UPPER BOUNDS FOR STEKLOV EIGENVALUES ON SURFACES

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Abstract. We give explicit isoperimetric upper bounds for all Steklov eigenvalues of a compact orientable surface with boundary, in terms of the genus, the length of the boundary, and the number of boundary components. Our estimates generalize a recent result of Fraser–Schoen, as well as the classical inequalites obtained by Hersch–Payne–Schiffer, whose approach is used in the present paper.

1. Introduction

1.1. **Steklov spectrum.** Let Σ be a compact orientable surface with boundary, and let Δ be the Laplace–Beltrami operator associated with a Riemannian metric on Σ. The Steklov eigenvalue problem on Σ is given by:

$$\Delta u = 0 \text{ in } \Sigma, \quad \partial_n u = \sigma u \text{ on } \partial\Sigma,$$

where ∂_n denotes the outward normal derivative. The spectrum of the Steklov problem is discrete and its eigenvalues form a sequence

$$0 = \sigma_0 < \sigma_1 \le \sigma_2 \le \sigma_3 \le \cdots \nearrow \infty,$$

where each eigenvalue is repeated according to its multiplicity [2]. The eigenfunctions ϕ_k, $k = 0, 1, 2\dots$ can be chosen to form an orthogonal basis of $L^2(\partial\Sigma)$. Note that the eigenfunction ϕ_0 corresponding to $\sigma_0 = 0$ is constant.

The Steklov eigenvalues coincide with the eigenvalues of the Dirichlet-to-Neumann map Λ. If the boundary $\partial\Sigma$ is smooth, it is a pseudo-differential elliptic operator $\Lambda : C^\infty(\partial\Sigma) \to C^\infty(\partial\Sigma)$ of order one [20], defined by

$$\Lambda(f) = \partial_n Hf,$$

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where Hf is the harmonic extension of f to the interior of Σ (i.e. $\Delta(Hf) = 0$ on Σ). The Dirichlet-to-Neumann map has important applications to inverse problems [6, 19].

1.2. **Main results.** Isoperimetric inequalities for Steklov eigenvalues have been actively studied for more than fifty years [21, 22, 3, 15, 12]. In particular, a number of recent papers are concerned with the Steklov spectrum on manifolds with boundary [9, 10, 14, 7]. The following estimate on the first nontrivial Steklov eigenvalue on a surface with boundary was proved by Fraser and Schoen [10]:

$$\sigma_1 L \leq 2\pi(\gamma + l). \tag{1.1}$$

Here L is the length of the boundary, γ is the genus of the surface and l the number of boundary components. For simply connected planar domains, inequality (1.1) is sharp and was proved by Weinstock in [21].

The goal of this note is to generalize (1.1) to higher eigenvalues. We prove

Theorem 1.2. *Let Σ be a compact orientable surface of genus γ, such that the boundary $\partial\Sigma$ has l connected components of total length L. Then*

$$\sigma_k L \leq 2\pi(\gamma + l)\, k \tag{1.3}$$

for any integer $k \geq 1$.

In fact, Theorem 1.2 is a special case (set $p = q$ below) of the following result:

Theorem 1.4. *Under the assumptions of Theorem 1.2,*

$$\sigma_p\sigma_q L^2 \leq \begin{cases} \pi^2(\gamma+l)^2(p+q)^2 & \text{if } p+q \text{ is even,} \\ \pi^2(\gamma+l)^2(p+q-1)^2 & \text{if } p+q \text{ is odd,} \end{cases} \tag{1.5}$$

for any pair of integers $p, q \geq 1$.

1.3. **Discussion.** It follows from Weyl's law for eigenvalues of the Dirichlet-to-Neumann operator that the linear dependence on k in (1.3) is optimal. For simply connected planar domains, the inequalities (1.5) were obtained by Hersch, Payne, and Schiffer in [16]. In [12] we proved that in this case (here $\gamma = 0$, $l = 1$) the estimates (1.3) are sharp for all $k \geq 1$. We do not expect (1.3) to be sharp for other values of γ and l (cf. [10, Theorem 2.5]); see also Question 1.8 below.

The proof of Theorem 1.4 combines the methods of [10] and [16]. Following [10], we use a version of Ahlfors Theorem [1] proved by Gabard [11], according to which any Riemannian surface of genus γ with l boundary components can be represented as a proper conformal branched cover of a disk $\mathbb{D}$ with degree at most $\gamma+l$. Properness of the covering map implies that the boundary $\partial\Sigma$ is mapped to the circle S^1. It is essential in this proof since the test functions for the variational characterization of the eigenvalues σ_k are built from the eigenfunctions of a certain one–dimensional problem on S^1. This approach was suggested by Hersch, Payne and Schiffer in [16].

The analogue of the estimate (1.1) for the first nonzero Laplace eigenvalue λ_1 on a closed surface Σ (without boundary) is the Yang–Yau inequality [23] :

$$\lambda_1 \operatorname{Area}(\Sigma) \leq 8\pi d, \tag{1.6}$$

where d is the degree of a conformal branch covering of Σ over a sphere. It was observed in [8] that one could take $d \leq [\frac{\gamma+3}{2}]$, where $[\cdot]$ denotes the integer part.

For higher eigenvalues of the Laplacian on surfaces, no explcit estimates like (1.3) are known. However, with an implicit constant such a bound was proved by Korevaar in [18] using a different approach. The analogue of Korevaar's result for Steklov eigenvalues on surfaces was obtained in [7] (see also [13, Section 5.3] and [17, Example 1.3]): there exists a universal constant C such that

$$\sigma_k L \leq C\,(\gamma+1)k, \quad k = 1, 2, 3, \ldots \tag{1.7}$$

Note that the bound (1.7) does not depend on the number of boundary components of $\partial\Sigma$, which makes it a sharper estimate than (1.3) for l large enough. Another interesting development of Korevaar's method for both Laplace and Steklov eigenvalues can be found in [14] where λ_k and σ_k are bounded by a linear combination of k and γ (instead of its product). However, the constants in [14] are also implicit.

Let us conclude by an open question. It was proved in [5] that there exists a sequence of closed surfaces Σ_n of genera $\gamma_n \to \infty$ such that

$$\lim_{n\to\infty} \lambda_1(\Sigma_n) \operatorname{Area}(\Sigma_n) = \infty.$$

Moreover, it was subsequently shown in [4] that one can choose a sequence of surfaces with $\gamma_n = n$ and $\lambda_1(\Sigma_n) \operatorname{Area}(\Sigma_n)$ growing linearly as $n \nearrow \infty$. Therefore, the dependence on the genus γ in the Yang–Yau inequality (1.6) is optimal up to a multiplicative constant.

Question 1.8. Is there a sequence Σ_n of surfaces with boundary of genera $\gamma_n \to \infty$ such that $\sigma_1(\Sigma_n)L(\partial\Sigma_n) \to \infty$ as $n \to \infty$? If yes, is it possible to achieve linear growth?

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2. Proof of Theorem 1.4

2.1. **Reduction to the circle.** Let $(\phi_k)_{k=0}^{\infty} \subset L^2(\partial\Sigma)$ be a complete orthonormal system of eigenfunctions of the Dirichlet–to–Neumann map. It is well known that if a function $f \in C^\infty(\Sigma)$ satisfies

$$\int_{\partial\Sigma} f\phi_j \quad \text{for } j = 0, 1, 2 \ldots, k-1, \tag{2.1}$$

then

$$\sigma_k \leq R_\Sigma(f) := \frac{\int_\Sigma |\nabla f|^2}{\int_{\partial\Sigma} f^2}. \tag{2.2}$$

The proof of Theorem 1.4 is based on the approach of [16]. We construct test functions using linear combinations of harmonic oscillators on S^1, extend them harmonically to the disk and then lift to a branched cover representation of Σ. Using sufficiently many harmonic oscillators, one can ensure the existence of a linear combination satisfying the orthogonality conditions (2.1).

As was shown in [11], there exists a proper conformal branched cover

$$\psi : \Sigma \to \mathbb{D}$$

of degree $d \leq \gamma + l$. Because ψ is proper, it takes the boundary $\partial\Sigma$ to the circle $S^1 = \partial\mathbb{D}$. The restriction of ψ to each connected component of $\partial\Sigma$ is a covering map of S^1. Let ds be the Riemannian measure on $\partial\Sigma$. We define the push-forward measure $d\mu = \psi_* ds$ on the circle S^1, and introduce the "mass parameter"

$$m(\theta) = \int_0^\theta d\mu(\theta).$$

In particular, $d\mu = m'(\theta)d\theta$ is absolutely continuous with respect to the Lebesgue measure $d\theta$, and the length of the boundary $\partial\Sigma$ is given by

$$L = m(2\pi) = \int_{S^1} d\mu.$$

Given a smooth periodic function $h : \mathbb{R} \to \mathbb{R}$ with period L, define $f : S^1 \to \mathbb{R}$ by

$$f(\theta) = h(m(\theta)).$$

The function f admits a unique harmonic extension u to the disk $\mathbb{D}$. Because the disk is simply connected, this function has a unique harmonic conjugate v normalized in such a way that

$$\int_{S^1} v\, d\mu = 0. \tag{2.3}$$

Let the functions $\alpha, \beta : \Sigma \to \mathbb{R}$ be defined by

$$\alpha = u \circ \psi \quad \text{and} \quad \beta = v \circ \psi.$$

Recall that the map ψ is a d-fold conformal branched covering of $\mathbb{D}$. It follows from conformal invariance of the Dirichlet energy in two dimensions (see also [23]) that

$$\int_\Sigma |\nabla \alpha|^2 = d \int_{\mathbb{D}} |\nabla u|^2, \quad \int_\Sigma |\nabla \beta|^2 = d \int_{\mathbb{D}} |\nabla v|^2. \tag{2.4}$$

Moreover, the Cauchy–Riemann equations imply that these two quantities are equal. Integration by parts gives

$$\int_{\mathbb{D}} |\nabla u|^2 = \int_{\mathbb{D}} |\nabla v|^2 = \int_{S^1} v\, \partial_r v. \tag{2.5}$$

Multiplying the two equations in (2.4) and using (2.5), we get

$$\int_\Sigma |\nabla \alpha|^2 \int_\Sigma |\nabla \beta|^2 = d^2 \left(\int_{S^1} v \partial_r v \right)^2. \tag{2.6}$$

The Cauchy–Riemann equations also imply the pointwise equality

$$\partial_r v = -\partial_\theta u = -f'(\theta) = -h'(m(\theta)) m'(\theta).$$

Applying the Cauchy–Schwarz inequality to the measure $d\mu = m'(\theta) d\theta$ leads to :

$$\begin{aligned} \left(\int_{S^1} v \partial_r v \right)^2 &= \left(\int_{S^1} v(\theta) h'(m(\theta)) \overbrace{m'(\theta)) d\theta}^{d\mu(\theta)} \right)^2 \\ &\leq \int_{S^1} v^2(\theta)\, d\mu(\theta) \int_{S^1} h'(m(\theta))^2\, d\mu(\theta). \end{aligned} \tag{2.7}$$

At the same time,

$$\int_{\partial\Sigma} \alpha^2\, d_{\partial\Sigma} = \int_{S^1} f^2\, d\mu \quad \text{and} \quad \int_{\partial\Sigma} \beta^2\, d_{\partial\Sigma} = \int_{S^1} v^2\, d\mu. \tag{2.8}$$

Estimating the product of the Rayleigh quotients $R_\alpha := R_\Sigma(\alpha)$ and $R_\beta := R_\Sigma(\beta)$ using the relations (2.6), (2.7) and (2.8), we notice that $\int_{S^1} v^2(\theta)\, d\mu(\theta)$ cancels out on the right–hand side. This is the key trick in the method introduced in [16]. Namely, we obtain the following bound:

$$R(\alpha)R(\beta) \leq d^2 \frac{\int_{S^1} h'(m(\theta))^2 \, d\mu(\theta)}{\int_{S^1} h(m(\theta)^2 \, d\mu(\theta)} = d^2 R_L(h).$$

Here

$$R_L(h) := \frac{\int_0^L h'(m)^2 \, dm}{\int_0^L h(m)^2 \, dm}$$

is the Rayleigh quotient of a uniform circular string of length L. Its eigenmodes are well known. Let $h_k : \mathbb{R} \to \mathbb{R}$, $k = 0, 1, 2 \ldots$, be defined by $h_0 = 1$ and

$$h_k(m) = \begin{cases} \cos\left(\frac{2n\pi m}{L}\right) & \text{if } k = 2n-1, \\ \sin\left(\frac{2n\pi m)}{L}\right) & \text{if } k = 2n. \end{cases}$$

for $k \geq 1$. Clearly,

$$R_L(h_k) = \left(\frac{2\pi n}{L}\right)^2 \quad \text{for } k = 2n \text{ or } k = 2n-1.$$

This leads to

$$R(\alpha)R(\beta) \leq \left(\frac{\pi d}{L}\right)^2 \begin{cases} k^2 & \text{if } k = 2n, \\ (k+1)^2 & \text{if } k = 2n-1. \end{cases} \tag{2.9}$$

2.2. **Construction of test-functions.** The rest of the argument is almost exactly the same as in [16]. We present it below for the sake of completeness. Let $N = p + q - 1$. Consider a function

$$f = \sum_{k=1}^{N} c_k f_k, \quad (c_k \in \mathbb{R}), \tag{2.10}$$

where the functions $f_k : \mathbb{S}^1 \to \mathbb{R}$ are defined by $f_k(\theta) = h_k(m(\theta)$. The functions f_k are $d\mu$–orthogonal to each other, and hence linearly independent. The harmonic extensions u_k of f_k are also linearly independent, because taking the harmonic extension is a linear and injective operation. For the same reason, the harmonic conjugates v_k are linearly independent as well. Moreover, since by definition $f_0 = 1$, f_k are $d\mu$–orthogonal to constants for all $k = 1, 2, 3, \ldots$, and hence $\int_{\partial\Sigma} \alpha_k = 0$ for all $k \geq 1$, where $\alpha_k = u_k \circ \psi$. At the same time, by the normalization (2.3), $\int_{\partial\Sigma} \beta_k = 0$ for all $k \geq 1$, where $\beta_k = v_k \circ \psi$. Let

$$u = \sum_{k=1}^{N} c_k u_k \quad \text{and} \quad v = \sum_{k=1}^{N} c_k v_k.$$

As before, these functions are lifted to $\alpha = u \circ \psi$ and $\beta = v \circ \psi$.

In order to use u and v in the variational characterization (2.2) for σ_p and σ_q respectiveley, they have to satisfy the orthogonality conditions (2.1) :

$$\int_{\partial\Sigma} \alpha\phi_k = 0 \quad \text{for } k = 1, \cdots, p-1$$
$$\int_{\partial\Sigma} \beta\phi_k = 0 \quad \text{for } k = 1, \cdots, q-1$$

These $N-1$ linear constraints can be resolved for some choice of N constants $c_1, \dots, c_N$. It follows from (2.9) that

$$\sigma_p\sigma_q \leq R(\alpha)R(\beta) \leq d^2 R_L(h),$$

where $h = \sum_{k=1}^N c_k h_k$. We conclude by observing that

$$\begin{aligned} R_L(h) \leq R_L(h_N) &= \left(\frac{\pi d}{L}\right)^2 \begin{cases} N^2 & \text{if } N \text{ is even,} \\ (N+1)^2 & \text{if } N \text{ is odd.} \end{cases} \\ &= \left(\frac{\pi d}{L}\right)^2 \begin{cases} (p+q-1)^2 & \text{if } p+q \text{ is odd,} \\ (p+q)^2 & \text{if } p+q \text{ is even.} \end{cases} \end{aligned}$$

Recalling that $d \leq \gamma + l$ completes the proof of Theorem 1.4. □

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