(\mathcal{W}_{p^k})}{\mu(\mathcal{W})} \geqslant 1 - \frac{10^{40}}{p} \right\}$$

and

$$\mathcal{R}^\flat(G) := \mathcal{R}(G) \setminus \mathcal{R}^\sharp(G).$$

(d) The *quality* of G is defined by

$$\begin{aligned} q(G) &:= \delta^{10}\mu(\mathcal{V})\mu(\mathcal{W}) \prod_{p \in \mathcal{P}} \frac{p^{|f(p)-g(p)|}}{(1 - \mathbb{1}_{f(p)=g(p)\geqslant 1}/p)^2 (1 - 1/p^{31/30})^{10}} \\ &= \frac{\mu(\mathcal{E})^{10}}{\mu(\mathcal{V})^9 \mu(\mathcal{W})^9} \prod_{p \in \mathcal{P}} \frac{p^{|f(p)-g(p)|}}{(1 - \mathbb{1}_{f(p)=g(p)\geqslant 1}/p)^2 (1 - 1/p^{31/30})^{10}}. \end{aligned}$$

As mentioned in Section 3, there are two natural candidates for a quantity to increment; either δ or $\mu(\mathcal{V})\mu(\mathcal{W}) \prod_{p \in \mathcal{P}} p^{|f(p)-g(p)|}$ (this is the natural generalization to non-squarefree integers). One should essentially think of the quality as a 'hybrid' of the two quantities, but with some additional factors which are included for technical reasons. The factor

$$\prod_{p\in\mathcal{P}} \frac{1}{(1-1/p^{31/30})^{10}}$$

always lies in the interval $[1, \zeta(31/30)^{10}]$, and so is always of constant size. This factor is included merely for convenience, and allows us to have a quality increment even if there is a tiny loss in our arguments in terms of p. The factor

$$\prod_{p\in\mathcal{P}} \frac{1}{(1-\mathbb{1}_{f(p)=g(p)\geqslant 1}/p)^2}$$

is crucial for the proof of a quality increment in Lemma 14.1 and Proposition 8.2. This is related to the technical point that it is vital that $\varphi(q)/q$ factors appear in this problem; this feature is discussed in more detail in Section 15.

We will repeatedly make use of two trivial properties of GCD graphs, given by Lemma 6.5 below, without further comment.

Lemma 6.5 (Basic properties of GCD graphs)**.** *Let G_1, G_2, G_3 be GCD graphs.*

(a) The property of being a GCD subgraph is transitive: If $G_1 \preceq G_2$ and $G_2 \preceq G_3$, then $G_1 \preceq G_3$
(b) If $G_1 \preceq G_2$, then $\mathcal{R}(G_1) \subseteq \mathcal{R}(G_2)$.

Proof. Both statements are immediate from the definition of GCD subgraphs. □

Remark 6.1. It is not necessarily the case that $\mathcal{R}^\flat(G') \subseteq \mathcal{R}^\flat(G)$ or that $\mathcal{R}^\sharp(G') \subseteq \mathcal{R}^\sharp(G)$.

Proposition 6.6 (Edge set bound)**.** *Let $\psi : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$, $t \geqslant 1$ and μ be the measure $\mu(v) = \psi(v)\varphi(v)/v$. Let $\mathcal{V} \subseteq \mathbb{Z}$ satisfy $0 < \mu(\mathcal{V}) \ll 1$. Let $G = (\mu, \mathcal{V}, \mathcal{V}, \mathcal{E}, \emptyset, f_\emptyset, g_\emptyset)$ be a bipartite GCD graph with measure μ, vertex sets $\mathcal{V}$, trivial set of primes, and edge set $\mathcal{E} \subseteq \mathcal{E}_t$, where $\mathcal{E}_t$ is defined as in Proposition 5.4. Then*

$$\mu(\mathcal{E}) \ll 1/t.$$

Proof of Proposition 5.4 assuming Proposition 6.6. Recall the notation ψ, $M(q,r)$, $L_t(a,b)$ and $\mathcal{E}_t$ of Proposition 5.4. We wish to show that

$$\sum_{(v_1,v_2)\in\mathcal{E}_t} \frac{\varphi(v_1)\psi(v_1)}{v_1} \cdot \frac{\varphi(v_2)\psi(v_2)}{v_2} \ll \frac{1}{t}. \tag{6.1}$$

Let μ be the measure on $\mathbb{N}$ defined by $\mu(v) := \psi(v)\varphi(v)/v$ and let

$$\mathcal{V} := \{q \in [X,Y] : \psi(q) \neq 0\},$$

so that

$$\mu(\mathcal{V}) = \sum_{q\in\mathcal{V}} \frac{\varphi(q)\psi(q)}{q} = \sum_{q\in[X,Y]} \frac{\varphi(q)\psi(q)}{q} \in [1,2].$$

Now define $\mathcal{E} = \mathcal{E}_t$ to be as in Proposition 5.4. We see that $(\mathcal{V}, \mathcal{V}, \mathcal{E})$ forms a bipartite graph with vertex sets two copies of $\mathcal{V}$ and edge set $\mathcal{E}$. We now turn this bipartite graph into a GCD graph $G = (\mu, \mathcal{V}, \mathcal{V}, \mathcal{E}, \emptyset, f_\emptyset, g_\emptyset)$ by attaching trivial multiplicative data to the bipartite graph (here $f_\emptyset$ and $g_\emptyset$ are viewed as two copies of the function of the empty set to $\mathbb{Z}_{\geqslant 0}$).

Since $0 < \mu(\mathcal{V}) \ll 1$, Proposition 6.6 now applies, showing that

$$\mu(\mathcal{E}_t) \ll 1/t.$$

This completes the proof. □

Thus we are left to establish Proposition 6.6.

7. Reduction to a good GCD subgraph

In this section, we reduce the proof of Proposition 6.6 (and hence of Theorem 1) to finding a 'good' GCD subgraph as described in Proposition 7.1 below. This reduction utilizes some results showing that few integers have lots of fairly small prime factors (based on 'the anatomy of integers').

Proposition 7.1 (Existence of a good GCD subgraph). *Let $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \emptyset, f_\emptyset, g_\emptyset)$ be a GCD graph with trivial set of primes and edge density δ. Assume further that*

$$\mathcal{E} \subseteq \{(v,w) \in \mathbb{N}^2 : L_t(v,w) \geqslant 10\}$$

for some t satisfying

$$t \geqslant 10\delta^{-1/50} \quad \textit{and} \quad t > 10^{2000}.$$

Then there is a GCD subgraph $G' = (\mu, \mathcal{V}', \mathcal{W}', \mathcal{E}', \mathcal{P}', f', g')$ of G with edge density δ' such that:

- *(a) $\mathcal{R}(G') = \emptyset$;*
- *(b) For all $v \in \mathcal{V}'$, we have $\mu(\Gamma_{G'}(v)) \geqslant (9\delta'/10)\mu(\mathcal{W}')$;*
- *(c) For all $w \in \mathcal{W}'$, we have $\mu(\Gamma_{G'}(w)) \geqslant (9\delta'/10)\mu(\mathcal{V}')$;*
- *(d) One of the following holds:*
    - *(i) $q(G') \gg \delta t^{50} q(G)$;*
    - *(ii) $q(G') \gg q(G)$, and if $(v,w) \in \mathcal{E}'$ and we write them as $v = v' \prod_{p\in\mathcal{P}'} p^{f'(p)}$ and $w = w' \prod_{p\in\mathcal{P}'} p^{g'(p)}$, then $L_t(v',w') \geqslant 4$.*

Our task is to prove how Proposition 7.1 implies Proposition 6.6. To do so, we need a couple of preparatory lemmas that exploit the condition that $L_t(v',w') \geqslant 4$ in Case (d)-(ii) of Proposition 7.1.

Lemma 7.2 (Bounds on multiplicative functions). *Let f be a non-negative multiplicative function that satisfies*

$$\sum_{p\leqslant y} f(p)\log p \leqslant Ay \quad (y \geqslant 1), \quad \textit{and} \quad \sum_p \sum_{\nu\geqslant 2} \frac{f(p^\nu)\log(p^\nu)}{p^\nu} \leqslant B$$

for some constants A, $B > 0$. Then

$$\sum_{n\leqslant x} f(n) \leqslant (A+B+1)\frac{x}{\log x}\sum_{n\leqslant x}\frac{f(n)}{n}.$$

Proof. This is [15, Theorem III.3.5, p. 456]. □

Lemma 7.3 (Few numbers with many prime factors). *For $x, t, c \geqslant 1$, we have*

$$\#\Big\{n \leqslant x : \sum_{\substack{p\mid n\\ p\geqslant t}} \frac{1}{p} \geqslant c\Big\} \ll x\exp\{-t^{e^{c-1}}\};$$

the implied constant is absolute.

Proof. We may assume that t is large enough, since the result is trivial when t is bounded. Set $T = t^{e^{c-1}}$, so that $\sum_{t\leqslant p\leqslant T} 1/p \leqslant c - 1/2$. Hence

$$\#\Big\{n \leqslant x : \sum_{\substack{p|n \\ p\geqslant t}} \frac{1}{p} \geqslant c\Big\} \leqslant \#\Big\{n \leqslant x : \sum_{\substack{p|n \\ p\geqslant T}} \frac{1}{p} \geqslant 1/2\Big\} \leqslant e^{-T} \sum_{n\leqslant x} \prod_{\substack{p|n \\ p\geqslant T}} e^{2T/p}. \tag{7.1}$$

We wish to apply Lemma 7.2 when f is the multiplicative function with $f(p^\nu) = e^{2T/p}$ for $p \geqslant T$ and all $\nu \geqslant 1$, and $f(p^\nu) = 0$ for $p < T$. With this choice of f we have $f(p^\nu) \leqslant e^2$, so

$$\sum_{p\leqslant y} f(p)\log p \leqslant e^2 \sum_{p\leqslant y} \log p \ll y,$$

and

$$\sum_p \sum_{\nu\geqslant 2} \frac{f(p^\nu)\log(p^\nu)}{p^\nu} \leqslant e^2 \sum_p \sum_{\nu\geqslant 2} \frac{\log(p^\nu)}{p^\nu} \ll 1.$$

Thus f satisfies both the required bounds for suitable absolute constants A and B. As a consequence,

$$\sum_{n\leqslant x} \prod_{p|n,\ p\geqslant T} e^{2T/p} \ll \frac{x}{\log x} \sum_{n\leqslant x} \frac{f(n)}{n}.$$

Since f is non-negative, we see that the right hand side is

$$\leqslant \frac{x}{\log x} \prod_{p\leqslant x} \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \cdots\right) = \frac{x}{\log x} \prod_{T\leqslant p\leqslant x} \left(1 + \frac{e^{T/2p}}{p-1}\right).$$

Since $e^{2T/p} = 1 + O(T/p)$ for $p \geqslant T$, and $\sum_{p\geqslant T} T/p^2 \ll 1$, the right hand side above is $O(x)$. Thus, combining this with (7.1), we find

$$\#\Big\{n \leqslant x : \sum_{\substack{p|n \\ p\geqslant t}} \frac{1}{p} \geqslant c\Big\} \ll e^{-T} x.$$

Recalling that $T = t^{e^{c-1}}$, we see that this gives the result. □

Proof of Proposition 6.6 assuming Proposition 7.1. Fix $t \geqslant 1$ and let G be the GCD graph of Proposition 6.6 with set of edges $\mathcal{E} \subseteq \mathcal{E}_t$ (where $\mathcal{E}_t$ is defined in Proposition 5.4), weight $\mu(v) = \varphi(v)/v$ and edge density $\delta = \mu(\mathcal{E})/\mu(\mathcal{V})^2$.

If $\delta \ll 1/t$, then $\mu(\mathcal{E}) \ll 1/t$ and so we are done. Therefore we may assume that

$$\delta \geqslant 1/t \quad \text{and} \quad t > 10^{2000}.$$

Note that this implies that

$$t \geqslant 10\delta^{-1/50}.$$

We apply Proposition 7.1 to G to find a GCD subgraph $G' = (\mu, \mathcal{V}', \mathcal{W}', \mathcal{E}', \mathcal{P}', f', g')$ of edge density δ' satisfying either case (d)-(i) or (d)-(ii) of its statement. In addition, we have that:

(a) $\mathcal{R}(G') = \emptyset$;
(b) $\mu(\Gamma_{G'}(v)) \geqslant (9\delta'/10)\mu(\mathcal{W}')$ for all $v \in \mathcal{V}'$;
(c) $\mu(\Gamma_{G'}(w)) \geqslant (9\delta'/10)\mu(\mathcal{V}')$ for all $w \in \mathcal{W}'$.

Set

$$a := \prod_{p\in\mathcal{P}'} p^{f'(p)} \quad \text{and} \quad b := \prod_{p\in\mathcal{P}'} p^{g'(p)}.$$

The definition of a GCD graph implies that

$$a|v \quad \text{for all } v\in\mathcal{V}', \qquad b|w \quad \text{for all } w\in\mathcal{W}'.$$

Moreover, since $\mathcal{R}(G')=\emptyset$, and $p^{\min\{f'(p),g'(p)\}}\|\gcd(v,w)$ for all $(v,w)\in\mathcal{E}'$, we have that

$$\gcd(v,w)=\gcd(a,b) \quad \text{for all} \quad (v,w)\in\mathcal{E}'.$$

Now, note that

$$\prod_{p\in\mathcal{P}'} p^{|f'(p)-g'(p)|} = \prod_{p\in\mathcal{P}'} p^{\max\{f'(p),g'(p)\}-\min\{f'(p),g'(p)\}} = \frac{\operatorname{lcm}[a,b]}{\gcd(a,b)} = \frac{ab}{\gcd(a,b)^2},$$

as well as

$$\prod_{p\in\mathcal{P}'} \frac{1}{(1-\mathbb{1}_{f'(p)=g'(p)\geqslant 1}/p)^2(1-1/p^{31/30})^{10}} \ll \prod_{p\in\mathcal{P}'} \frac{1}{(1-\mathbb{1}_{f'(p)=g'(p)\geqslant 1}/p)^2} \leqslant \frac{ab}{\varphi(a)\varphi(b)}.$$

Consequently,

$$\begin{aligned} q(G') &:= (\delta')^{10}\mu(\mathcal{V}')\mu(\mathcal{W}') \prod_{p\in\mathcal{P}'} \frac{p^{|f'(p)-g'(p)|}}{(1-\mathbb{1}_{f'(p)=g'(p)\geqslant 1}/p)^2(1-1/p^{31/30})^{10}} \\ &\ll (\delta')^{10}\mu(\mathcal{V}')\mu(\mathcal{W}')\frac{ab}{\gcd(a,b)^2}\cdot\frac{ab}{\varphi(a)\varphi(b)} \\ &= (\delta')^{9}\mu(\mathcal{E}')\frac{ab}{\gcd(a,b)^2}\cdot\frac{ab}{\varphi(a)\varphi(b)}. \end{aligned} \tag{7.2}$$

Proposition 7.1 offers a lower bound on $q(G')/q(G)$. Since

$$q(G)=\delta^{10}\mu(\mathcal{V})\mu(\mathcal{W})=\delta^9\mu(\mathcal{E}),$$

we can obtain an upper bound on the size of $\mu(\mathcal{E})$ by estimating $q(G')$ from above.

Note that

$$\mathcal{E}'\subseteq\mathcal{E}\subseteq\{(v,w)\in\mathcal{V}\times\mathcal{W} : M(v,w)\leqslant t\cdot\gcd(v,w)\},.$$

where we recall that $M(v,w)=\max\{v\psi(w),w\psi(v)\}$. Since $\gcd(v,w)=\gcd(a,b)$ for all $(v,w)\in\mathcal{E}'$, we infer that

$$\psi(v)\leqslant\frac{t\cdot\gcd(a,b)}{w} \quad \text{and} \quad \psi(w)\leqslant\frac{t\cdot\gcd(a,b)}{v} \quad \text{for all} \quad (v,w)\in\mathcal{E}'.$$

The vertex sets $\mathcal{V}',\mathcal{W}'$ are finite sets of positive integers. For each $v\in\mathcal{V}'$, let $w_{\max}(v)$ be the largest integer in $\mathcal{W}'$ such that $(v,w_{\max}(v))\in\mathcal{E}'$. (We emphasise to the reader that 'largest' refers to the size of elements as positive integers, and does not depend on the measure μ.) Similarly, for each $w\in\mathcal{W}'$, let $v_{\max}(w)$ be the largest element of $\mathcal{V}'$ such that $(v_{\max}(w),w)\in\mathcal{E}'$. Consequently,

$$\psi(v)\leqslant\frac{t\cdot\gcd(a,b)}{w_{\max}(v)} \quad \text{and} \quad \psi(w)\leqslant\frac{t\cdot\gcd(a,b)}{v_{\max}(w)} \quad \text{whenever} \quad (v,w)\in\mathcal{E}'. \tag{7.3}$$

Now, let w_0 be the largest integer in $\mathcal{W}'$ and $\mathcal{E}'' = \{(v,w) \in \mathcal{E}' : (v,w_0) \in \mathcal{E}'\}$. Since G' satisfies conditions (b) and (c) in the statement of Proposition 6.6, we have

$$\mu(\mathcal{E}'') = \sum_{v\in\Gamma_{G'}(w_0)} \mu(v)\mu(\Gamma_{G'}(v)) \geqslant \mu(\Gamma_{G'}(w_0)) \cdot \frac{9\delta'\mu(\mathcal{W}')}{10} \geqslant \left(\frac{9\delta'}{10}\right)^2 \mu(\mathcal{V}')\mu(\mathcal{W}') \geqslant \frac{\delta'\mu(\mathcal{E}')}{2}.$$

Substituting this bound into (7.2), we find

$$q(G') \ll (\delta')^8\mu(\mathcal{E}'')\frac{ab}{\varphi(a)\varphi(b)} \cdot \frac{ab}{\gcd(a,b)^2} \leqslant \mu(\mathcal{E}'')\frac{ab}{\gcd(a,b)^2} \cdot \frac{ab}{\varphi(a)\varphi(b)}. \tag{7.4}$$

Here we used the trivial bound $\delta' \leqslant 1$ in the second inequality. In addition,

$$\mu(\mathcal{E}'') = \sum_{(v,w)\in\mathcal{E}''} \frac{\psi(v)\varphi(v)}{v} \cdot \frac{\psi(w)\varphi(w)}{v}.$$

Since $a|v$ and $b|w$, we have $\varphi(v)/v \leqslant \varphi(a)/a$ and $\varphi(w)/w \leqslant \varphi(b)/b$. Therefore

$$\mu(\mathcal{E}'') \leqslant \frac{\varphi(a)\varphi(b)}{ab} \sum_{(v,w)\in\mathcal{E}''} \psi(v)\psi(w).$$

Together with (7.3) and (7.4), this implies that

$$q(G') \ll t^2 ab \sum_{(v,w)\in\mathcal{E}''} \frac{1}{v_{\max}(w)w_0}. \tag{7.5}$$

We now split our argument depending on whether (d)-(i) or (d)-(ii) of Proposition 7.1 holds.

Case 1: (d)-(i) of Proposition 7.1 holds

In this case we have $q(G') \gg \delta t^{50} q(G)$. Writing $v = v'a$ and $w = w'b$, we find that

$$\begin{aligned}\sum_{(v,w)\in\mathcal{E}''} \frac{1}{v_{\max}(w)w_0} &\leqslant \sum_{w'\leqslant w_0/b} \frac{1}{w_0 v_{\max}(bw')} \sum_{v'\leqslant v_{\max}(bw')/a} 1 \\ &\leqslant \sum_{w'\leqslant w_0/b} \frac{1}{w_0 v_{\max}(bw')} \cdot \frac{v_{\max}(bw')}{a} \\ &\leqslant \frac{1}{ab}.\end{aligned}$$

Together with (7.5), this implies that

$$q(G') \ll t^2.$$

Since $q(G') \gg \delta t^{50} q(G)$ in this case, and since $\delta \geqslant 1/t$, this gives

$$\mu(\mathcal{E}) = \delta^{-9} q(G) \ll \delta^{-10} t^{-50} q(G') \ll \frac{1}{\delta^{10} t^{48}} \ll \frac{1}{t}.$$

This establishes Proposition 6.6 in this case.

Case 2: (d)-(ii) of Proposition 7.1 holds

Write $v = v'a$ and $w = w'b$. In this case

$$L_t(v',w') \geqslant 4 \quad \text{whenever} \quad (v,w) \in \mathcal{E}''. \tag{7.6}$$

We also have $q(G') \gg q(G)$.

From (7.6), we see that either

$$\sum_{p|v',\, p\geqslant t}\frac{1}{p}\geqslant 2\quad\text{or}\quad\sum_{p|w',\, p\geqslant t}\frac{1}{p}\geqslant 2$$

whenever $(v,w)\in\mathcal{E}'$. Consequently,

$$\begin{aligned}\sum_{(v,w)\in\mathcal{E}''}\frac{1}{v_{\max}(w)w_0}&\leqslant\sum_{\substack{w'\leqslant w_0/b,\ v'\leqslant v_{\max}(bw')/a\\ \sum_{p|v'w'}1/p\geqslant 4}}\sum\frac{1}{v_{\max}(bw')w_0}\\&\leqslant S_1+S_2,\end{aligned}$$

where

$$S_1=\sum_{\substack{w'\leqslant w_0/b,\ v'\leqslant v_{\max}(bw')/a\\ \sum_{p|v'}1/p\geqslant 2}}\sum\frac{1}{v_{\max}(bw')w_0},$$

$$S_2=\sum_{\substack{w'\leqslant w_0/b,\ v'\leqslant v_{\max}(bw')/a\\ \sum_{p|w'}1/p\geqslant 2}}\sum\frac{1}{v_{\max}(bw')w_0}.$$

For S_1, we note that

$$\begin{aligned}S_1&\leqslant\sum_{w'\leqslant w_0/b}\frac{1}{w_0v_{\max}(bw')}\sum_{\substack{v'\leqslant v_{\max}(bw')/a\\ \sum_{p|v'}1/p\geqslant 2}}1\\&\ll\sum_{w'\leqslant w_0/b}\frac{1}{w_0v_{\max}(bw')}\cdot\frac{v_{\max}(bw')/a}{t^2e^t}\\&\leqslant\frac{1}{abt^2e^t}\end{aligned}$$

by Lemma 7.3, since $\exp(t^e)\gg\exp(t)t^2$. Similarly for S_2, we find that

$$\begin{aligned}S_2&\leqslant\sum_{\substack{w'\leqslant w_0/b\\ \sum_{p|w'}1/p\geqslant 2}}\frac{1}{w_0v_{\max}(bw')}\sum_{v'\leqslant v_{\max}(bw')/a}1\\&\leqslant\sum_{\substack{w'\leqslant w_0/b\\ \sum_{p|w'}1/p\geqslant 2}}\frac{1}{aw_0}\\&\ll\frac{1}{abt^2e^t},\end{aligned}$$

by applying Lemma 7.3 once again. Substituting these bounds into (7.5), we conclude that

$$q(G')\ll e^{-t}.$$

Since we have $q(G)\gg q(G')$ and $\delta\geqslant 1/t$, this gives

$$\mu(\mathcal{E})=\delta^{-9}q(G)\ll t^9q(G')\ll t^9e^{-t}\ll 1/t.$$

This establishes Proposition 6.6 in all cases. □

Thus we are left to prove Proposition 7.1.

8. Reduction of Proposition 7.1 to three iterative propositions

We will prove Proposition 7.1 by an iterative argument, where we repeatedly find GCD subgraphs with progressively nicer properties. In this section we reduce the proof to five technical iterative statements, given by three key propositions (Propositions 8.1-8.3) and two auxiliary lemmas (Lemmas 8.4-8.5) given below.

Proposition 8.1 (Iteration when $\mathcal{R}^\flat(G) \neq \emptyset$)**.** *Let $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ be a GCD graph of edge density δ such that*

$$\mathcal{R}(G) \subseteq \{p > 10^{2000}\} \quad \text{and} \quad \mathcal{R}^\flat(G) \neq \emptyset.$$

Then there is a GCD subgraph G' of G with edge density δ' and multiplicative data $(\mathcal{P}', f', g')$ such that

$$\mathcal{P} \subsetneq \mathcal{P}' \subseteq \mathcal{P} \cup \mathcal{R}(G), \quad \mathcal{R}(G') \subsetneq \mathcal{R}(G), \quad \min\{1, \delta'/\delta\} \cdot q(G') \geqslant 2^N q(G),$$

where $N = \#\{p \in \mathcal{P}' \setminus \mathcal{P} : f'(p) \neq g'(p)\}$.

Proposition 8.2 (Iteration when $\mathcal{R}^\flat(G) = \emptyset$)**.** *Let $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ be a GCD graph such that*

$$\mathcal{R}(G) \subseteq \{p > 10^{2000}\}, \quad \mathcal{R}^\flat(G) = \emptyset, \quad \mathcal{R}^\sharp(G) \neq \emptyset.$$

Then there is a GCD subgraph G' of G such that

$$\mathcal{P} \subsetneq \mathcal{P}' \subseteq \mathcal{P} \cup \mathcal{R}(G), \quad \mathcal{R}(G') \subsetneq \mathcal{R}(G), \quad q(G') \geqslant q(G).$$

Propositions 8.1 and 8.2 deal with large primes. We need a complementary result that handles the small primes.

Proposition 8.3 (Bounded quality loss for small primes)**.** *Let $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \emptyset, f_\emptyset, g_\emptyset)$ be a GCD graph with edge density δ and trivial set of primes. Then there is a GCD subgraph $G' = (\mu, \mathcal{V}', \mathcal{W}', \mathcal{E}', \mathcal{P}', f, g)$ of G with edge density δ' such that*

$$\mathcal{P}' \subseteq \{p \leqslant 10^{2000}\}, \quad \mathcal{R}(G') \subseteq \{p > 10^{2000}\}, \quad \min\left\{1, \frac{\delta'}{\delta}\right\} \cdot \frac{q(G')}{q(G)} \geqslant \frac{1}{10^{10^{3000}}}.$$

Finally, we need two further technical estimates. The first one strengthens the quality of the inequality $L_t(v,w) \geqslant 10$ under certain assumptions, whereas the second allows one to pass to a subgraph where all vertices have high degree.

Lemma 8.4 (Removing the effect of $\mathcal{R}(G)$ from $L_t(v,w)$)**.** *Let t be a sufficiently large real. Let $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ be a GCD graph of edge density δ such that*

$$\mathcal{R}^\flat(G) = \emptyset, \quad \delta \geqslant (10/t)^{50}, \quad \mathcal{E} \subseteq \{(v,w) \in \mathcal{V} \times \mathcal{W} : L_t(v,w) \geqslant 10\}.$$

Then there exists a GCD subgraph $G' = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}', \mathcal{P}, f, g)$ of G such that

$$q(G') \geqslant \frac{q(G)}{2} \quad \text{and} \quad \mathcal{E}' \subseteq \left\{(v,w) \in \mathcal{V}' \times \mathcal{W}' : \sum_{\substack{p \mid vw/\gcd(v,w)^2 \\ p \geqslant t,\ p \notin \mathcal{R}(G)}} \frac{1}{p} \geqslant 5\right\}.$$

Lemma 8.5 (Subgraph with high-degree vertices)**.** *Let $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ be a GCD graph of edge density δ. Then there is a GCD subgraph $G' = (\mu, \mathcal{V}', \mathcal{W}', \mathcal{E}', \mathcal{P}, f, g)$ with edge density δ' such that:*

(a) $q(G') \geqslant q(G)$;
(b) $\delta' \geqslant \delta$;

(c) For all $v \in \mathcal{V}'$ and for all $w \in \mathcal{W}'$, we have

$$\mu(\Gamma_{G'}(v)) \geqslant \frac{9\delta'}{10}\mu(\mathcal{W}') \quad \textit{and} \quad \mu(\Gamma_{G'}(w)) \geqslant \frac{9\delta'}{10}\mu(\mathcal{V}').$$

Proof of Proposition 7.1 assuming Propositions 8.1-8.3 and Lemmas 8.4-8.5. We will construct the required subgraph G' in several stages. It suffices to produce a GCD subgraph G' of G satisfying only conclusions (a) and (d) of Proposition 7.1, since an application of Lemma 8.5 then produces a GCD subgraph satisfying all the conclusions.

Stage 1: Obtaining a GCD subgraph $G^{(1)}$ with $\mathcal{R}(G^{(1)}) \subseteq \{p > 10^{2000}\}$.

Since G has set of primes equal to the empy set, we may apply Proposition 8.3 to G to produce a GCD subgraph $G^{(1)} = (\mu, \mathcal{V}^{(1)}, \mathcal{W}^{(1)}, \mathcal{E}^{(1)}, \mathcal{P}^{(1)}, f^{(1)}, g^{(1)})$ of G with edge density $\delta^{(1)}$ and for which

$$\mathcal{R}(G^{(1)}) \subseteq \{p > 10^{2000}\}, \quad q(G^{(1)}) \geqslant \frac{q(G)}{10^{10^{3000}}} \quad \text{and} \quad \delta^{(1)}q(G^{(1)}) \geqslant \frac{\delta \cdot q(G)}{10^{10^{3000}}}. \tag{8.1}$$

In particular, we have

$$\mathcal{R}(H) \subseteq \mathcal{R}(G^{(1)}) \subseteq \{p > 10^{2000}\} \tag{8.2}$$

for any $H \preceq G^{(1)}$.

Stage 2: Obtaining a GCD subgraph $G^{(2)}$ with $\mathcal{R}^\flat(G^{(2)}) = \emptyset$.

If $\mathcal{R}^\flat(G^{(1)}) \neq \emptyset$, then $G^{(1)}$ satisfies the conditions of Proposition 8.1. We then repeatedly apply Proposition 8.1 to produce a sequence of GCD subgraphs of $G^{(1)}$ given by

$$G^{(1)} =: G_1^{(1)} \succeq G_2^{(1)} \succeq \cdots$$

until we obtain a GCD graph $G^{(2)}$ of $G^{(1)}$ which does not satisfy the conditions of Proposition 8.1. Since $\mathcal{R}(G_{i+1}^{(1)}) \subsetneq \mathcal{R}(G_i^{(1)})$ and $\mathcal{R}(G^{(1)})$ is a finite set, this process must terminate. Let us denote by $G^{(2)} = (\mu, \mathcal{V}^{(2)}, \mathcal{W}^{(2)}, \mathcal{E}^{(2)}, \mathcal{P}^{(2)}, f^{(2)}, g^{(2)})$ the graph we obtain at the end of this process. We know that it does not satisfy the conditions of Proposition 8.1. Since $\mathcal{R}(G^{(2)}) \subseteq \{p > 10^{2000}\}$ by (8.2), it must be the case that

$$\mathcal{R}^\flat(G^{(2)}) = \emptyset.$$

In addition, Proposition 8.1 implies that

$$q(G^{(2)}) \geqslant 2^N q(G^{(1)}) \quad \text{and} \quad \delta^{(2)} q(G^{(2)}) \geqslant 2^N \delta^{(1)} q(G^{(1)}),$$

where

$$N = \#\{p \in \mathcal{P}^{(2)} \setminus \mathcal{P}^{(1)} : f^{(2)}(p) \neq g^{(2)}(p)\}.$$

Together with (8.1), this yields that

$$q(G^{(2)}) \geqslant \frac{2^N}{10^{10^{3000}}} \cdot q(G) \quad \text{and} \quad \delta^{(2)} q(G^{(2)}) \geqslant \frac{2^N}{10^{10^{3000}}} \cdot \delta \cdot q(G). \tag{8.3}$$

On the other hand, if $\mathcal{R}^\flat(G^{(1)}) = \emptyset$, then we simply take $G^{(2)} = G^{(1)}$ and note that (8.3) is trivially satisfied by (8.1).

This completes Stage 2. The remaining part of the proof deviates according to whether the ratio $q(G^{(2)})/q(G)$ is larger or smaller than $(t/10)^{50}\delta/10^{10^{3000}}$.

Case (a): $q(G^{(2)})/q(G) \geqslant (t/10)^{50}\delta/10^{10^{3000}}$.

In this case we do not need to keep track of the condition that $L_t(v,w) \geqslant 10$ because we have a very large gain in the quality of the new graph. The next stage of the argument is then:

Stage 3a: Obtaining a GCD subgraph with $\mathcal{R}(G^{(3a)}) = \emptyset$.

Notice if $H \preceq G^{(2)}$, then $\mathcal{R}(H) \subseteq \{p > 10^{2000}\}$ by (8.2). Consequently, if $\mathcal{R}(H) \neq \emptyset$, then either Proposition 8.1 or Proposition 8.2 is applicable to H, thus producing a GCD subgraph H' of H such that

$$\mathcal{R}(H') \subsetneq \mathcal{R}(H) \quad \text{and} \quad q(H') \geqslant q(H). \tag{8.4}$$

Since $\mathcal{R}(G^{(2)})$ is finite, starting with $H_1 = G^{(2)}$ and iterating the above fact, we can construct a finite sequence of GCD graphs

$$G^{(2)} = H_1 \succeq H_2 \succeq \cdots \succeq H_J =: G^{(3a)}$$

such that

$$\mathcal{R}(G^{(3a)}) = \emptyset \quad \text{and} \quad q(G^{(3a)}) \geqslant q(G^{(2)}).$$

Applying the assumption that $q(G^{(2)})/q(G) \geqslant (t/10)^{50}\delta/10^{10^{3000}}$, we infer that

$$q(G^{(3a)}) \geqslant \Big(\frac{t}{10}\Big)^{50} \frac{\delta}{10^{10^{3000}}} \cdot q(G).$$

We can therefore take $G' = G^{(3a)}$ which satisfies condition (a) and condition (d)-(i) of Proposition 7.1, giving the result in this case.

In order to complete the proof of Proposition 6.6, it remains to consider:

Case (b): $q(G^{(2)})/q(G) < (t/10)^{50}\delta/10^{10^{3000}}$.

In this case, the quality increment is not large enough and we must make sure not to lose track of the condition $L_t(v,w) \geqslant 10$. For this reason, we perform some cosmetic surgery to our graph before applying Proposition 8.2. This consists of Stage 3(b) that we present below.

Stage 3b: Removing the effect of primes in $\mathcal{R}(G^{(2)})$ from the anatomical condition $L_t(v,w) \geqslant 10$.

Note that (8.3) implies that

$$\delta^{(2)} \geqslant \frac{\delta}{10^{10^{3000}}} \cdot \frac{q(G)}{q(G^{(2)})} \geqslant \Big(\frac{10}{t}\Big)^{50},$$

and that

$$2^N \leqslant 10^{10^{3000}} \cdot \frac{q(G^{(2)})}{q(G)} \leqslant \Big(\frac{t}{10}\Big)^{50}\delta \leqslant t^{50}, \tag{8.5}$$

where we recall that

$$N = \#\{p \in \mathcal{P}^{(2)} \setminus \mathcal{P}^{(1)} : f^{(2)}(p) \neq g^{(2)}(p)\}$$

(here we used the trivial bound $\delta \leqslant 1$).

Since $\delta^{(2)} \geqslant (10/t)^{50}$, $\mathcal{R}^\flat(G^{(2)}) = \emptyset$, and $L_t(v,w) \geqslant 10$ for all $(v,w) \in \mathcal{E}^{(2)}$, it is the case that $G^{(2)}$ satisfies the conditions of Lemma 8.4. Consequently, there exists a GCD subgraph $G^{(3b)} = (\mu, \mathcal{V}^{(3b)}, \mathcal{W}^{(3b)}, \mathcal{E}^{(3b)}, \mathcal{P}^{(3b)}, f^{(3b)}, g^{(3b)})$ of $G^{(2)}$ with

$$\mathcal{P}^{(3b)} = \mathcal{P}^{(2)}, \tag{8.6}$$

$$q(G^{(3b)}) \geqslant \frac{q(G^{(2)})}{2}, \tag{8.7}$$

and such that

$$\sum_{\substack{p\mid vw/\gcd(v,w)^2\\ p\geqslant t,\ p\notin\mathcal{R}(G^{(2)})}}\frac{1}{p}\geqslant 5\quad\text{whenever}\quad(v,w)\in\mathcal{E}^{(3\mathrm{b})},\tag{8.8}$$

We claim that an inequality of the form (8.8) holds even if we remove from consideration the primes lying in the set

$$\mathcal{P}^{(2)}_{\mathrm{diff}}:=\{p\in\mathcal{P}^{(2)}:\ f^{(2)}(p)\neq g^{(2)}(p)\}.$$

It turns out that we can do this rather crudely, starting from the estimate

$$\sum_{\substack{p\mid vw/\gcd(v,w)^2\\ p\geqslant t,\ p\in\mathcal{P}^{(2)}_{\mathrm{diff}}}}\frac{1}{p}\leqslant\frac{\#(\mathcal{P}^{(2)}_{\mathrm{diff}}\cap\{p\geqslant t\})}{t}.$$

Recalling that $t>10^{2000}$ and $\mathcal{P}^{(1)}\subseteq\{p\leqslant 10^{2000}\}$, we deduce that

$$\sum_{\substack{p\mid vw/\gcd(v,w)^2\\ p\geqslant t,\ p\in\mathcal{P}^{(2)}_{\mathrm{diff}}}}\frac{1}{p}\leqslant\frac{\#(\mathcal{P}^{(2)}_{\mathrm{diff}}\setminus\mathcal{P}^{(1)})}{t}=\frac{N}{t}.\tag{8.9}$$

Since $t\geqslant 10^{2000}$, relation (8.5) implies that $N\leqslant 2\log(t^{50})=100\log t<t$, that is to say the right hand side of (8.9) is $\leqslant 1$. As a consequence,

$$\sum_{\substack{p\mid vw/\gcd(v,w)^2\\ p\geqslant t,\ p\notin\mathcal{R}(G^{(2)})\cup\mathcal{P}^{(2)}_{\mathrm{diff}}}}\frac{1}{p}\geqslant 4\quad\text{whenever}\quad(v,w)\in\mathcal{E}^{(3\mathrm{b})}.\tag{8.10}$$

Having removed the effect to the condition $L_t(v,w)\geqslant 10$ of primes from the sets $\mathcal{R}(G^{(2)})\cup\mathcal{P}^{(2)}_{\mathrm{diff}}$, we are ready to complete the construction of G' in Case (b):

Stage 4b: Obtaining a GCD subgraph with $\mathcal{R}(G^{(4\mathrm{b})})=\emptyset$.

We argue as in Stage 3a: for each $H\preceq G^{(3\mathrm{b})}$, we have $\mathcal{R}(H)\subseteq\{p>10^{2000}\}$ by (8.2). Hence, if $\mathcal{R}(H)\neq\emptyset$, then either Proposition 8.1 or Proposition 8.2 is applicable to H, thus producing a GCD subgraph H' of H such that

$$\mathcal{R}(H')\subsetneq\mathcal{R}(H),\quad\mathcal{P}_{H'}\subseteq\mathcal{R}(H)\cup\mathcal{P}_H,\quad\text{and}\quad q(H')\geqslant q(H),\tag{8.11}$$

where $\mathcal{P}_H$ and $\mathcal{P}_{H'}$ denote the set of primes of H and of H', respectively. Since $\mathcal{R}(G^{(3\mathrm{b})})$ is finite, starting with $H_1=G^{(3\mathrm{b})}$ and iterating the above fact, we can construct a finite sequence of GCD graphs

$$G^{(3\mathrm{b})}=H_1\succeq H_2\succeq\cdots\succeq H_J=:G^{(4\mathrm{b})}$$

such that

$$\mathcal{R}(G^{(4\mathrm{b})})=\emptyset\quad\text{and}\quad q(G^{(4\mathrm{b})})\geqslant q(G^{(3\mathrm{b})}).$$

In addition, note that

$$P^{(4\mathrm{b})}\subseteq\mathcal{R}(G^{(3\mathrm{b})})\cup\mathcal{P}^{(3\mathrm{b})}\subseteq\mathcal{R}(G^{(2)})\cup\mathcal{P}^{(2)},$$

where the second relation follows by fact (8.6) that $\mathcal{P}^{(3\mathrm{b})}=\mathcal{P}^{(2)}$. We now verify that if we let

$$G'=G^{(4b)},$$

then condition (d)-(ii) of Proposition 7.1 is satisfied. Clearly G' satisfies condition (a), and so this suffices for the completion of the proof.

First of all, note that by (8.7), (8.3) and (8.1) and $q(G^{(4b)}) \geqslant q(G^{(3b)})$, we have

$$q(G') = q(G^{(4b)}) \geqslant q(G^{(3b)}) \geqslant \frac{q(G^{(2)})}{2} \geqslant \frac{q(G)}{2 \cdot 10^{10^{3000}}}.$$

Let $(v, w) \in \mathcal{E}^{(4b)}$. It remains to check that $L_t(v', w') \geqslant 4$, where v' and w' are defined by the relations

$$v = v' \prod_{p \in \mathcal{P}^{(4b)}} p^{f^{(4b)}(p)} \quad \text{and} \quad w = w' \prod_{p \in \mathcal{P}^{(4b)}} p^{g^{(4b)}(p)}.$$

By the definition of the set $\mathcal{R}(G^{(4b)})$ and since $\mathcal{R}(G^{4b}) = \emptyset$, all primes factors of $\gcd(v, w)$ belong to $\mathcal{P}^{(4b)}$. But for each prime $p \in \mathcal{P}^{(4b)}$ we have $p^{\min\{f^{(4b)}(p), g^{(4b)}(p)\}} \| \gcd(v, w)$. Thus

$$\gcd(v, w) = \prod_{p \in \mathcal{P}^{(4b)}} p^{\min\{f^{(4b)}(p), g^{(4b)}(p)\}}.$$

In particular, we must have that

$$\gcd(v', w') = 1.$$

Now, let p be a prime such that

$$p \Big| \frac{vw}{\gcd(v, w)^2} \quad \text{and} \quad p \nmid v'w'.$$

Since $p \nmid v'w'$, we have $p \in \mathcal{P}^{(4b)}$, and so $p^{\min\{f^{(4b)}(p), g^{(4b)}(p)\}} \| \gcd(v, w)$. But then p divides $vw/\gcd(v, w)^2$ only if $f^{(4b)}(p) \neq g^{(4b)}(p)$. If $p \in \mathcal{P}^{(2)}$, we infer that $p \in \mathcal{P}^{(2)}_{\text{diff}}$. On the other hand, if $p \notin \mathcal{P}^{(2)}$, then the inclusion $\mathcal{P}^{(4b)} \subseteq \mathcal{P}^{(2)} \cup \mathcal{R}(G^{(2)})$ implies that $p \in \mathcal{R}(G^{(2)})$. In either case, we have that $p \in \mathcal{P}^{(2)}_{\text{diff}} \cup \mathcal{R}(G^{(2)})$. Thus, since $\mathcal{E}^{(4b)} \subseteq \mathcal{E}^{(3b)}$, we may use the bound (8.10), which gives

$$L_t(v', w') = \sum_{\substack{p | v'w'/\gcd(v', w')^2 \\ p \geqslant t}} \frac{1}{p} \geqslant \sum_{\substack{p | vw/\gcd(v, w)^2 \\ p \geqslant t,\ p \notin \mathcal{R}(G^{(2)}) \cup \mathcal{P}^{(2)}_{\text{diff}}}} \frac{1}{p} \geqslant 4.$$

In particular, $G' = G^{(4b)}$ satisfies the conditions of case (d)-(ii) of Proposition 7.1. This completes the proof of Proposition 7.1 in Case (b) too. □

Thus we are left to establish Propositions 8.1-8.3 and Lemmas 8.4-8.5. We begin with Lemma 8.4 and Lemma 8.5 since these are easier to establish.

9. Proof of Lemma 8.4

In this section we establish Lemma 8.4 directly.

For brevity, let

$$S(v, w) = \sum_{\substack{p | vw/\gcd(v, w)^2 \\ p \in \mathcal{R}(G),\ p \geqslant t^{50}}} \frac{1}{p}.$$

We have

$$\sum_{(v,w) \in \mathcal{E}} \mu(v)\mu(w) S(v, w) = \sum_{\substack{p \in \mathcal{R}(G) \\ p \geqslant t^{50}}} \mu(\{(v, w) \in \mathcal{E} : p | vw/\gcd(w, v)^2\})$$

Fix for the moment a prime $p \in \mathcal{R}(G)$. Since we have $\mathcal{R}^\flat(G) = \emptyset$, it must be the case that $p \in \mathcal{R}^\sharp(G)$, that is to say there exists some $k \in \mathbb{Z}_{\geqslant 0}$ such that

$$\mu(\mathcal{V}_{p^k}) \geqslant \Big(1 - \frac{10^{40}}{p}\Big)\mu(\mathcal{V}) \quad \text{and} \quad \mu(\mathcal{W}_{p^k}) \geqslant \Big(1 - \frac{10^{40}}{p}\Big)\mu(\mathcal{W}).$$

Now we note that if $p|vw/\gcd(v,w)^2$, then $p^j\|v$ and $p^\ell\|w$ for some $j \neq \ell$. In particular we cannot have $p^k\|v$ and $p^k\|w$. Thus

$$\begin{aligned}\mu\Big(\Big\{(v,w) \in \mathcal{E} : p|\frac{vw}{\gcd(w,v)^2}\Big\}\Big) &\leqslant \mu((\mathcal{V}\setminus\mathcal{V}_{p^k})\times\mathcal{W}) + \mu(\mathcal{V}\times(\mathcal{W}\setminus\mathcal{W}_{p^k}))\\ &\leqslant 2\cdot\frac{10^{40}}{p}\cdot\mu(\mathcal{V})\mu(\mathcal{W}).\end{aligned}$$

Thus we conclude that

$$\begin{aligned}\sum_{(v,w)\in\mathcal{E}} \mu(v)\mu(w)S(v,w) &\leqslant \sum_{p \geqslant t^{50}} \frac{2\cdot 10^{40}\mu(\mathcal{V})\mu(\mathcal{W})}{p^2}\\ &\leqslant \frac{2\cdot 10^{40}\mu(\mathcal{V})\mu(\mathcal{W})}{t^{50}}\\ &< \frac{\mu(\mathcal{E})}{100},\end{aligned}$$

where in the final line we used the fact that $\delta = \mu(\mathcal{E})/\mu(\mathcal{V})\mu(\mathcal{W}) \geqslant (10/t)^{50}$. We now define

$$\mathcal{E}' := \{(v,w) \in \mathcal{E} : S(v,w) \leqslant 1\}.$$

Markov's inequality implies that

$$\mu(\mathcal{E}\setminus\mathcal{E}') \leqslant \sum_{(v,w)\in\mathcal{E}} \mu(v)\mu(w)S(v,w) < \frac{\mu(\mathcal{E})}{100}.$$

Thus $\mu(\mathcal{E}') \geqslant 99\mu(\mathcal{E})/100$. We then take $G' := (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}', \mathcal{P}, f, g)$ and note that

$$\frac{q(G')}{q(G)} = \Big(\frac{\mu(\mathcal{E}')}{\mu(\mathcal{E})}\Big)^{10} \geqslant \frac{1}{2}.$$

Finally, we note that by Mertens' theorem for t sufficiently large we have

$$\sum_{t \leqslant p \leqslant t^{50}} \frac{1}{p} \leqslant 4,$$

and so if $(v', w') \in \mathcal{E}'$ then

$$\sum_{\substack{p|v'w'/\gcd(v',w')^2\\ p\in\mathcal{R}(G)\\ p\geqslant t}} \frac{1}{p} \leqslant \sum_{\substack{p|v'w'/\gcd(v',w')^2\\ p\in\mathcal{R}(G)\\ p\geqslant t^{50}}} \frac{1}{p} + 4 \leqslant 5.$$

Thus, since $\mathcal{E}' \subseteq \mathcal{E}$ and if $(v,w) \in \mathcal{E}$ then $L_t(v,w) \geqslant 10$, we have for any $(v',w') \in \mathcal{E}'$

$$\sum_{\substack{p|v'w'/\gcd(v',w')^2\\ p\notin\mathcal{R}(G)\\ p\geqslant t}} \frac{1}{p} \geqslant L_t(v',w') - 5 \geqslant 5.$$

This completes the proof of Lemma 8.4. ☐

We are left to establish Propositions 8.1-8.3 and Lemma 8.5.

10. Proof of Lemma 8.5

In this section we establish Lemma 8.5. We begin with an auxiliary lemma.

Lemma 10.1 (Quality increment or all vertices have high degree)**.** *Let $G=(\mu,\mathcal{V},\mathcal{W},\mathcal{E},\mathcal{P},f,g)$ be a GCD graph of edge density δ. For each $v\in\mathcal{V}$ and for each $w\in\mathcal{W}$, we let*

$$\Gamma_G(v):=\{w\in\mathcal{W}:\ (v,w)\in\mathcal{E}\}\quad\text{and}\quad\Gamma_G(w):=\{v\in\mathcal{V}:\ (v,w)\in\mathcal{E}\}$$

be the sets of their neighbours. Then one of the following holds:

(a) For all $v\in\mathcal{V}$ and for all $w\in\mathcal{W}$, we have

$$\mu(\Gamma_G(v))\geqslant\frac{9\delta}{10}\mu(\mathcal{W})\quad\text{and}\quad\mu(\Gamma_G(w))\geqslant\frac{9\delta}{10}\mu(\mathcal{V}).$$

(b) There is a GCD subgraph $G'=(\mu,\mathcal{V}',\mathcal{W}',\mathcal{E}',\mathcal{P},f,g)$ of G with edge density $\delta'>\delta$ and quality $q(G')>q(G)$.

Proof. Assume that (a) fails. Then either its first or its second inequality fails. Assume that the first one fails for some $v\in\mathcal{V}$; the other case is entirely analogous. Let $\mathcal{E}'$ be the set of edges between the vertex sets $\mathcal{V}\setminus\{v\}$ and $\mathcal{W}$. We then consider $G'=(\mu,\mathcal{V}\setminus\{v\},\mathcal{W},\mathcal{E}',\mathcal{P},f,g)$, which is a GCD subgraph of G. We claim that $q(G')>q(G)$.

Indeed, we have

$$\mu(\mathcal{E}')=\mu(\mathcal{E})-\mu(v_1)\mu(\Gamma_G(v))>\delta\mu(\mathcal{V})\mu(\mathcal{W})-\frac{9\delta}{10}\mu(v)\mu(\mathcal{W}).$$

On the one hand, this implies that

$$\mu(\mathcal{E}')>\delta\mu(\mathcal{V})\mu(\mathcal{W})>0.$$

In particular, $\mathcal{E}'\neq\emptyset$. On the other hand, we have that

$$\mu(\mathcal{E}')>\delta\big(\mu(\mathcal{V})-\mu(v)\big)\mu(\mathcal{W})\cdot\Big(1+\frac{\mu(v)/10}{\mu(\mathcal{V})-\mu(v)}\Big).$$

Thus G' has edge density δ' satisfying

$$\delta'=\frac{\mu(\mathcal{E}')}{\mu(\mathcal{V}\setminus\{v\})\mu(\mathcal{W})}=\frac{\mu(\mathcal{E}')}{\big(\mu(\mathcal{V})-\mu(v)\big)\mu(\mathcal{W})}>\delta\cdot\Big(1+\frac{\mu(v)/10}{\mu(\mathcal{V})-\mu(v)}\Big).$$

Thus we see that $\delta'>\delta$, and that

$$(\delta')^{10}\mu(\mathcal{V}\setminus\{v\})\mu(\mathcal{W}')>\delta^{10}\big(\mu(\mathcal{V})-\mu(v)\big)\mu(\mathcal{W})\Big(1+\frac{\mu(v)}{\mu(\mathcal{V})-\mu(v)}\Big)=\delta^{10}\mu(\mathcal{V})\mu(\mathcal{W}).$$

This proves our claim that $q(G')>q(G)$ too, thus completing the proof of the lemma. □

Proof of Lemma 8.5. If G satisfies conclusion (a) of Lemma 8.5, then we are done by taking $G'=G$. We note that conclusion (a) of Lemma 8.5 is the same as conclusion (a) of Lemma 10.1. Thus, if G does not satisfy conclusion (a) of Lemma 8.5, then we may repeatedly apply Lemma 10.1 to produce a sequence of GCD subgraphs

$$G=:G_1\succeq G_2\succeq\cdots$$

until we arrive at a GCD subgraph of G which satisfies conclusion (a) of Lemma 10.1. This process must terminate after a finite number of steps since the edge density δ_i of G_i satisfies $\delta_{i+1}>\delta_i$, and at least one vertex must be removed at each stage for the edge density to increase. Let the process terminate at G_J, which satisfies conclusion (a) of Lemma 10.1. Since $\delta_{i+1} > \delta_i$ and $q(G_{i+1}) > q(G_i)$ by Lemma 10.1, we have that

$$\delta_J > \delta_{J-1} > \cdots > \delta_1 = \delta, \quad \text{and} \quad q(G_J) > q(G_{J-1}) > \cdots > q(G_1) = q(G).$$

Since the multiplicative data is also maintained at each iteration, we see that taking $G' = G_J$ gives the result. $\square$

Thus we are left to establish Propositions 8.1-8.3.

11. Preparatory Lemmas on GCD graphs

Our remaining task is to prove Propositions 8.1-8.3. Before we attack these directly, we establish various preliminary results about GCD graphs in this section, which we will then use in the remaining sections to prove Propositions 8.1-8.3.

Lemma 11.1 (Quality variation for special GCD subgraphs). *Let $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ be a GCD graph, $p \in \mathcal{R}(G) \setminus \mathcal{P}$ and $k, \ell \in \mathbb{Z}_{\geqslant 0}$. If G_{p^k,p^ℓ} is as in Definition 6.3, then G_{p^k,p^ℓ} is a GCD subgraph of G whose quality satisfies the relation*

$$\frac{q(G_{p^k,p^\ell})}{q(G)} = \left(\frac{\mu(\mathcal{E}_{p^k,p^\ell})}{\mu(\mathcal{E})}\right)^{10} \left(\frac{\mu(\mathcal{V})}{\mu(\mathcal{V}_{p^k})}\right)^{9} \left(\frac{\mu(\mathcal{W})}{\mu(\mathcal{W}_{p^\ell})}\right)^{9} \frac{p^{|k-\ell|}}{(1-\mathbb{1}_{k=\ell\geqslant 1}/p)^2(1-1/p^{31/30})^{10}}.$$

Proof. This follows directly from the definitions. $\square$

Lemma 11.2 (One subgraph must have limited quality loss). *Let $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ be a GCD graph with edge density δ, and let $\mathcal{V} = \mathcal{V}_1 \sqcup \cdots \sqcup \mathcal{V}_I$ and $\mathcal{W} = \mathcal{W}_1 \sqcup \cdots \sqcup \mathcal{W}_J$ be partitions of $\mathcal{V}$ and $\mathcal{W}$. Then there is a GCD subgraph $G' = (\mu, \mathcal{V}', \mathcal{W}', \mathcal{E}', \mathcal{P}, f, g)$ of G with edge density δ' such that*

$$q(G') \geqslant \frac{q(G)}{(IJ)^{10}}, \qquad \delta' \geqslant \frac{\delta}{IJ},$$

and with $\mathcal{V}' \in \{\mathcal{V}_1, \dots, \mathcal{V}_I\}$, $\mathcal{W}' \in \{\mathcal{W}_1, \dots, \mathcal{W}_J\}$, and $\mathcal{E}' = \mathcal{E} \cap (\mathcal{V}' \times \mathcal{W}')$.

Proof. For brevity let $\mathcal{E}_{i,j} = \mathcal{E} \cap (\mathcal{V}_i \times \mathcal{W}_j)$ be the edges between $\mathcal{V}_i$ and $\mathcal{W}_j$ for $i \in \{1, \dots, I\}$ and $j \in \{1, \dots, J\}$. Since the partitions of $\mathcal{V}$ and $\mathcal{W}$ induce an partition $\mathcal{E} = \sqcup_{i=1}^I \sqcup_{j=1}^J \mathcal{E}_{i,j}$ of $\mathcal{E}$, we have

$$\mu(\mathcal{E}) = \sum_{i=1}^I \sum_{j=1}^J \mu(\mathcal{E}_{i,j}).$$

Thus, by the pigeonhole principle, there is a choice of i_0 and j_0 such that $\mu(\mathcal{E}_{i_0,j_0}) \geqslant \mu(\mathcal{E})/IJ$. We then let $G' = (\mu, \mathcal{V}_{i_0}, \mathcal{W}_{j_0}, \mathcal{E}_{i_0,j_0}, \mathcal{P}, f, g)$, which is clearly a GCD subgraph of G. We see that

$$\frac{\delta'}{\delta} = \Big(\frac{\mu(\mathcal{E}_{i_0,j_0})}{\mu(\mathcal{E})}\Big)\Big(\frac{\mu(\mathcal{V})}{\mu(\mathcal{V}_{i_0})}\Big)\Big(\frac{\mu(\mathcal{W})}{\mu(\mathcal{W}_{j_0})}\Big) \geqslant \frac{\mu(\mathcal{E}_{i_0,j_0})}{\mu(\mathcal{E})} \geqslant \frac{1}{IJ}$$

and

$$\frac{q(G')}{q(G)} = \left(\frac{\mu(\mathcal{E}_{i_0,j_0})}{\mu(\mathcal{E})}\right)^{10} \left(\frac{\mu(\mathcal{V})}{\mu(\mathcal{V}_{i_0})}\right)^{9} \left(\frac{\mu(\mathcal{W})}{\mu(\mathcal{W}_{j_0})}\right)^{9} \geqslant \left(\frac{\mu(\mathcal{E}_{i_0,j_0})}{\mu(\mathcal{E})}\right)^{10} \geqslant \frac{1}{(IJ)^{10}}.$$

This gives the result. $\square$

Lemma 11.3 (Few edges between unbalanced sets, I)**.** *Let $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ be a GCD graph. Let p be a prime, $r \in \mathbb{Z}_{\geqslant 1}$ and $k \in \mathbb{Z}_{\geqslant 0}$ be such that $p^r > 10^{2000}$ and*[2]

$$\frac{\mu(\mathcal{W}_{p^k})}{\mu(\mathcal{W})} \geqslant 1 - \frac{10^{40}}{p},$$

and set $\mathcal{L}_{k,r} = \{\ell \in \mathbb{Z}_{\geqslant 0} : |\ell - k| \geqslant r+1\}$. If δ_{p^k,p^ℓ} denotes the edge density of the graph G_{p^k,p^ℓ}, then one of the following holds:

(a) There is $\ell \in \mathcal{L}_{k,r}$ such that $q(G_{p^k,p^\ell}) > 2q(G)$ and $\delta_{p^k,p^\ell} q(G_{p^k,p^\ell}) > 2\delta q(G)$.

(b) $\sum_{\ell \in \mathcal{L}_{k,r}} \mu(\mathcal{E}_{p^k,p^\ell}) \leqslant \mu(\mathcal{E})/(4p^{31/30})$.

Proof. Assume that $\sum_{\ell \in \mathcal{L}_{k,r}} \mu(\mathcal{E}_{p^k,p^\ell}) > \mu(\mathcal{E})/(4p^{31/30})$. Then there must exist some $\ell \in \mathcal{L}_{k,r}$ such that

$$\mu(\mathcal{E}_{p^k,p^\ell}) > \frac{\mu(\mathcal{E})}{300 \cdot 2^{|k-\ell|/20} p^{31/30}},$$

where we used that $\sum_{|j| \geqslant 0} 2^{-|j|/20} \leqslant 2/(1-2^{-1/20}) \leqslant 60$. Since $\mu(\mathcal{E}_{p^k,p^\ell}) \leqslant \mu(\mathcal{V}_{p^k})\mu(\mathcal{W}_{p^\ell})$, this certainly implies that $\mu(\mathcal{V}_{p^k}), \mu(\mathcal{W}_{p^\ell}) > 0$. Since $\mu(\mathcal{W}_{p^k}) \geqslant (1 - 10^{40}/p)\mu(\mathcal{W})$, we also have that $\mu(\mathcal{W}_{p^\ell}) \leqslant 10^{40}\mu(\mathcal{W})/p$. Consequently,

$$\begin{aligned}
\frac{q(G_{p^k,p^\ell})}{q(G)} &= \left(\frac{\mu(\mathcal{E}_{p^k,p^\ell})}{\mu(\mathcal{E})}\right)^{10} \left(\frac{\mu(\mathcal{V})}{\mu(\mathcal{V}_{p^k})}\right)^{9} \left(\frac{\mu(\mathcal{W})}{\mu(\mathcal{W}_{p^\ell})}\right)^{9} \frac{p^{|k-\ell|}}{(1-1/p^{31/30})^{10}} \\
&\geqslant \left(\frac{1}{300 \cdot 2^{|k-\ell|/20} p^{31/30}}\right)^{10} \left(\frac{p}{10^{40}}\right)^{9} p^{|k-\ell|} \\
&\geqslant \frac{p^{-4/3}(p/2^{1/2})^{|k-\ell|}}{10^{25} 10^{40 \cdot 9}}.
\end{aligned}$$

Since $|k-\ell| \geqslant r+1 \geqslant r/2 + 3/2$, we have

$$p^{-4/3}(p/2^{1/2})^{|k-\ell|} \geqslant 2^{-3/4}(p/2^{1/2})^{r/2}.$$

In addition, note that $(p/2^{1/2}) \geqslant p^{1/2}$ for all primes. Therefore

$$\frac{q(G_{p^k,p^\ell})}{q(G)} \geqslant \frac{p^{r/4}}{2^{3/4} \cdot 10^{385}} > 2$$

by our assumption that $p^r > 10^{2000}$.

Similarly, we have

$$\begin{aligned}
\frac{\delta_{p^k,p^\ell}}{\delta} \cdot \frac{q(G_{p^k,p^\ell})}{q(G)} &= \left(\frac{\mu(\mathcal{E}_{p^k,p^\ell})}{\mu(\mathcal{E})}\right)^{11} \left(\frac{\mu(\mathcal{V})}{\mu(\mathcal{V}_{p^k})}\right)^{10} \left(\frac{\mu(\mathcal{W})}{\mu(\mathcal{W}_{p^\ell})}\right)^{10} \frac{p^{|k-\ell|}}{(1-1/p^{31/30})^{10}} \\
&\geqslant \left(\frac{1}{300 \cdot 2^{|k-\ell|/20} p^{31/30}}\right)^{11} \left(\frac{p}{10^{40}}\right)^{10} p^{|k-\ell|} \\
&\geqslant \frac{p^{-41/30}(p/2^{11/20})^{|k-\ell|}}{10^{406}} \\
&\geqslant \frac{(p/2^{11/20})^{r/2}}{2^{33/40} \cdot 10^{406}}.
\end{aligned}$$

[2]If $p \leqslant 10^{40}$, this hypothesis is vacuous.

Since $p/2^{11/20} \geqslant p^{9/20}$ and $p^r > 10^{2000}$, we conclude that

$$\frac{\delta_{p^k,p^\ell}}{\delta} \cdot \frac{q(G_{p^k,p^\ell})}{q(G)} > 2.$$

This completes the proof of the lemma. □

The symmetric version of Lemma 11.3 to the above one also clearly holds:

Lemma 11.4 (Few edges between unbalanced sets, II)**.** *Let $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ be a GCD graph. Le p be a prime, $r \in \mathbb{Z}_{\geqslant 1}$ and $\ell \in \mathbb{Z}_{\geqslant 0}$ be such that $p^r > 10^{2000}$ and*

$$\frac{\mu(\mathcal{V}_{p^\ell})}{\mu(\mathcal{V})} \geqslant 1 - \frac{10^{40}}{p},$$

and set $\mathcal{K}_{\ell,r} = \{k \in \mathbb{Z}_{\geqslant 0} : |\ell - k| \geqslant r + 1\}$. If δ_{p^k,p^ℓ} denotes the edge density of the graph G_{p^k,p^ℓ}, then one of the following holds:

(a) There is $k \in \mathcal{K}_{\ell,r}$ such that $q(G_{p^k,p^\ell}) > 2q(G)$ and $\delta_{p^k,p^\ell} q(G_{p^k,p^\ell}) > 2\delta q(G)$.
(b) $\sum_{k \in \mathcal{K}_{\ell,r}} \mu(\mathcal{E}_{p^k,p^\ell}) \leqslant \mu(\mathcal{E})/(4p^{31/30})$.

Lemma 11.5 (Few edges between small sets)**.** *Let $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ be a GCD graph. Let $p > 10^{500}$, $\mathcal{A} \subseteq \mathcal{V}$, and $\mathcal{B} \subseteq \mathcal{W}$ satisfy*

$$\frac{\mu(\mathcal{A})}{\mu(\mathcal{V})} \leqslant \frac{10^{40}}{p} \quad \textit{and} \quad \frac{\mu(\mathcal{B})}{\mu(\mathcal{W})} \leqslant \frac{10^{40}}{p},$$

and let $G' = (\mu, \mathcal{A}, \mathcal{B}, \mathcal{E}', \mathcal{P}, f, g)$, where $\mathcal{E}' = \mathcal{E}(\mathcal{A}, \mathcal{B})$. If δ' denotes the edge density of G', then one of the following holds:

(a) $q(G') > 2q(G)$ and $\delta' q(G') > 2\delta\, q(G)$.
(b) $\mu(\mathcal{E}') \leqslant \mu(\mathcal{E})/(2p^{3/2})$.

Proof. Assume that $\mu(\mathcal{E}') > \mu(\mathcal{E})/(2p^{3/2})$. We then have that

$$\begin{aligned} \frac{q(G')}{q(G)} &= \left(\frac{\mu(\mathcal{E}')}{\mu(\mathcal{E})}\right)^{10} \left(\frac{\mu(\mathcal{V})}{\mu(\mathcal{A})}\right)^{9} \left(\frac{\mu(\mathcal{W})}{\mu(\mathcal{B})}\right)^{9} \\ &> \frac{1}{(2p^{3/2})^{10}} \left(\frac{p}{10^{40}}\right)^{18} \\ &= \frac{p^3}{2^{10} 10^{40\cdot 18}} > 2 \end{aligned}$$

by our assumption that $p > 10^{500}$. Similarly,

$$\begin{aligned} \frac{\delta'}{\delta} \cdot \frac{q(G')}{q(G)} &= \left(\frac{\mu(\mathcal{E}')}{\mu(\mathcal{E})}\right)^{11} \left(\frac{\mu(\mathcal{V})}{\mu(\mathcal{A})}\right)^{10} \left(\frac{\mu(\mathcal{W})}{\mu(\mathcal{B})}\right)^{10} \\ &> \frac{1}{(2p^{3/2})^{11}} \left(\frac{p}{10^{40}}\right)^{20} \\ &= \frac{p^{7/2}}{2^{11} 10^{40\cdot 20}} > 2. \end{aligned}$$

This completes the proof of the lemma. □

12. PROOF OF PROPOSITION 8.1

In this section we prove Proposition 8.1, which is the iteration procedure for 'generic' primes. This section is essentially self-contained (relying only on the notation of Section 6 and the trivial Lemma 11.1), and serves as a template for the proof of the harder Propositions 8.2 and 8.3.

Lemma 12.1 (Bounds on edge sets)**.** *Consider a GCD graph $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ and a prime $p \in \mathcal{R}(G)$. For each $k, \ell \in \mathbb{Z}_{\geqslant 0}$, let*

$$\alpha_k = \frac{\mu(\mathcal{V}_{p^k})}{\mu(\mathcal{V})} \quad \text{and} \quad \beta_\ell = \frac{\mu(\mathcal{W}_{p^\ell})}{\mu(\mathcal{W})}.$$

Then there exist $k, \ell \in \mathbb{Z}_{\geqslant 0}$ such that

$$\frac{\mu(\mathcal{E}_{p^k,p^\ell})}{\mu(\mathcal{E})} \geqslant \begin{cases} (\alpha_k\beta_k)^{9/10} & \text{if } k = \ell, \\ \dfrac{\alpha_k(1-\beta_k) + \beta_k(1-\alpha_k) + \alpha_\ell(1-\beta_\ell) + \beta_\ell(1-\alpha_\ell)}{2^{|k-\ell|/20} \times 1000} & \text{otherwise}. \end{cases}$$

Proof. Assume the claimed inequality does not hold for any k, ℓ. Then, since $\sum_{k,\ell\geqslant 0} \mu(\mathcal{E}_{p^k,p^\ell}) = \mu(\mathcal{E})$, we have

$$1 = \sum_{k,\ell\geqslant 0} \frac{\mu(\mathcal{E}_{p^k,p^\ell})}{\mu(\mathcal{E})} < S_1 + S_2,$$

where

$$S_1 := \sum_{k=0}^{\infty} (\alpha_k\beta_k)^{9/10}$$

and

$$S_2 := \sum_{\substack{k,\ell\geqslant 0\\ k\neq \ell}} \frac{\alpha_k(1-\beta_k) + \beta_k(1-\alpha_k) + \alpha_\ell(1-\beta_\ell) + \beta_\ell(1-\alpha_\ell)}{2^{|k-\ell|/20} \times 1000}.$$

Thus, to arrive at a contradiction, it suffices to show that

$$S_1 + S_2 \leqslant 1.$$

First of all, note that $\sum_{|j|\geqslant 1} 2^{-|j|/20} = 2/(2^{1/20} - 1) \leqslant 100$, whence

$$\begin{aligned} S_2 &\leqslant \frac{1}{10}\Bigg(\sum_{k=0}^{\infty} \alpha_k(1-\beta_k) + \sum_{k=0}^{\infty} \beta_k(1-\alpha_k) + \sum_{\ell=0}^{\infty} \alpha_\ell(1-\beta_\ell) + \sum_{\ell=0}^{\infty} \beta_\ell(1-\alpha_\ell) \Bigg) \\ &= \frac{1}{5}\Bigg(\sum_{k=0}^{\infty} \alpha_k(1-\beta_k) + \sum_{\ell=0}^{\infty} \beta_\ell(1-\alpha_\ell) \Bigg). \end{aligned}$$

Observing that

$$1 - \beta_k = \sum_{\ell\geqslant 0,\ \ell\neq k} \beta_\ell \quad \text{and} \quad 1 - \alpha_\ell = \sum_{k\geqslant 0,\ k\neq \ell} \alpha_k,$$

we conclude that

$$S_2 \leqslant \frac{2}{5} \sum_{\substack{k,\ell\geqslant 0\\ k\neq \ell}} \alpha_k\beta_\ell.$$

Since α_k, β_ℓ are non-negative reals which sum to 1, there exists some $k_0 \geqslant 0$ such that

$$\gamma := \max_{k\geqslant 0} \alpha_k\beta_k = \alpha_{k_0}\beta_{k_0}.$$

We thus find that

$$S_1 = \sum_{k=0}^{\infty}(\alpha_k\beta_k)^{9/10} \leqslant \gamma^{2/5}\sum_{k=0}^{\infty}(\alpha_k\beta_k)^{1/2} \leqslant \gamma^{2/5}\Big(\sum_{k=0}^{\infty}\alpha_k\Big)^{1/2}\Big(\sum_{\ell=0}^{\infty}\beta_\ell\Big)^{1/2} = \gamma^{2/5}$$

where we used the Cauchy-Schwarz inequality to bound $\sum_k(\alpha_k\beta_k)^{1/2}$ from above. We also find that

$$\frac{5S_2}{2} \leqslant \sum_{\substack{k,\ell\geqslant 0\\ k\neq\ell}} \alpha_k\beta_\ell = 1 - \sum_{k=0}^{\infty}\alpha_k\beta_k \leqslant 1-\gamma.$$

As a consequence,

$$S_1 + S_2 \leqslant \gamma^{2/5} + \frac{2}{5}(1-\gamma).$$

The function $x \mapsto x^{2/5} + 2(1-x)/5$ is increasing for $0 \leqslant x \leqslant 1$, and so maximized at $x = 1$. Thus we infer that $S_1 + S_2 \leqslant 1$ as required, completing the proof of the lemma. □

Lemma 12.2 (Quality increment unless a prime power divides almost all)**.** *Consider a GCD graph $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ and a prime $p \in \mathcal{R}(G)$. with $p > 10^{40}$. Then one of the following holds:*

(a) Then there is a GCD subgraph G' of G with multiplicative data $(\mathcal{P}', f', g')$ and edge density δ' such that

$$\mathcal{P}' = \mathcal{P}\cup\{p\}, \quad \mathcal{R}(G') = \mathcal{R}(G)\setminus\{p\}, \quad \min\Big\{1, \frac{\delta'}{\delta}\Big\}\cdot\frac{q(G')}{q(G)} \geqslant 2^{\mathbb{1}_{f'(p)\neq g'(p)}}.$$

(b) There is some $k \in \mathbb{Z}_{\geqslant 0}$ such that

$$\frac{\mu(\mathcal{V}_{p^k})}{\mu(\mathcal{V})} \geqslant 1 - \frac{10^{40}}{p} \quad \text{and} \quad \frac{\mu(\mathcal{W}_{p^k})}{\mu(\mathcal{W})} \geqslant 1 - \frac{10^{40}}{p}.$$

Proof. Let α_k and β_ℓ be defined as in the statement of Lemma 12.1. Consequently, there are $k, \ell \in \mathbb{Z}_{\geqslant 0}$ such that

$$\frac{\mu(\mathcal{E}_{p^k,p^\ell})}{\mu(\mathcal{E})} \geqslant \begin{cases} (\alpha_k\beta_k)^{9/10} & \text{if } k=\ell,\\[2ex] \dfrac{\alpha_k(1-\beta_k)+\beta_k(1-\alpha_k)+\alpha_\ell(1-\beta_\ell)+\beta_\ell(1-\alpha_\ell)}{2^{|k-\ell|/20}\times 1000} & \text{otherwise.}\end{cases}$$

We separate two cases, according to whether $k = \ell$ or not.

Case 1: $k = \ell$

Let $G' = G_{p^k,p^k}$. Lemma 11.1 and our lower bound $\mu(\mathcal{E}_{p^k,p^k}) \geqslant (\alpha_k\beta_k)^{9/10}$ imply that

$$\frac{q(G')}{q(G)} = \Big(\frac{\mu(\mathcal{E}_{p^k,p^k})}{\mu(\mathcal{E})}\Big)^{10}(\alpha_k\beta_k)^{-9}\frac{1}{(1-\mathbb{1}_{k\geq 1}/p)^2(1-1/p^{31/30})^{10}} \geqslant 1.$$

In addition,

$$\frac{\delta'}{\delta} = \frac{\mu(\mathcal{E}_{p^k,p^k})}{\mu(\mathcal{E})}\cdot\frac{\mu(\mathcal{V})\mu(\mathcal{W})}{\mu(\mathcal{V}_{p^k})\mu(\mathcal{W}_{p^k})} \geqslant (\alpha_k\beta_k)^{9/10}\frac{1}{\alpha_k\beta_k} \geqslant 1.$$

This establishes conclusion (a) in this case, noting that $f'(p) = g'(p) = k$ so $\mathbb{1}_{f'(p)\neq g'(p)} = 0$.

Case 2: $k \neq \ell$

As before, we let $G' = G_{p^k,p^\ell}$, and use Lemma 11.1 and our lower bound on $\mathcal{E}_{p^k,p^\ell}$ to find that

$$\frac{q(G')}{q(G)} = \left(\frac{\mu(\mathcal{E}_{p^k,p^\ell})}{\mu(\mathcal{E})}\right)^{10}(\alpha_k\beta_\ell)^{-9}\frac{p^{|k-\ell|}}{(1-1/p^{31/30})^{10}}$$
$$\geqslant \frac{S^{10}}{1000^{10}(\alpha_k\beta_\ell)^9}\cdot\left(\frac{p}{2^{1/2}}\right)^{|k-\ell|},$$

where

$$S = \alpha_k(1-\beta_k) + \beta_k(1-\alpha_k) + \alpha_\ell(1-\beta_\ell) + \beta_\ell(1-\alpha_\ell).$$

In addition, we have

$$\frac{\delta'}{\delta}\cdot\frac{q(G')}{q(G)} = \left(\frac{\mu(\mathcal{E}_{p^k,p^\ell})}{\mu(\mathcal{E})}\right)^{11}(\alpha_k\beta_\ell)^{-10}\frac{p^{|k-\ell|}}{(1-1/p^{31/30})^{10}}$$
$$\geqslant \frac{S^{11}}{1000^{11}(\alpha_k\beta_\ell)^{10}}\cdot\left(\frac{p}{2^{1/2}}\right)^{|k-\ell|}.$$

Note that

$$S \geqslant \alpha_k(1-\beta_k) \geqslant \alpha_k\beta_\ell. \tag{12.1}$$

Indeed, this follows by our assumption that $k \neq \ell$, which implies that $\beta_k + \beta_\ell \leqslant \sum_{j\geqslant 0}\beta_j = 1$. Combining the above, we conclude that

$$\min\left\{\frac{q(G')}{q(G)}, \frac{\delta'}{\delta}\cdot\frac{q(G')}{q(G)}\right\} \geqslant \frac{S^2}{1000^{11}\alpha_k\beta_\ell}\cdot\left(\frac{p}{2^{1/2}}\right)^{|k-\ell|}. \tag{12.2}$$

Now, assume that conclusion (a) of the lemma does not hold, so that the left hand side of (12.2) is $\leqslant 2$. We must then have that

$$S \leqslant \frac{S^2}{\alpha_k\beta_\ell} \leqslant 10^{33}\left(\frac{2^{1/2}}{p}\right)^{|k-\ell|} \leqslant \frac{10^{34}}{p} \leqslant \frac{1}{5},$$

where we used our assumption that $k \neq \ell$ for the second to last inequality, and our assumption that $p \geqslant 10^{40}$ for the last inequality. In particular, this gives

$$S \leqslant \frac{10^{34}}{p} \quad \text{and} \quad \frac{S^2}{\alpha_k\beta_\ell} \leqslant \frac{1}{5}. \tag{12.3}$$

We note that

$$S \geqslant \alpha_k(1-\beta_k) + \beta_\ell(1-\alpha_\ell) \geqslant (\alpha_k+\beta_\ell)(1-\max\{\alpha_\ell,\beta_k\}). \tag{12.4}$$

Thus by the arithmetic-geometric mean inequality, and relations (12.4) and (12.3), we have

$$(1-\max\{\alpha_\ell,\beta_k\})^2 \leqslant \frac{(\alpha_k+\beta_\ell)^2}{4\alpha_k\beta_\ell}(1-\max\{\alpha_\ell,\beta_k\})^2 \leqslant \frac{S^2}{4\alpha_k\beta_\ell} \leqslant \frac{1}{20}.$$

In particular, $\max\{\alpha_\ell,\beta_k\} \geqslant 1/2$.

We consider the case when $\beta_k \geqslant 1/2$; the case with $\alpha_\ell \geqslant 1/2$ is entirely analogous with the roles of β and α swapped, and the roles of k and ℓ swapped. Thus, to complete the proof of the lemma, it suffices to show that

$$\alpha_k, \beta_k \geqslant 1 - \frac{10^{40}}{p}. \tag{12.5}$$

The first inequality of (12.3) states that

$$\alpha_k(1-\beta_k)+\beta_k(1-\alpha_k)+\alpha_\ell(1-\beta_\ell)+\beta_\ell(1-\alpha_\ell)\leqslant\frac{10^{34}}{p}.$$

Since $\beta_k\geqslant 1/2$, we infer that

$$1-\alpha_k\leqslant 2\beta_k(1-\alpha_k)\leqslant\frac{2\cdot 10^{34}}{p}\leqslant\frac{10^{35}}{p}.$$

In particular, $\alpha_k\geqslant 1-10^{40}/p$ and $\alpha_k\geqslant 1/2$, whence

$$1-\beta_k\leqslant 2\alpha_k(1-\beta_k)\leqslant\frac{2\cdot 10^{34}}{p}\leqslant\frac{10^{40}}{p}.$$

This completes the proof of (12.5) and hence of the lemma. □

Proof of Proposition 8.1. This follows almost immediately from Lemma 12.2. Since $\mathcal{R}(G)\subseteq\{p>10^{2000}\}$ by assumption, if $p\in\mathcal{R}(G)$ then $p>10^{2000}$. We have also assumed that $\mathcal{R}^\flat(G)\neq\emptyset$. Consequently, there is a prime $p\in\mathcal{R}^\flat(G)$ with $p>10^{2000}>10^{40}$. We now apply Lemma 12.2 with this choice of p. By definition of $\mathcal{R}^\flat(G)$, conclusion (b) cannot hold, and so conclusion (a) must hold. This then gives the result. □

We are left to establish Proposition 8.3 and Proposition 8.2.

13. Proof of Proposition 8.3

In this section we prove Proposition 8.3, which is the iteration procedure for small primes. This section relies on the notation of Section 6, Lemma 10.1, the Lemmas 11.1-11.3 from Section 11 and Lemma 12.2. The basic idea of the proof is similar to that of Proposition 8.1, but we can no longer ensure a quality increment when the primes are small; instead we show that there is only a bounded loss.

Lemma 13.1 (Small quality loss or prime power divides positive proportion)**.** *Consider a GCD graph $G=(\mu,\mathcal{V},\mathcal{W},\mathcal{E},\mathcal{P},f,g)$ and a prime $p\in\mathcal{R}(G)$. Then one of the following holds:*

(a) There is a GCD subgraph G' of G with multiplicative data $(\mathcal{P}',f',g')$ and edge density δ' such that

$$\mathcal{P}'=\mathcal{P}\cup\{p\},\quad \mathcal{R}(G')=\mathcal{R}(G)\setminus\{p\},\quad \min\left\{1,\frac{\delta'}{\delta}\right\}\cdot\frac{q(G')}{q(G)}\geqslant\frac{1}{10^{40}}.$$

(b) There is some $k\in\mathbb{Z}_{\geqslant 0}$ such that

$$\frac{\mu(\mathcal{V}_{p^k})}{\mu(\mathcal{V})}\geqslant\frac{9}{10}\quad\textit{and}\quad\frac{\mu(\mathcal{W}_{p^k})}{\mu(\mathcal{W})}\geqslant\frac{9}{10}.$$

Proof. Assume that conclusion (a) does not hold, so we intend to establish (b). Let $\mu(\mathcal{V}_{p^j})=\alpha_j\mu(\mathcal{V})$ and $\mu(\mathcal{W}_{p^\ell})=\beta_\ell\mu(\mathcal{W})$. We begin by an identical argument to that in the proof of Lemma 12.2 leading up to (12.2). (We note that this argument requires no assumption on the size of p.) Since conclusion (a) does not hold, we find that there are non-negative integers $k\neq\ell$ such that

$$\frac{1}{10^{40}}\geqslant\min\left\{\frac{q(G')}{q(G)},\frac{\delta'}{\delta}\cdot\frac{q(G')}{q(G)}\right\}\geqslant\frac{S^2}{1000^{11}\alpha_k\beta_\ell}\cdot\left(\frac{p}{2^{1/2}}\right)^{|k-\ell|}\geqslant\frac{S^2}{1000^{11}\alpha_k\beta_\ell},$$

where $G'=G_{p^k,p^\ell}$, and

$$S=\alpha_k(1-\beta_k)+\beta_k(1-\alpha_k)+\alpha_\ell(1-\beta_\ell)+\beta_\ell(1-\alpha_\ell).$$

Therefore we have that

$$S \leqslant \frac{S^2}{\alpha_k \beta_\ell} \leqslant \frac{1}{10^7}.$$

Since $S \geqslant (\alpha_k + \beta_\ell)(1 - \max\{\alpha_\ell, \beta_k\})$, we have

$$(1 - \max\{\alpha_\ell, \beta_k\})^2 \leqslant \frac{(\alpha_k + \beta_\ell)^2}{4\alpha_k \beta_\ell}(1 - \max\{\alpha_\ell, \beta_k\})^2 \leqslant \frac{S^2}{4\alpha_k \beta_\ell} \leqslant \frac{1}{10},$$

so $\max\{\alpha_\ell, \beta_k\} \geqslant 9/10$. We deal with the case when $\beta_k \geqslant 9/10$; the case with $\alpha_\ell \geqslant 9/10$ is entirely analogous with the roles of k and ℓ, and the roles of α and β swapped.

Since $\beta_k \geqslant 9/10$, we have

$$1 - \alpha_k \leqslant 2\beta_k(1 - \alpha_k) \leqslant 2S \leqslant \frac{2}{10^7}$$

In particular, $\alpha_k \geqslant 9/10$. Therefore

$$1 - \beta_k \leqslant 2\alpha_k(1 - \beta_k) \leqslant 2S \leqslant \frac{2}{10^7}.$$

Thus, we have proven that $\alpha_k, \beta_k \geqslant 9/10$, as needed. □

Lemma 13.2 (Adding small primes to $\mathcal{P}$)**.** *Let $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ be a GCD graph with edge density δ. Let $p \in \mathcal{R}(G)$ be a prime with $p \leqslant 10^{2000}$.*

Then there is a GCD subgraph G' of G with set of primes $\mathcal{P}'$ and edge density δ' such that

$$\mathcal{P}' = \mathcal{P} \cup \{p\}, \quad \mathcal{R}(G') \subseteq \mathcal{R}(G) \setminus \{p\}, \quad \min\left\{1, \frac{\delta'}{\delta}\right\} \cdot \frac{q(G')}{q(G)} \geqslant \frac{1}{10^{50}}.$$

Proof. We first repeatedly apply Lemma 10.1 until we arrive at a GCD subgraph

$$G^{(1)} = (\mu, \mathcal{V}^{(1)}, \mathcal{W}^{(1)}, \mathcal{E}^{(1)}, \mathcal{P}, f, g)$$

of G with edge density $\delta^{(1)}$ such that

$$\delta^{(1)} \geqslant \delta \quad \text{and} \quad q(G^{(1)}) \geqslant q(G),$$

as well as

$$\mu(\Gamma_{G^{(1)}}(v)) \geqslant \frac{9\delta^{(1)}}{10} \cdot \mu(\mathcal{W}^{(1)}) \quad \text{for all} \quad v \in \mathcal{V}^{(1)}.$$

(We must eventually arrive at such a subgraph since the vertex sets are strictly decreasing at each stage but can never become empty.)

We now apply Lemma 13.1 to $G^{(1)}$. If conclusion (a) of Lemma 13.1 holds, then there is a GCD subgraph $G^{(2)}$ of $G^{(1)}$ satisfying the required conditions, so we are done by taking $G' = G^{(2)}$. Therefore we may assume that instead conclusion (b) of Lemma 13.1 holds, so there is a $k \in \mathbb{Z}_{\geqslant 0}$ such that

$$\frac{\mu(\mathcal{V}^{(1)}_{p^k})}{\mu(\mathcal{V}^{(1)})} \geqslant \frac{9}{10} \quad \text{and} \quad \frac{\mu(\mathcal{W}^{(1)}_{p^k})}{\mu(\mathcal{W}^{(1)})} \geqslant \frac{9}{10}.$$

We now would like to apply Lemma 11.3. If $p \leqslant 10^{40}$ then we certainly have $\mu(\mathcal{V}^{(1)}_{p^k})/\mu(\mathcal{V}^{(1)}) \geqslant 1 - 10^{40}/p$ (and similarly for $\mathcal{W}^{(1)}_{p^k}$). If instead $p > 10^{40}$ then we apply Lemma 12.2. If conclusion (a) of Lemma 12.2 holds, then there is a GCD subgraph $G^{(3)}$ of $G^{(1)}$ satisfying the required conditions, so we are done by taking $G' = G^{(3)}$. Therefore we may assume that conclusion (b) of Lemma 12.2 holds, and so regardless of the size of p we have that

$$\frac{\mu(\mathcal{V}_{p^k}^{(1)})}{\mu(\mathcal{V}^{(1)})} \geqslant \max\Big(\frac{9}{10}, 1 - \frac{10^{40}}{p}\Big) \quad \text{and} \quad \frac{\mu(\mathcal{W}_{p^k}^{(1)})}{\mu(\mathcal{W}^{(1)})} \geqslant \max\Big(\frac{9}{10}, 1 - \frac{10^{40}}{p}\Big).$$

Let $r \leqslant 6644$ be such that $p^r > 10^{2000}$ (such an integer exists because $2^{6644} > 10^{2000}$). With this choice of parameters we may now apply Lemma 11.3. We separate two cases.

If conclusion (a) of Lemma 11.3 holds, then we take $G' = G^{(1)}_{p^k, p^\ell}$, whose quality satisfies

$$q(G') \geqslant 2q(G^{(1)}) \geqslant 2q(G)$$

and whose edge density δ' satisfies

$$\delta' q(G') \geqslant 2\delta^{(1)} q(G^{(1)}) \geqslant 2\delta q(G).$$

This completes the proof in this case.

Thus we may assume that conclusion (b) of Lemma 11.3 holds, so that

$$\sum_{\ell \in \mathcal{L}_{k,r}} \mu(\mathcal{E}^{(1)}_{p^k, p^\ell}) \leqslant \frac{\mu(\mathcal{E}^{(1)})}{4p^{31/30}} < \frac{\mu(\mathcal{E}^{(1)})}{4},$$

where we recall the notation $\mathcal{L}_{k,r} := \{\ell \in \mathbb{Z}_{\geqslant 0} : |\ell - k| \geqslant r + 1\}$. Let

$$\widetilde{\mathcal{W}}^{(1)} = \bigcup_{\substack{\ell \geqslant 0 \\ |\ell - k| \leqslant r}} \mathcal{W}^{(1)}_{p^\ell}$$

and let

$$\mathcal{E}^{(2)} = \mathcal{E}^{(1)} \cap (\mathcal{V}^{(1)}_{p^k} \times \widetilde{\mathcal{W}}^{(1)}) \subseteq \mathcal{E}^{(1)}$$

be the set of edges between $\mathcal{V}^{(1)}_{p^k}$ and $\widetilde{\mathcal{W}}^{(1)}$ in $G^{(1)}$. Since $\mu(\mathcal{V}^{(1)}_{p^k}) \geqslant 9\mu(\mathcal{V}^{(1)})/10$ and $\mu(\Gamma_{G^{(1)}}(v)) \geqslant 9\delta^{(1)}\mu(\mathcal{W}^{(1)})/10$ for all $v \in \mathcal{V}^{(1)}_{p^k}$, we have

$$\begin{aligned}
\mu(\mathcal{E}^{(2)}) \geqslant \mu(\mathcal{E}^{(1)} \cap (\mathcal{V}^{(1)}_{p^k} \times \mathcal{W}^{(1)})) - \sum_{\ell \in \mathcal{L}_{k,r}} \mu(\mathcal{E}^{(1)}_{p^k, p^\ell}) &\geqslant \sum_{v \in \mathcal{V}^{(1)}_{p^k}} \mu(v)\mu(\Gamma_{G^{(1)}}(v)) - \frac{\mu(\mathcal{E}^{(1)})}{4} \\
&\geqslant \frac{9\delta^{(1)}}{10}\mu(\mathcal{V}^{(1)}_{p^k})\mu(\mathcal{W}^{(1)}) - \frac{\mu(\mathcal{E}^{(1)})}{4} \\
&\geqslant \frac{56}{100}\mu(\mathcal{E}^{(1)}).
\end{aligned}$$

Let $G^{(2)} = (\mu, \mathcal{V}^{(1)}_{p^k}, \widetilde{\mathcal{W}}^{(1)}, \mathcal{E}^{(2)}, \mathcal{P}, f, g)$ be the GCD subgraph of $G^{(1)}$ formed by restricting to $\mathcal{V}^{(1)}_{p^k}$ and $\widetilde{\mathcal{W}}^{(1)}$. If $\delta^{(2)}$ denotes its edge density, then

$$\frac{\delta^{(2)}}{\delta^{(1)}} = \left(\frac{\mu(\mathcal{E}^{(2)})}{\mu(\mathcal{E}^{(1)})}\right)\left(\frac{\mu(\mathcal{V}^{(1)})}{\mu(\mathcal{V}^{(1)}_{p^k})}\right)\left(\frac{\mu(\mathcal{W}^{(1)})}{\mu(\widetilde{\mathcal{W}}^{(1)})}\right) \geqslant \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}.$$

In addition, we have that

$$\frac{q(G^{(2)})}{q(G^{(1)})} = \left(\frac{\mu(\mathcal{E}^{(2)})}{\mu(\mathcal{E}^{(1)})}\right)^{10}\left(\frac{\mu(\mathcal{V}^{(1)})}{\mu(\mathcal{V}^{(1)}_{p^k})}\right)^{9}\left(\frac{\mu(\mathcal{W}^{(1)})}{\mu(\widetilde{\mathcal{W}}^{(1)})}\right)^{9} \geqslant \left(\frac{1}{2}\right)^{10} \cdot 1^9 \cdot 1^9 = \frac{1}{2^{10}}.$$

Finally, we apply Lemma 11.2 to the partition

$$\widetilde{\mathcal{W}}^{(1)} = \bigsqcup_{|\ell-k|\leqslant r} \mathcal{W}^{(1)}_{p^\ell}$$

of $\widetilde{\mathcal{W}}^{(1)}$ into $\leqslant 2\cdot 6644+1 \leqslant 15000$ subsets. This produces a GCD subgraph

$$G^{(3)} = (\mu, \mathcal{V}^{(1)}_{p^k}, \mathcal{W}^{(1)}_{p^\ell}, \mathcal{E}^{(1)}_{p^k,p^\ell}, \mathcal{P}, f, g)$$

of $G^{(2)}$ for some $\ell \geqslant 0$ with $|\ell - k| \leqslant r$ such that

$$q(G^{(3)}) \geqslant \frac{q(G^{(2)})}{15000^{10}} \geqslant \frac{q(G^{(1)})}{15000^{10}\cdot 2^{10}} \geqslant \frac{q(G)}{10^{50}}.$$

In addition, Lemma 11.2 implies that the density of $G^{(3)}$, call it $\delta^{(3)}$, satisfies

$$\delta^{(3)} q(G^{(3)}) \geqslant \frac{\delta^{(2)}}{15000}\cdot\frac{q(G^{(2)})}{15000^{10}} \geqslant \frac{\delta^{(1)}}{15000\cdot 2}\cdot\frac{q(G^{(1)})}{15000^{10}\cdot 2^{10}} \geqslant \frac{\delta\, q(G)}{10^{50}}.$$

Finally, we note that $G^{(2)}_{p^k,p^\ell}$ is a GCD subgraph of $G^{(3)}$ with set of primes $\mathcal{P}\cup\{p\}$ and $q(G^{(2)}_{p^k,p^\ell}) \geqslant q(G^{(3)})$. Taking $G' = G^{(2)}_{p^k,p^\ell}$ then gives the result. □

Proof of Proposition 8.3. If $\mathcal{R}(G)\cap\{p\leqslant 10^{2000}\}=\emptyset$, then we can simply take $G' = G$.

If $\mathcal{R}(G)\cap\{p\leqslant 10^{2000}\}\neq\emptyset$, then we can choose a prime $p\in\mathcal{R}(G)\cap\{p\leqslant 10^{2000}\}$ and apply Lemma 13.2. We do this repeatedly to produce a sequence of GCD subgraphs

$$G =: G_1 \succeq G_2 \succeq \cdots .$$

At each stage $\mathcal{R}(G_{i+1})\cap\{p\leqslant 10^{2000}\} \subsetneq \mathcal{R}(G_i)\cap\{p\leqslant 10^{2000}\}$ is strictly decreasing, so after at most 10^{2000} steps we arrive at a GCD subgraph $G^{(1)} = (\mu, \mathcal{V}^{(1)}, \mathcal{W}^{(1)}, \mathcal{E}^{(1)}, \mathcal{P}^{(1)}, f^{(1)}, g^{(1)})$ of G with

$$\mathcal{R}(G^{(1)})\cap\{p\leqslant 10^{2000}\}=\emptyset.$$

Let G_i have set of primes $\mathcal{P}_i$. Then we see that

$$\mathcal{P}_i\subseteq\mathcal{P}_{i+1} \quad\text{and}\quad \mathcal{P}_{i+1}\cup\mathcal{R}(G_{i+1})\subseteq\mathcal{P}_i\cup\mathcal{R}(G_i),$$

so

$$\mathcal{P}\subseteq\mathcal{P}^{(1)}\subseteq\mathcal{P}\cup\mathcal{R}(G).$$

Writing δ_i for the edge density of G_i, we have that

$$\frac{q(G_{i+1})}{q(G_i)}\geqslant\frac{1}{10^{50}} \quad\text{and}\quad \frac{\delta_{i+1}q(G_{i+1})}{\delta_i q(G_i)}\geqslant\frac{1}{10^{50}}$$

for each i. As a consequence, if $\delta^{(1)}$ denotes the edge density of the end graph $G^{(1)}$, we find that

$$\frac{q(G^{(1)})}{q(G)}\geqslant\frac{1}{(10^{50})^{10^{2000}}}\geqslant\frac{1}{10^{10^{3000}}} \quad\text{and}\quad \frac{\delta^{(1)}q(G^{(1)})}{\delta\, q(G)}\geqslant\frac{1}{(10^{50})^{10^{2000}}}\geqslant\frac{1}{10^{10^{3000}}}.$$

Thus, taking $G' = G^{(1)}$ gives the result. □

Thus we are just left to establish Proposition 8.2.

14. Proof of Proposition 8.2

Finally, in this section we prove Proposition 8.2, and hence complete the proof of Theorem 1. The proof is similar to that of Proposition 8.1, but more care is required when dealing with the primes coming from $\mathcal{R}^\sharp(G)$.

Lemma 14.1 (Quality increment when a prime power divides almost all)**.** *Consider the GCD graph $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$, a prime $p \in \mathcal{R}(G)$ with $p \geqslant 10^{2000}$, and an integer $k \in \mathbb{Z}_{\geqslant 0}$ such that*

$$\frac{\mu(\mathcal{V}_{p^k})}{\mu(\mathcal{V})} \geqslant 1 - \frac{10^{40}}{p} \quad \textit{and} \quad \frac{\mu(\mathcal{W}_{p^k})}{\mu(\mathcal{W})} \geqslant 1 - \frac{10^{40}}{p}.$$

Then there is a GCD subgraph G' of G with set of primes $\mathcal{P}' = \mathcal{P} \cup \{p\}$ such that

$$\mathcal{R}(G') \subseteq \mathcal{R}(G) \setminus \{p\}, \quad q(G') \geqslant q(G).$$

Proof. Set

$$\widetilde{\mathcal{V}}_{p^k} = \mathcal{V}_{p^{k-1}} \cup \mathcal{V}_{p^k} \cup \mathcal{V}_{p^{k+1}} \quad \text{and} \quad \widetilde{\mathcal{W}}_{p^k} = \mathcal{W}_{p^{k-1}} \cup \mathcal{W}_{p^k} \cup \mathcal{W}_{p^{k+1}},$$

with the convention that $\mathcal{V}_{p^{-1}} = \emptyset = \mathcal{W}_{p^{-1}}$. In view of Lemmas 11.5, 11.3 and 11.4, we may assume that

$$\mu(\mathcal{E}(\mathcal{V} \setminus \mathcal{V}_{p^k}, \mathcal{W} \setminus \mathcal{W}_{p^k})) \leqslant \frac{\mu(\mathcal{E})}{2p^{3/2}},$$

$$\mu\big(\mathcal{E}(\mathcal{V} \setminus \widetilde{\mathcal{V}}_{p^k}, \mathcal{W}_{p^k})\big) = \sum_{\substack{i \geqslant 0 \\ |i-k| \geqslant 2}} \mu(\mathcal{E}(\mathcal{V}_{p^i}, \mathcal{W}_{p^k})) \leqslant \frac{\mu(\mathcal{E})}{4p^{31/30}},$$

and

$$\mu\big(\mathcal{E}(\mathcal{V}_{p^k}, \mathcal{W} \setminus \widetilde{\mathcal{W}}_{p^k})\big) = \sum_{\substack{j \geqslant 0 \\ |j-k| \geqslant 2}} \mu(\mathcal{E}(\mathcal{V}_{p^k}, \mathcal{W}_{p^j})) \leqslant \frac{\mu(\mathcal{E})}{4p^{31/30}}.$$

Hence, if we let

$$\mathcal{E}^* = \mathcal{E}(\mathcal{V}_{p^k}, \widetilde{\mathcal{W}}_{p^k}) \cup \mathcal{E}(\widetilde{\mathcal{V}}_{p^k}, \mathcal{W}_{p^k}),$$

then

$$\frac{\mu(\mathcal{E}^*)}{\mu(\mathcal{E})} \geqslant 1 - \frac{1}{2p^{31/30}} - \frac{1}{2p^{3/2}} \geqslant \Big(1 - \frac{1}{p^{31/30}}\Big)^{2/3},$$

where we used our assumption that $p > 10^{2000}$ and the inequality $(1-x)^{2/3} \leqslant 1 - 2x/3$ for $x \in [0,1]$ that follows from Taylor's theorem. We then consider the GCD subgraph $G^* = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}^*, \mathcal{P}, f, g)$ of G formed by restricting the edge set to $\mathcal{E}^*$. Note that

$$\frac{q(G^*)}{q(G)} = \Big(\frac{\mu(\mathcal{E}^*)}{\mu(\mathcal{E})}\Big)^{10} \geqslant \Big(1 - \frac{1}{p^{31/30}}\Big)^{20/3}. \tag{14.1}$$

Now, let $(v,w) \in \mathcal{E}^*$. We have the following five possibilities:

(a) $v \in \mathcal{V}_{p^k}$ and $w \in \mathcal{W}_{p^k}$, in which case $p^k \| v, w$ and $p^k \| \gcd(v,w)$;
(b) $v \in \mathcal{V}_{p^k}$ and $w \in \mathcal{W}_{p^{k+1}}$, in which case $p^k | v, w$ and $p^k \| \gcd(v,w)$;
(c) $v \in \mathcal{V}_{p^{k+1}}$ and $w \in \mathcal{W}_{p^k}$, in which case $p^k | v, w$ and $p^k \| \gcd(v,w)$;
(d) $v \in \mathcal{V}_{p^k}$ and $w \in \mathcal{W}_{p^{k-1}}$, in which case $p^k \| v$, $p^{k-1} \| w$ and $p^{k-1} \| \gcd(v,w)$;
(e) $v \in \mathcal{V}_{p^{k-1}}$ and $w \in \mathcal{W}_{p^k}$, in which case $p^{k-1} \| v$, $p^k \| w$ and $p^{k-1} \| \gcd(v,w)$.

Now, let $G^+ = (\mu, \mathcal{V}^+, \mathcal{W}^+, \mathcal{E}^+, \mathcal{P}\cup\{p\}, f^+, g^+)$, where:

$$\mathcal{V}^+ = \mathcal{V}_{p^k}\cup\mathcal{V}_{p^{k+1}}, \quad \mathcal{W}^+ = \mathcal{W}_{p^k}\cup\mathcal{W}_{p^{k+1}}, \quad \mathcal{E}^+ = \mathcal{E}^*\cap(\mathcal{V}^+\times\mathcal{W}^+),$$

as well as

$$f^+\big|_{\mathcal{P}} = f, \quad f^+(p) = k, \quad g^+\big|_{\mathcal{P}} = g, \quad g^+(p) = k.$$

It is easy to check that G^+ is a GCD subgraph of G^* (and hence of G). In addition, its quality satisfies the relation

$$\frac{q(G^+)}{q(G^*)} = \left(\frac{\mu(\mathcal{E}^+)}{\mu(\mathcal{E}^*)}\right)^{10}\left(\frac{\mu(\mathcal{V}^*)}{\mu(\mathcal{V}^+)}\right)^{9}\left(\frac{\mu(\mathcal{W}^*)}{\mu(\mathcal{W}^+)}\right)^{9}\frac{1}{(1-\mathbb{1}_{k\geqslant 1}/p)^2(1-1/p^{31/30})^{10}}.$$

We separate two cases.

Case 1: $k=0$.

In this case $\mathcal{V}_{p^{k-1}} = \mathcal{W}_{p^{k-1}} = \emptyset$, so all parameters of G^+ are the same as those of G^* except that the set of primes of G^+ is $\mathcal{P}\cup\{p\}$ instead of $\mathcal{P}$ and f,g have been extended to take the value 0 at p. As a consequence,

$$\frac{q(G^+)}{q(G^*)} = \frac{1}{(1-1/p^{31/30})^{10}}.$$

In particular, by (14.1) we have

$$q(G^+) = \frac{q(G^*)}{(1-1/p^{31/30})^{10}} \geqslant \frac{(1-1/p^{31/30})^{20/3}}{(1-1/p^{31/30})^{10}}q(G) \geqslant q(G).$$

Thus the lemma follows by taking $G' = G^+$.

Case 2: $k\geqslant 1$.

We then have that

$$\frac{q(G^+)}{q(G^*)} = \left(\frac{\mu(\mathcal{E}^+)}{\mu(\mathcal{E}^*)}\right)^{10}\left(\frac{\mu(\mathcal{V})}{\mu(\mathcal{V}^+)}\right)^{9}\left(\frac{\mu(\mathcal{W})}{\mu(\mathcal{W}^+)}\right)^{9}\frac{1}{(1-1/p)^2(1-1/p^{31/30})^{10}}.$$

We also consider the GCD subgraphs $G_{p^k,p^{k-1}}$ and G_{p^{k-1},p^k} of G. By Lemma 11.1, we have

$$\frac{q(G_{p^k,p^{k-1}})}{q(G^*)} = \left(\frac{\mu(\mathcal{E}_{p^k,p^{k-1}})}{\mu(\mathcal{E}^*)}\right)^{10}\left(\frac{\mu(\mathcal{V})}{\mu(\mathcal{V}_{p^k})}\right)^{9}\left(\frac{\mu(\mathcal{W})}{\mu(\mathcal{W}_{p^{k-1}})}\right)^{9}\frac{p}{(1-1/p^{31/30})^{10}}$$

and

$$\frac{q(G_{p^{k-1},p^k})}{q(G^*)} = \left(\frac{\mu(\mathcal{E}_{p^{k-1},p^k})}{\mu(\mathcal{E}^*)}\right)^{10}\left(\frac{\mu(\mathcal{V})}{\mu(\mathcal{V}_{p^{k-1}})}\right)^{9}\left(\frac{\mu(\mathcal{W})}{\mu(\mathcal{W}_{p^k})}\right)^{9}\frac{p}{(1-1/p^{31/30})^{10}}.$$

Since $\mu(\mathcal{V}_{p^k}) \geqslant (1-10^{40}/p)\mu(\mathcal{V})$, we have that $\mu(\mathcal{V}_{p^{k-1}}) \leqslant 10^{40}\mu(\mathcal{V})/p$. Similarly, we have that $\mu(\mathcal{W}_{p^{k-1}}) \leqslant 10^{40}\mu(\mathcal{W})/p$. To this end, let $0\leqslant A, B\leqslant 10^{40}$ be such that

$$\frac{\mu(\mathcal{V}_{p^{k-1}})}{\mu(\mathcal{V})} = \frac{A}{p} \quad\text{and}\quad \frac{\mu(\mathcal{W}_{p^{k-1}})}{\mu(\mathcal{W})} = \frac{B}{p}.$$

We note that this implies that

$$\frac{\mu(\mathcal{V}^+)}{\mu(\mathcal{V})} \leqslant 1-\frac{A}{p} \quad\text{and}\quad \frac{\mu(\mathcal{W}^+)}{\mu(\mathcal{W})} \leqslant 1-\frac{B}{p}.$$

Together with (14.1), this gives

$$\frac{q(G^+)}{q(G)} \geqslant \left(\frac{\mu(\mathcal{E}^+)}{\mu(\mathcal{E}^*)}\right)^{10} \frac{1}{(1-A/p)^9(1-B/p)^9(1-1/p)^2(1-1/p^{31/30})^{10/3}},$$
$$\frac{q(G_{p^k,p^{k-1}})}{q(G)} \geqslant \left(\frac{\mu(\mathcal{E}_{p^k,p^{k-1}})}{\mu(\mathcal{E}^*)}\right)^{10} \frac{p^{10}}{B^9(1-1/p^{31/30})^{10/3}},$$
$$\frac{q(G_{p^{k-1},p^k})}{q(G)} \geqslant \left(\frac{\mu(\mathcal{E}_{p^{k-1},p^k})}{\mu(\mathcal{E}^*)}\right)^{10} \frac{p^{10}}{A^9(1-1/p^{31/30})^{10/3}}.$$

Therefore, the lemma will follow if we can show that one of the following inequalities holds:

$$\frac{\mu(\mathcal{E}^+)}{\mu(\mathcal{E}^*)} \geqslant (1-A/p)^{9/10}(1-B/p)^{9/10}(1-1/p)^{2/10}(1-1/p^{31/30})^{1/3};$$
$$\frac{\mu(\mathcal{E}_{p^k,p^{k-1}})}{\mu(\mathcal{E}^*)} \geqslant \frac{B^{9/10}(1-1/p^{31/30})^{1/3}}{p};$$
$$\frac{\mu(\mathcal{E}_{p^{k-1},p^k})}{\mu(\mathcal{E}^*)} \geqslant \frac{A^{9/10}(1-1/p^{31/30})^{1/3}}{p}.$$

Since $\mu(\mathcal{E}^+) + \mu(\mathcal{E}_{p^k,p^{k-1}}) + \mu(\mathcal{E}_{p^{k-1},p^k}) = \mu(\mathcal{E}^*)$, it suffices to prove that

$$S := \left((1-A/p)^{\frac{9}{10}}(1-B/p)^{\frac{9}{10}}(1-1/p)^{\frac{1}{5}} + \frac{B^{\frac{9}{10}}}{p} + \frac{A^{\frac{9}{10}}}{p}\right)\left(1-\frac{1}{p^{31/30}}\right)^{1/3} \leqslant 1.$$

Using three times the inequality $1-x \leqslant e^{-x}$, we find that

$$S \leqslant \left(\exp\Big(-\frac{9A+9B+2}{10p}\Big) + \frac{B^{\frac{9}{10}}}{p} + \frac{A^{\frac{9}{10}}}{p}\right)\left(1-\frac{1}{p^{31/30}}\right)^{1/3}.$$

Since we also have that $e^{-x} \leqslant 1-x+x^2/2$ for $x \geqslant 0$, as well as $0 \leqslant A, B \leqslant 10^{40}$, we conclude that

$$S \leqslant \left(1 - \frac{9A+9B+2}{10p} + \frac{10^{81}}{p^2} + \frac{B^{\frac{9}{10}}}{p} + \frac{A^{\frac{9}{10}}}{p}\right)\left(1-\frac{1}{p^{31/30}}\right)^{1/3}.$$

By the arithmetic-geometric mean inequality, we have that $(9A+1)/10 \geqslant A^{9/10}$ and $(9B+1)/10 \geqslant B^{9/10}$, whence

$$S \leqslant \left(1+\frac{10^{81}}{p^2}\right)\left(1-\frac{1}{p^{31/30}}\right)^{1/3}.$$

Since $(1-x)^{1/3} \leqslant 1-x/3$, we must have that $S \leqslant 1$ for $p \geqslant 10^{2000}$, thus completing the proof of the lemma. □

Proof of Proposition 8.2. This follows almost immediately from Lemma 14.1. Our assumptions that

$$\mathcal{R}(G) \subseteq \{p > 10^{2000}\} \quad \text{and} \quad \mathcal{R}^\flat(G) = \emptyset$$

imply that if $p \in \mathcal{R}(G)$, then $p > 10^{2000}$ and $p \in \mathcal{R}^\sharp(G)$. Since $\mathcal{R}(G) \neq \emptyset$, such a prime p exists. In addition, the fact that $p \in \mathcal{R}^\sharp(G)$ is equivalent to the existence of an integer $k \geqslant 0$ such that

$$\frac{\mu(\mathcal{V}_{p^k})}{\mu(\mathcal{V})} \geqslant 1 - \frac{10^{40}}{p} \quad \text{and} \quad \frac{\mu(\mathcal{W}_{p^k})}{\mu(\mathcal{W})} \geqslant 1 - \frac{10^{40}}{p}.$$

Thus we can apply Lemma 14.1 with this choice of p and k. This then gives the result. □

This completes the proof of Proposition 8.2, and hence Theorem 1.

15. Concluding remarks and approximate counterexamples

It is a vital feature of our proof that all vertices v are weighted by a factor $\varphi(v)/v$, as naturally arises from the setup of the Duffin-Schaeffer conjecture. This allows our proof to (just) work, but in particular the proof presented here fails to answer the model question from Section 3 corresponding to vertices being weighted with weights 1 (i.e. without the $\varphi(q)/q$ factor). This point may appear to be a mere technicality, but it allows us to sidestep several 'approximate counterexamples' which would otherwise require additional input to handle.

First, let us see where the proof breaks down without the $\varphi(q)/q$ factors. Although most of the argument holds for a general measure μ, in Proposition 6.6 we specialize to the measure $\mu(v) = \psi(v)\varphi(v)/v$. In the proof of Proposition 6.6 (in particular (7.5)), the $\varphi(v)\varphi(w)/vw$ factor cancels out the factor $ab/\varphi(a)\varphi(b)$ coming from

$$\prod_{p\in\mathcal{P}}(1-\mathbb{1}_{f(p)=g(p)\geqslant 1}/p)^{-2}$$

in the definition of quality. Without this the proof of Proposition 6.6 would fail. If we did not have the factor above in the definition of the quality, then instead the proof of Lemma 14.1 would break down and we would not obtain a quality increment when there are many primes dividing a proportion of $1-1/p$ of each vertex set. Thus the argument we present fails without the $\varphi(q)/q$ weights.

Now, let use explain why the presence of the weight $\varphi(q)/q$ is essential for the kind of argument we have given to work. The crucial thing to notice is that at nowhere in the iterations did we make use of the (trivial) fact that there are $o(y)$ primes of size y; instead we just worked prime-by-prime without regard to previous iterations. If one doesn't make use of this feature, however, then one can construct counterexamples to the model question of Section 3 where each vertex is weighted by 1, but which are not counterexamples when one has the $\varphi(q)/q$ weights. Therefore any argument proving the model problem would need to take this feature into account, whereas we do not need to in our situation thanks to the $\varphi(q)/q$ weights.

To construct the alleged counterexamples, let $\mathcal{P}$ being a set of k primes in $[y, 2y]$, d_0 a fixed integer of size x^{1-c}/y^{k-2r} and

$$\mathcal{S} = \{d_0 m p_1\cdots p_{k-r} \in [x,2x] :\ m\in\mathbb{N},\ p_1,\dots,p_{k-r}\in\mathcal{P},\ p_i\neq p_j\ \forall i\neq j\}.$$

We see that if $n_1 = d_0 m_1 p_1\cdots p_{k-r}$ and $n_2 = d_0 m_2 p_1'\cdots p_{k-r}'$ are two elements of $\mathcal{S}$, then the relation $\#A\cap B = \#A + \#B - \#A\cup B$ implies that

$$\#\{p_1,\dots,p_{k-r}\}\cap\{p_1',\dots,p_{k-r}'\} \geqslant (k-r)+(k-r)-\#\mathcal{P} = k-2r.$$

Therefore $\gcd(p_1\cdots p_{k-r}, p_1'\cdots p_{k-r}') \geqslant y^{k-2r}$. Since n_1 and n_2 are both also a multiple of d_0, we see that

$$\gcd(n_1,n_2)\geqslant d_0 y^{k-2r}\geqslant x^{1-c}$$

for all pairs $n_1, n_2\in\mathcal{S}$. On the other hand, there are $\binom{k}{r}\approx (k/r)^r$ choices of the primes $p_1,\dots,p_{k-r}$, the integer d_0 is uniquely determined, and there are $x/d_0p_1\dots p_{k-r}\geqslant x^c/(2y)^r$ choices of m. Thus

$$\#\mathcal{S}\approx\Big(\frac{k}{r}\Big)^r\frac{x^c}{(2y)^r} = x^c\Big(\frac{k}{2ry}\Big)^r.$$

However, it is straightforward to check that if k is large compared with r, then no integer $d \gg x^{1-c}$ divides a positive proportion of elements of $\mathcal{S}$. In particular, we see that this construction would give a counterexample to the model question raised in Section 3 if we could take k to be larger than ry. Thus we would need to make use of the fact that $k = o(y)$ to avoid this counterexample.

If one instead counted integers with the $\mu(q) = \varphi(q)/q$ weights, then we see that

$$\mu(\mathcal{S}) \approx \Big(1 - \frac{1}{y}\Big)^{k-r} \Big(\frac{k}{r}\Big)^r \frac{x^c}{(2y)^r} \approx x^c \Big(\frac{k}{2ry}\Big)^r \exp\Big(-\frac{k}{y}\Big).$$

Thus even if $k \geqslant yr$, the additional factor $\exp(-k/y)$ would imply that $\mu(\mathcal{S}) = o(x^c)$, so the above construction is no longer a counterexample. This means it is no longer necessary to use the fact that $k = o(y)$ in the proof, which allows us to perform the iterations prime-by-prime separately from one another.

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