Shape optimization for Neumann and Steklov eigenvalues

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References

Alexandre Girouard, Nikolai Nadirashvili, I.P., *Maximization of the second positive Neumann eigenvalue for planar domains*, *J. Diff. Geometry 83, no. 3 (2009), 637–662.*

Alexandre Girouard, I.P., *On the Hersch-Payne-Schiffer inequalities for Steklov eigenvalues*, *Func. Anal. Appl. 44, no. 2 (2010), 106–117.*

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The spectra of Neumann and Steklov problems are **discrete**:

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Both Neumann and Steklov spectra start with the simple eigenvalue

$$\mu_0 = \sigma_0 = 0,$$

and the corresponding eigenfunctions are **constant**.

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Here U_k and E_k denote k–dimensional subspaces of the Sobolev space $H^1(\Omega)$, such that U_k are orthogonal to constants on Ω and E_k are orthogonal to constants on $\partial\Omega$.

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How large can μ_k and σ_k be on a membrane of a given mass?

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Two years later, **Weinberger** generalized this result to arbitrary (not necessarily simply–connected) domains in any dimension.

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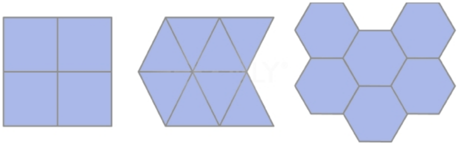

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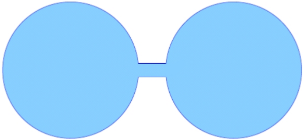

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Corollary Pólya's conjecture holds for $k = 2$ on simply–connected planar domains with Neumann boundary conditions.

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Weinstock inequality '54:

$$\sigma_1(\Omega)\, L(\partial\Omega) \leq 2\pi\sigma_1(\mathbf{D}) = 2\pi.$$

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$$\sigma_k(\Omega)\,\sigma_n(\Omega)\,L(\partial\Omega)^2 \leq \begin{cases} (k+n-1)^2\,\pi^2 & \text{if } k+n \text{ is odd,} \\ (k+n)^2\,\pi^2 & \text{if } k+n \text{ is even.} \end{cases}$$

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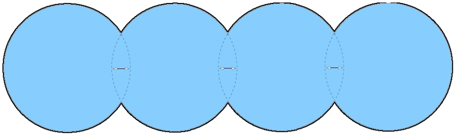
In fact, a much stronger statement holds!

Theorem 2 (**Girouard–P.** '10) There exists a family of simply-connected domains, degenerating to the disjoint union of n identical disks as $\varepsilon \to 0+$, such that



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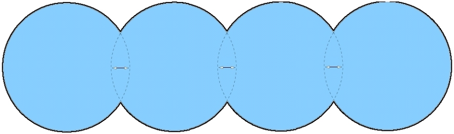

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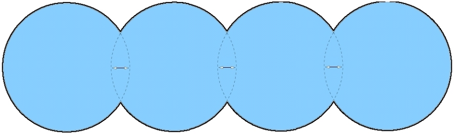

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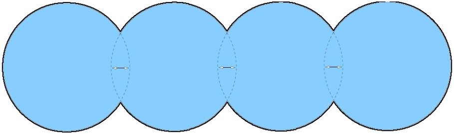

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In particular, the Hersch–Payne–Schiffer inequalities are sharp for

all $n = k$ and $n = k + 1$, $k \geq 1$.

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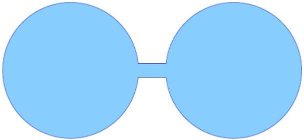
One could ask why in Theorem 2 we do not apply a more natural construction used in Theorem 1:



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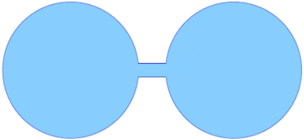

The reason is that convergence of Steklov eigenvalues in this case could be quite unexpected.

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According to the variational principle,

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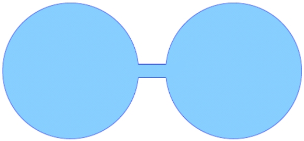

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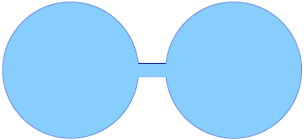

Consider k pairwise orthogonal test–functions *vanishing* in the disks and equal to $\sin \frac{2\pi\, m\, x}{s}$, $m = 1, \ldots, k$, inside the passage.

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For every fixed m, the numerator in the Rayleigh quotient tends to zero as $\varepsilon \to 0+$, while the denominator does not. This implies

$$\lim_{\varepsilon \to 0+} \sigma_k(\varepsilon) = 0$$

for **all** $k = 1, 2, \ldots !$

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Conjecture The product of the first two Neumann eigenvalues attains its maximum on a disk among all domains of a given area:

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For simply–connected domains, it follows from Szegő inequality and Theorem 1 that

$$\mu_1(\Omega)\mu_2(\Omega)\,\mathrm{area}(\Omega)^2 < 2\,\mu_1(\mathbf{D})^2\,\pi^2.$$

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Since the Dirichlet energy is conformally invariant,

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Infimum is taken over all two–dimensional subspaces $E \subset H^1(\mathbf{D})$ such that

$$\int_{\mathbf{D}} f \, d\rho = 0 \quad \text{for all } f \in E.$$

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Set

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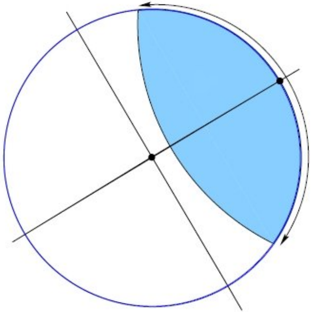
Hyperbolic caps



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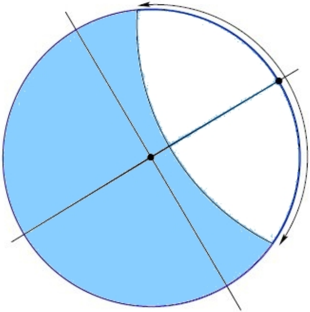

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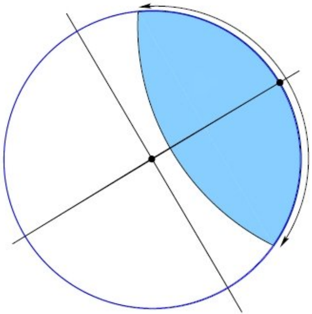

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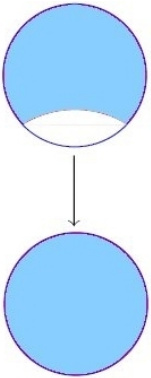
The **lift** of $u : a \to \mathbf{R}$ is $\tilde{u}(z) = \begin{cases} u(z) & \text{if } z \in a, \\ u(\tau z) & \text{if } z \in a^* = \tau(a). \end{cases}$

Test functions for $\mu_2(\Omega)$



Conformal equivalence $\phi_a : a \to \mathbf{D}$

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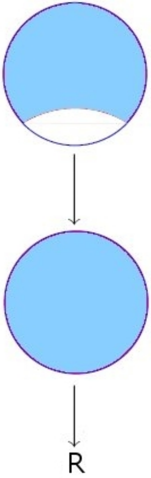

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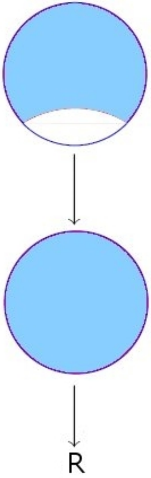

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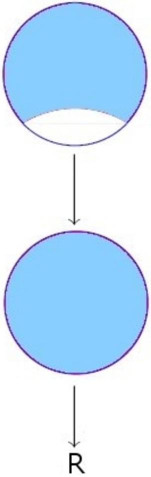

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Hersch's lemma allows to choose ϕ_a so that $\int_{\mathbf{D}} \tilde{u}_a^t \, d\rho = 0$.

Using $E = E_a = \left\{ \tilde{u}_a^t \mid t \in \mathbf{R}^2 \right\}$ in

$$\mu_2(\Omega) = \inf_{E} \sup_{0 \neq f \in E} \frac{\int_{\mathbf{D}} |\nabla f|^2 \, dz}{\int_{\mathbf{D}} f^2 \, d\rho}$$

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Lemma

$$\int_{\mathbf{D}} |\nabla \tilde{u}_a^t|^2 \, dz = 2\mu_1(\mathbf{D})K.$$

The proof uses conformal invariance of the Dirichlet energy. The factor 2 appears because of the **lift**.

Previous lemma implies:

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Indeed, since Ω is **planar**, the Gaussian curvature is zero, and hence $\Delta \log \delta = 0$.

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Since f is increasing, for any $s, t \in \mathbf{S}^1$ with $s \perp t$ we have:

$$\int_{\mathbf{D}} \left(\left(\tilde{u}_a^s\right)^2 + \left(\tilde{u}_a^t\right)^2\right) \, d\rho = \int_{\mathbf{D}} (\overbrace{X_s^2 + X_t^2}^{f^2(|z|)})\, \delta(z)dz \ge \int_{\mathbf{D}} f^2(|z|)\, dz.$$

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Since for each cap $a \subset \mathbf{D}$, the integral

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is quadratic in t, there exists a unique maximizing direction $\pm t \in \mathbf{S}^1$ such that for each $s \neq \pm t$

$$\int_{\mathbf{D}} \left(\tilde{u}_a^t\right)^2 \, d\rho > \int_{\mathbf{D}} \left(\tilde{u}_a^s\right)^2 \, d\rho.$$

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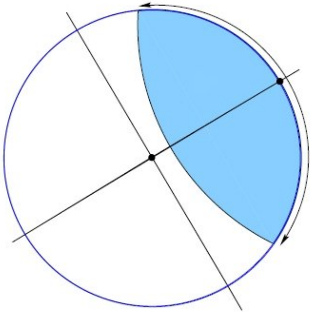

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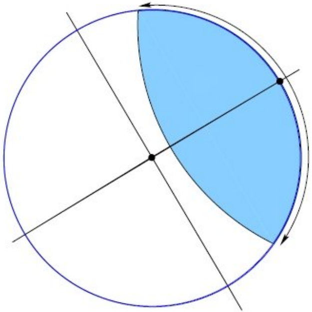

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first coordinate: midpoint of $\partial a \cap \partial \mathbf{D}$,
second coordinate: length of $\partial a \cap \partial \mathbf{D}$.

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$$\mu_2(\Omega) \leq \sup_{t \in \mathbf{S}^1} \frac{2\mu_1(\mathbf{D})K}{\int_{\mathbf{D}} \left(\tilde{u}_a^t\right)^2 \, d\rho} \leq 2\mu_1(\mathbf{D}). \qquad \square$$

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Understanding the geometry of maximizers for higher Neumann eigenvalues is an interesting open problem.