Problem 1 (10 points). Let $\mathcal{P}$ be a set of primes.

(a) Prove that
$$\sum_{a \leq x \atop p|a \Rightarrow p \in \mathcal{P}} \frac{1}{a} \sum_{b \leq x \atop p|b \Rightarrow p \not\in \mathcal{P}} \frac{1}{b} \geq \sum_{n \leq x} \frac{1}{n}.$$ 

(b) Prove that
$$\sum_{n \leq x \atop p|n} \frac{1}{n} \asymp \prod_{p \in \mathcal{P} \cap [1, x]} \left(1 + \frac{1}{p}\right).$$

Problem 2. [15 points] An integer $n$ is called $y$-smooth if all of its prime factors are $\leq y$. Let $\Psi(x, y)$ be the number of $y$-smooth numbers in $[1, x]$.

(a) If $y \in [\sqrt{x}, x]$ and $u = \log x / \log y$, then show that
$$\Psi(x, y) = x(1 - \log u) + O(x / \log x).$$  

[Suggestion: Count, instead, $n \leq x$ that are not $y$-smooth.]

(b) Prove that if $x^\epsilon \leq y \leq x$, then $\Psi(x, y) \gg x$. [Suggestion: Write $\{ p \leq y \} = \mathcal{P} \cup \mathcal{P}'$, where $\mathcal{P} = \{ p \leq \sqrt{y} \}$ and $\mathcal{P}' = \{ \sqrt{y} < p \leq y \}$. Prove that if $w \in [\sqrt{x}, x]$ and $k$ is big enough, then $\sum_{p_1, \ldots, p_k \in \mathcal{P}', p_1 \cdots p_k \leq w} 1 \gg w / \log w$. Then use Problem 1(b) above.]

Problem 3 (15 points). Given an odd prime $p$, we let $n_p$ be the smallest positive integer such that $(\frac{n_p}{p}) = -1$, where $(\frac{n}{p})$ denotes the Legendre symbol mod $p$. This number is called the least quadratic nonresidue mod $p$.

(a) Prove that $n_p$ is well-defined and that it must be a prime number $\ll \sqrt{p} \log p$.

(b) Prove that $n_p \ll \epsilon p^{1/(2\sqrt{e})+\epsilon}$, for any fixed $\epsilon > 0$, which improves the estimate coming from part (a). [Suggestion: Use Problem 2(a) above.]

Problem 4 (15 points). Let $q \geq 3$.

(a) Prove that if $\chi_1, \chi_2$ are two distinct real, non-principal characters (mod $q$), then $\max\{L_q(1, \chi_1), L_q(1, \chi_2)\} \gg 1$. [Suggestion: Theorem 7.1.1]

(b) Show that if the Brun-Titchmarsh inequality can be improved to
$$\pi(x; q, a) \leq \frac{(2 - \epsilon)x}{\varphi(q) \log(x/q)} \quad (x \geq 2q, (a, q) = 1)$$
for some fixed $\epsilon > 0$, then $L_q(1, \chi) \gg 1$ for all real, non-principal characters $\chi$ (mod $q$).