Pólya’s conjecture for the disk: a computer-assisted proof

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Abstract

The celebrated Pólya’s conjecture (1954) in spectral geometry states that the eigenvalue counting functions of the Dirichlet and Neumann Laplacian on a bounded Euclidean domain can be estimated from above and below, respectively, by the leading term of Weyl’s asymptotics. Pólya’s conjecture is known to be true for domains which tile the space, and, in addition, for some special domains in higher dimensions. In this paper, we prove Pólya’s conjecture for the disk, making it the first non-tiling planar domain for which the conjecture is verified. Along the way, we develop the known links between the spectral problems in the disk and certain lattice counting problems. The key novel ingredient is the observation, made in recent work of the last named author, that the corresponding eigenvalue and lattice counting functions are related not only asymptotically, but in fact satisfy certain uniform bounds. Our proofs are purely analytic for the values of the spectral parameter above some large (but explicitly given) number. Below that number we give a rigorous computer-assisted proof which converges in a finite number of steps and uses only integer arithmetic.

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A Mathematica script and data used for a computer-assisted part of the paper are available for download at https://michaellevitin.net/polya.html


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§ 1. Weyl’s law and Pólya’s conjecture

Let \( \Omega \subset \mathbb{R}^d \) be a bounded domain. Consider the Dirichlet eigenvalue problem for the Laplacian

\[
-\Delta := -\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}
\]

in \( \Omega \):

\[
-\Delta u = \lambda u \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega.
\]  

(1.1)

It is well known that the spectrum of (1.1) is discrete and consists of isolated eigenvalues of finite multiplicity accumulating to \(+\infty\),

\[0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leq \cdots \leq \lambda_N(\Omega) \leq \ldots,\]

which we enumerate with account of multiplicities.

Similarly, assuming additionally that \( \partial \Omega \) is Lipschitz, consider the Neumann eigenvalue problem

\[
-\Delta u = \mu u \quad \text{in } \Omega, \\
\partial_n u = 0 \quad \text{on } \partial \Omega,
\]

(1.2)

where \( \partial_n u = \langle \nabla u, n \rangle |_{\partial \Omega} \) denotes the normal derivative of \( u \) with respect to the exterior unit normal \( n \) on the boundary. The spectrum of (1.2) again consists of isolated eigenvalues of finite multiplicity accumulating to \(+\infty\),

\[0 = \mu_1(\Omega) < \mu_2(\Omega) \leq \cdots \leq \mu_N(\Omega) \leq \ldots,\]

enumerated with account of multiplicities.

Let, for \( \lambda \in \mathbb{R} \),

\[
\mathcal{N}^D_\Omega(\lambda) := \# \{ n : \lambda_n(\Omega) \leq \lambda^2 \} \quad \text{and} \quad \mathcal{N}^N_\Omega(\lambda) := \# \{ n : \mu_n(\Omega) \leq \lambda^2 \}
\]

(1.3)

denote the counting functions\(^1\) of the Dirichlet and Neumann eigenvalue problems on \( \Omega \).\(^2\) It follows from the variational principles for (1.1) and (1.2) that

\[
\mathcal{N}^D_\Omega(\lambda) \leq \mathcal{N}^N_\Omega(\lambda)
\]

for any \( \lambda \geq 0 \).\(^3\)

Under the assumptions stated above, the leading term asymptotics of the counting functions is given by Weyl’s law,

\[
\mathcal{N}_\Omega(\lambda) = C_d |\Omega|_d \lambda^d + R(\lambda),
\]

(1.4)

\(^1\)Strictly speaking, we are counting the number of eigenvalues less than or equal to a given \( \lambda^2 \) but such normalisation will be convenient to us throughout.

\(^2\)One can also define the counting functions using strict inequalities; this does not affect any of the results below.

\(^3\)This in fact can be improved to \( \mathcal{N}^D_\Omega(\lambda) + 1 < \mathcal{N}^N_\Omega(\lambda) \), see [Fri91] and [Fil04].
where $N_\Omega(\lambda)$ denotes either $N^D_\Omega(\lambda)$ or $N^N_\Omega(\lambda)$, $|\cdot|_d$ denotes the $d$-dimensional volume, $R(\lambda) = o(\lambda^d)$ as $\lambda \to +\infty$, and

$$C_d := \frac{1}{(4\pi)^{\frac{d}{2}} \Gamma\left(\frac{d}{2} + 1\right)}$$

(1.5) is the so-called Weyl constant. We refer to [SafVas97] for a historical review, as well as numerous generalisations and improvements.

H. Weyl himself conjectured a sharper version of (1.4) taking into account the boundary conditions: for $\Omega \subset \mathbb{R}^d$ with a piecewise smooth boundary,

$$N_\Omega(\lambda) = C_d |\Omega|_d \lambda^d \mp C_{b,d} |\partial\Omega|_{d-1} \lambda^{d-1} + o\left(\lambda^{d-1}\right) \quad \text{as} \quad \lambda \to +\infty,$$

(1.6)

where the minus sign is taken for the Dirichlet boundary conditions and the plus sign for the Neumann ones, and

$$C_{b,d} = \frac{1}{2^{d+1} \pi^{\frac{d+1}{2}} \Gamma\left(\frac{d+1}{2}\right)}.$$  

(1.7)

We note that for planar domains (1.6) takes the particularly simple form

$$N_\Omega(\lambda) = \frac{\text{Area}(\Omega)}{4\pi} \lambda^2 \mp \frac{\text{Length}(\partial\Omega)}{4\pi} \lambda + o(\lambda).$$

(1.8)

The two-term Weyl’s law (1.6) remains open in full generality. It has been proved by V. Ivrii [Ivr80] under the condition that the set of periodic billiard trajectories in $\Omega$ has measure zero. While this condition is conjectured to be satisfied for all Euclidean domains, it has been verified only for a few classes, such as convex analytic domains and polygons, see [SafVas97] and references therein. Specifically for a disk, it was proved by N. Kuznetsov and B. Fedosov in [KuzFed65].

Assuming that the two-term Weyl’s asymptotics (1.6) holds for a domain $\Omega \subset \mathbb{R}^d$, we immediately obtain that for sufficiently large but unspecified $\lambda$ we have

$$N^D_\Omega(\lambda) \leq C_d |\Omega|_d \lambda^d \leq N^N_\Omega(\lambda).$$

(1.9)

We refer also to [Mel80] for results of the same kind in the Riemannian setting.

In 1954, G. Pólya [Pól54] conjectured that the inequalities (1.9) hold for all $\lambda \geq 0$. He later proved in [Pól61] this conjecture for tiling domains $\Omega$: that is, domains such that $\mathbb{R}^d$ can be covered, up to a set of measure zero, by a disjoint union of copies of $\Omega$. In fact, in the Neumann case, some additional assumptions were imposed in [Pól61] that have been removed in [Kel66], see also [Fil20]. It has been also shown that Pólya’s conjecture in the Dirichlet case holds for a Cartesian product of a tiling domain and a bounded set [Lap97, Theorem 2.8]. For general domains, somewhat weakened versions of (1.9) are known to hold, e.g. in the Dirichlet case

$$N^D_\Omega(\lambda) \leq \left(\frac{d+2}{d}\right)^{d/2} C_d |\Omega|_d \lambda^d,$$

see [Lap97], and [Krö92] for the best general bound in the Neumann case. We refer also to [Lin17, KLS19, FLP21] for some recent results and interesting links to other problems in spectral geometry.

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4In fact, Pólya’s original conjecture was only for the Dirichlet problem for planar domains, and in a slightly different form, see also [Fil20] for a brief historical review.
Remark 1.1. Pólya’s conjecture (1.9) can be equivalently restated as the inequalities for the eigenvalues (instead of the counting functions),

\[ \mu_{n+1}(\Omega) \leq (C_d|\Omega|_d)^{\frac{2}{d}} n^\frac{2}{d} \leq \lambda_n(\Omega) \]  

(1.10)

for all \( n \geq 1 \). It is known that inequalities (1.10) hold for any domain in any dimension for \( n = 1, 2 \). In particular, for \( n = 1 \) this follows from the celebrated Faber–Krahn and Szegő–Weinberger inequalities, and for \( n = 2 \) in the Dirichlet case from the Krahn–Szegő inequality, see [Hen06]. For \( n = 2 \) in the Neumann case, we refer to [GNP09, BucHen19]. These are the only eigenvalues for which it is known in full generality. We refer also to [Fre19] for further results on the validity of Pólya’s conjecture for low eigenvalues in higher dimensions.

Remarkably, since balls do not tile the space, Pólya’s conjecture has so far remained open for Euclidean balls, including planar disks.\(^5\) Note that by a simple rescaling argument it suffices to verify it for the unit disk. Although all the eigenvalues of the Dirichlet and Neumann Laplacians on the unit disk are explicitly known in terms of zeros of the Bessel functions or their derivatives, see § 2 below, in each case the spectrum is given by a two-parametric family, and rearranging it in a single monotone sequence appears to be an unfeasible task.

The main result of our paper is the following

**Theorem 1.2.** Pólya’s conjecture holds for the unit disk \( D \subset \mathbb{R}^2 \):

\[ \mathcal{N}_D^D(\lambda) < \frac{\lambda^2}{4} < \mathcal{N}_D^N(\lambda) \]

for all \( \lambda \geq 0 \).

More precisely, let us set

\[ \Lambda_1 = 84123. \]  

(1.11)

Effectively, we prove

**Theorem 1.3.** Pólya’s conjecture for the unit disk holds for all \( \lambda \geq \Lambda_1 \).

and

**Theorem 1.4.** Pólya’s conjecture for the unit disk holds for all \( \lambda \in [0, \Lambda_1) \).

We split the main result in two since the techniques used in the proofs of Theorems 1.3 and 1.4 are different: the proof of the former is completely analytic, whereas for the latter we give a rigorous computer-assisted proof. More specifically, it is based on a realisation of an algorithm which satisfies two fundamental principles.

**Principle 1.** The algorithm should complete in a finite number of steps.

**Principle 2.** The algorithm should operate only with integer or rational numbers, thus avoiding any use of floating-point arithmetic and any rounding errors.

\(^5\) As stated in [Lap12, p. 638]: “Remarkably this conjecture still remains open even for such a simple domain as the disc, where the eigenvalues of the Dirichlet Laplacians could be calculated via the roots of Bessel functions.” See also [FLW09, p. 1366] and [Laut, p. 66].
Remark 1.5. If we address Pólya’s conjecture in the form (1.10), see Remark 1.1, then Theorem 1.3 provides an analytic proof of (1.10) for

\[ n_1 \geq \left\lceil \frac{\Lambda_1^2}{4} \right\rceil = 1769169783. \]

We additionally have the following

**Corollary 1.6.** Pólya’s conjecture holds for any circular sector \( S_\ell \) with the angle \( \frac{2\pi}{\ell} \), where \( \ell \in \{2, 3, \ldots\} \).

Corollary 1.6 follows immediately from the following generalisation of Pólya’s result for tiling domains.

**Theorem 1.7.** Let \( \Omega \subset \mathbb{R}^d \) be a domain for which either the Dirichlet or the Neumann Pólya’s conjecture holds, and let \( \Omega' \) be a domain which tiles \( \Omega \). Then the same Pólya’s conjecture also holds for \( \Omega' \).

**Proof.** Assume that \( \Omega \) can be tiled by \( M \geq 2 \) congruent copies of \( \Omega' \), so that \( |\Omega|_d = M|\Omega'|_d \). We have, by bracketing and since the eigenvalues of all the congruent copies coincide with those of \( \Omega' \),

\[ M N_{\Omega}^D(\lambda) < N_{\Omega}^D(\lambda) \leq M N_{\Omega}^N(\lambda) < M N_{\Omega}^N(\lambda). \]

Assuming now (1.9) for all \( \lambda \geq 0 \), we get

\[ M N_{\Omega}^D(\lambda) < C_d|\Omega|_d \lambda^d = C_d M|\Omega'|_d \lambda^d < M N_{\Omega}^N(\lambda), \]

and the result follows by cancelling \( M \).

Finally, using [Lap97, Theorem 2.8], we immediately deduce

**Corollary 1.8.** The Dirichlet Pólya conjecture holds for any bounded cylinder \( \mathbb{D} \times (0, h) \subset \mathbb{R}^3 \), and, moreover, for any bounded domain \( \mathbb{D} \times \Omega' \subset \mathbb{R}^d \), \( d \geq 3 \), with \( \Omega' \subset \mathbb{R}^{d-2} \). The same is also true for bounded domains \( S_\ell \times \Omega' \), where \( S_\ell \) is any sector from Corollary 1.6.

**Plan of the paper.** In the next section we describe a lattice counting problem (2.8), originally introduced by N. Kuznetsov and B. Fedosov in [KuzFed65], that is closely linked to the Dirichlet and Neumann spectral counting problems on the disk. The key novel tool is Theorem 2.1, originally obtained in part in [She22], which gives a uniform bound between the eigenvalue and the spectral counting functions, as opposed to an asymptotic relation that was previously known. We provide a somewhat simpler proof of this result in § 3. Theorem 1.3 is proved in § 4. We make heavy use of the techniques due to R. Laugesen and S. Liu [LauLiu18], [Liu17] in order to analyse the lattice counting problem (2.8). See also Appendix A3 for the proof of an auxiliary Proposition 4.3. The computer-assisted proof of Theorem 1.4 is presented in § 5. The summary of the calculations can be found in Figures 10 and 11. In Appendices A1 and A2 we collect the pre-rational and rational approximations of various quantities used in the computer-assisted argument. They are necessary in order to use only integer arithmetic and thus keep the computer calculations rigorous.

**Remark 1.9.** Although we did not pursue their implementation in higher dimensions, our methods should extend to \( d \)-dimensional balls with \( d \geq 3 \). However, the analysis for each dimension will have to be done separately, and this will be investigated in a forthcoming student project. Similarly, our methods should be applicable to any particular sector with an arbitrary given angle. It would be interesting to extend them as well to ellipses, for which there also exists a link between spectral and lattice counting problems [Kuz66].

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\(^6\)This is a very, very, very large number but still small enough to allow for computer-assisted algorithm to converge in reasonable time in order to check Pólya’s conjecture for eigenvalues below \( \lambda_{n_1} \).
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§ 2. Dirichlet and Neumann eigenvalues of the disk and lattice count problems

It is well known that the eigenvalues of the Dirichlet Laplacian in the unit disk $D$ are given by the squares of the zeros of the cylindrical Bessel functions. Namely, we have the single eigenvalues

$$\lambda_{0,k} = (j_{0,k})^2, \quad k \in \mathbb{N},$$

and the double eigenvalues

$$\lambda_{m,k} = (j_{m,k})^2, \quad m, k \in \mathbb{N}.$$ 

Throughout, $J_m$ are the Bessel functions, and $j_{m,k}$ is the $k$th positive zero of $J_m$. We therefore have

$$N_D(\lambda) = \# \{k \in \mathbb{N} : j_{0,k} \leq \lambda\} + 2\# \{(m, k) \in \mathbb{N}^2 : j_{m,k} \leq \lambda\}. \quad (2.1)$$

Similarly, the eigenvalues of the Neumann Laplacian in the unit disk $D$ are given by the squares of the zeros of the derivatives of cylindrical Bessel functions. We have the single eigenvalues

$$\mu_{0,k} = (j'_{0,k})^2, \quad k \in \mathbb{N},$$

and the double eigenvalues

$$\mu_{m,k} = (j'_{m,k})^2, \quad m, k \in \mathbb{N},$$

where $j'_{m,k}$ is the $k$th positive zero of $j'_m$, with the exception that $j'_{0,1} = 0$. We therefore have

$$N_N(\lambda) = \# \{k \in \mathbb{N} : j'_{0,k} \leq \lambda\} + 2\# \{(m, k) \in \mathbb{N}^2 : j'_{m,k} \leq \lambda\}. \quad (2.2)$$

For illustrative purposes only, we show the graphs of the Dirichlet and Neumann eigenvalue counting functions for the disk in Figure 1.

We will be comparing the eigenvalue counting functions $N_D(\lambda)$ to the following lattice counting functions. Let

$$h(x) := \frac{1}{\pi} \left( \sqrt{1 - x^2} - x \arccos x \right), \quad x \in [-1, 1],$$

and let $P$ be a planar region given by

$$P := \{(x, y) : x \in [-1, 1], y \in \max(0, -x), h(x))\}, \quad (2.3)$$

see Figure 2.

We will further split $P$ into three parts, $P = P_- \cup P_+ \cup P_0$, which are the sub-regions lying to the left of the $y$-axis, to the right of the $y$-axis, and the interval of the $y$-axis, respectively.

Set, for $\lambda > 0$,

$$\mathcal{P}_D(\lambda) := \# \{(m, k) \in \mathbb{N} : \frac{1}{\lambda} \left( m, k - \frac{1}{4} \right) \in P\}, \quad (2.5)$$

and $\mathcal{P}_D(0) = 0$. We also set

$$\mathcal{P}_N(\lambda) := \# \{(m, k) \in \mathbb{N} : \frac{1}{\lambda} \left( m, k - \frac{3}{4} \right) \in P\}. \quad (2.6)$$
Figure 1: The Dirichlet eigenvalue counting function $N_D^D(\lambda)$ (blue), the Neumann eigenvalue counting function $N_D^N(\lambda)$ (red), and the function $\lambda^2/4$ (black). The plots are produced using the floating-point evaluation of zeros of the Bessel functions and their derivatives. If we were to assume (contrary to the philosophy of this paper) the validity of floating-point arithmetic, these plots would have presented a numerically assisted (as opposed to computer-assisted) "proof" of Pólya’s conjecture for the disk for $\lambda \leq 15$.

Figure 2: Regions $P_-$ and $P_+$, the interval $P_0$, and the lattice points appearing in the definitions of $P_D^D(\lambda)$ (blue) and $P_N^N(\lambda)$ (red), shown here for $\lambda = 17$.

and $P_N^N(0) = 1$. The reason for choosing this notation will become evident later.

We will further write

$$P_N^N(\lambda) = P_0^N(\lambda) + 2P_+^N(\lambda),$$

where $P_\oplus^N(\lambda), \oplus \in \{\ominus, +, 0\}$, is obtained by replacing $P$ in equations (2.5) and (2.6) by $P_\oplus$.

We observe that with either $\ominus \in \{D, N\}$, we have

$$P_\ominus^N(\lambda) = P_0^N(\lambda) + 2P_+^N(\lambda),$$

which is due to the fact that $P_N^N(\lambda) = P_+^N(\lambda)$ since both $P$ and the lattice $\mathbb{Z}^2$ are invariant with respect to the linear involution $(x, y) \mapsto (-x, y + x)$ [CdV10].

We re-write (2.5) and (2.6) with account of (2.7) as

$$P_N^N(\lambda) := 2\# \{(m, k) \in \mathbb{N}^2 : (m, k - s_N) \in P_\ominus \} + \# \{k \in \mathbb{N} : (0, k - s_N) \in P_{\ominus, 0} \},$$
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where

\[ s^D := \frac{1}{4}, \quad s^N := \frac{3}{4}, \quad (2.9) \]

\( P_\lambda \) is a dilation of \( P_+ \) with coefficient \( \lambda \) with respect to the origin,

\[ P_\lambda = \{(z, y) : 0 < z \leq \lambda, 0 \leq y \leq f_\lambda(z)\}, \quad (2.10) \]

that is, the region under the graph of

\[ f_\lambda(z) := \lambda h\left(\frac{z}{\lambda}\right) = \frac{1}{\pi} \left( \sqrt{\lambda^2 - z^2} - z \arccos \frac{z}{\lambda} \right), \quad (2.11) \]

and

\[ P_{\lambda,0} = \{0\} \times [0, f_\lambda(0)] = \{0\} \times \left[0, \frac{\lambda}{\pi}\right]. \]

It is well known that as \( \lambda \to +\infty \), the asymptotics of the lattice point counting function \( \mathcal{P}^D(\lambda) \) is intricately linked to the asymptotics of the disk eigenvalue counting function \( \mathcal{N}^D(\lambda) \), as was shown in [KuzFed65] and later re-discovered in [CdV10], see also [Gra07]. Namely, in some appropriate sense,

\[ \mathcal{N}^D(\lambda) \sim \mathcal{P}^D(\lambda) \quad \text{as} \quad \lambda \to +\infty. \]

This observation, together with asymptotic bounds on the difference between the two functions, has been used to great effect to estimate the remainder in Weyl's law for the unit disk. In particular, for the Dirichlet problem in the disk the two-term Weyl asymptotics (1.8) holds with an improved remainder estimate \( O(\lambda^{131/208}(\log \lambda)^{18627/8320}) \), see [GMWW21] (the remainder estimate \( O(\lambda^{2/3}) \) was already obtained in [KuzFed65], [CdV10]).

As has been recently found in [She22] in the Dirichlet case, there is a further simple non-asymptotic relation between the lattice point and the eigenvalue counting functions, which lies at the cornerstone of our proof of Theorem 1.2.

**Theorem 2.1.** For any \( \lambda \geq 0 \), we have

\[ \mathcal{N}^D(\lambda) \leq \mathcal{P}^D(\lambda) \leq \mathcal{N}^N(\lambda) \leq \mathcal{N}^D(\lambda). \quad (2.12) \]

**§ 3. Proof of Theorem 2.1**

We start by introducing some additional notation. Let, for \( m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) and \( \lambda \geq m \),

\[ \tilde{f}_m(\lambda) := \lambda h\left(\frac{m}{\lambda}\right) = \frac{1}{\pi} \left( \sqrt{\lambda^2 - m^2} - m \arccos \frac{m}{\lambda} \right), \quad (3.1) \]

with \( \tilde{f}_0(0) := 0 \). (Observe that (3.1) is just a re-labelling of (2.11): \( \tilde{f}_m(\lambda) = f_\lambda(m) \).) We note that \( \tilde{f}_m(m) = 0 \) for all \( m \).

Set, for \( \lambda \in [0, +\infty) \) and \( N \in \{D, N\} \),

\[ A^N_m(\lambda) := \begin{cases} \tilde{f}_m(\lambda) + s^N, & \text{if} \ \lambda \geq m, \\ s^N, & \text{if} \ 0 \leq \lambda < m, \end{cases} \quad (3.2) \]

where \( s^N \) is defined by (2.9). Some typical graphs of the functions \( A^N_m(\lambda) \) are shown in Figure 3.

Proceeding to the lattice point counting problems and comparing (2.4)–(2.6) with (3.1), (3.2), we immediately obtain

\[ \mathcal{P}^N(\lambda) = [A^N_0(\lambda)] + 2 \sum_{m=1}^{\infty} [A^N_m(\lambda)]. \quad (3.3) \]
Remark 3.1. Note that the terms with \( m > \lambda \) do not contribute into the sums in (3.3) since in that case \([A^R_m(\lambda)] = [s^R_m] = 0\).

Switching now to the eigenvalue counting functions, we recall the representations of the Bessel functions and their derivatives in terms of the so-called modulus functions \( M_\nu(z) \) and \( N_\nu(z) \) and the phase functions \( \theta_\nu(z) \) and \( \phi_\nu(z) \),

\[
J_\nu(z) = M_\nu(z) \cos \theta_\nu(z), \quad Y_\nu(z) = M_\nu(z) \sin \theta_\nu(z), \\
J'_\nu(z) = N_\nu(z) \cos \phi_\nu(z), \quad Y'_\nu(z) = N_\nu(z) \sin \phi_\nu(z) \tag{3.4}
\]

(see [DLMF, Eqs. 10.18.4–5]). We will be using various properties of the phase functions below; for a review of these properties see [Hor17]. We will be only considering the cases \( \nu \in \mathbb{N}_0 \) and \( z \geq 0 \) for which the moduli \( M_\nu(z) \) and \( N_\nu(z) \) are both positive.

Let us concentrate first on the Dirichlet eigenvalue counting function (2.1). We have \( J_m(z_0) = 0 \) if and only if \( \cos \theta_m(z_0) = 0 \), and so if and only if

\[
\frac{1}{\pi} \theta_m(z_0) + \frac{1}{2} \in \mathbb{Z}. \tag{3.5}
\]

Note that the phase function \( \theta_m(z) \) satisfies \( \theta_m(z) \to -\frac{\pi}{2} \) as \( z \to +0 \) [DLMF, Eq. 10.18.3], and that it is monotone increasing for \( z \in (0, +\infty) \) [Hor17, Theorem 1], therefore (3.5) can be replaced by

\[
B^D_m(z_0) \in \mathbb{N},
\]

where

\[
B^D_m(z) := \frac{1}{\pi} \theta_m(z) + \frac{1}{2}. \tag{3.6}
\]

As a result, (2.1) becomes

\[
\mathcal{N}^D(\lambda) = [B^D_0(\lambda)] + 2 \sum_{m=1}^{\infty} [B^D_m(\lambda)]. \tag{3.7}
\]

Remark 3.2. Similarly to Remark 3.1, the terms with \( m > \lambda \) do not contribute to (3.7). This is due to the fact that the Bessel function \( J_m(z) \) does not have zeros in the interval \((0, m)\).

In the same manner we have \( J'_m(z_0) = 0 \) if and only if

\[
\frac{1}{\pi} \phi_m(z_0) + \frac{1}{2} \in \mathbb{Z}. \tag{3.8}
\]

We note that the phase function \( \phi_m(z) \) satisfies \( \phi_m(z) \to \frac{\pi}{2} \) as \( z \to +0 \) [DLMF, Eq. 10.18.3]. Also, \( \phi_m(z) \) is monotone increasing for \( z \in (m, +\infty) \) and monotone decreasing for \( z \in (0, m) \) [Hor17, Theorem 1], with \( \phi_m(m) > -\frac{\pi}{2} \) [Hor17, formula (60)]. Thus, the condition (3.8) can be replaced by

\[
B^N_m(z_0) \in \mathbb{N},
\]

where

\[
B^N_m(z) := \frac{1}{\pi} \phi_m(z) + \frac{1}{2}. \tag{3.9}
\]

As a result, (2.2) becomes

\[
\mathcal{N}^N(\lambda) = [B^N_0(\lambda)] + 2 \sum_{m=1}^{\infty} [B^N_m(\lambda)]. \tag{3.10}
\]

Remark 3.3. Similarly to Remarks 3.1 and 3.2, the terms with \( m > \lambda \) do not contribute to (3.10), since there are no zeros of \( J'_m(z) \) in \((0, m)\).

The crucial contribution to the proof of Theorem 2.1 now comes form
Lemma 3.4. For any \( m \in \mathbb{N}_0 \) and any \( \lambda \in [0, +\infty) \) we have
\[
B_m^D(\lambda) < A_m^D(\lambda) < A_m^N(\lambda) < B_m^N(\lambda) \quad (3.11)
\]

Proof of Lemma 3.4. We have
\[
\tilde{f}_m(\lambda) = \frac{\lambda}{\pi} - \frac{m}{2} + O\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to +\infty. \quad (3.12)
\]

Using the asymptotics\(^7\) \([\text{Hor}17, \text{Eq. (21)}]\), \([\text{DLMF, Eq. 10.18.18}]\),
\[
\theta_m(\lambda) = \lambda - \frac{\pi}{2} \left( m + \frac{1}{2} \right) + O\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to +\infty,
\]
and \((3.6), (3.12)\), we obtain
\[
B_m^D(\lambda) - A_m^D(\lambda) \to 0 \quad \text{as } \lambda \to +\infty.
\]

Further,
\[
\frac{d}{d\lambda} B_m^D(\lambda) = \frac{1}{\pi} \theta'_m(\lambda) > \frac{\sqrt{\lambda^2 - m^2}}{\pi \lambda} \quad \frac{d}{d\lambda} A_m^D(\lambda)
\]
for \( \lambda \geq m \) by \([\text{Hor}17, \text{Eq. (56)}]\). As we additionally have
\[
\frac{d}{d\lambda} B_m^N(\lambda) < \frac{d}{d\lambda} A_m^N(\lambda)
\]
for \( \lambda \in (0, m] \), the function \( B_m^N(\lambda) - A_m^N(\lambda) \) is monotone increasing on \((0, +\infty)\) and tends to zero at infinity, therefore it is positive on \((0, +\infty)\).

Similarly, using the asymptotics \([\text{Hor}17, \text{Eq. (22)}]\), \([\text{DLMF, Eq. 10.18.21}]\),
\[
\phi_m(\lambda) = \lambda - \frac{\pi}{2} \left( m - \frac{1}{2} \right) + O\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to +\infty,
\]
and \((3.9), (3.12)\), we get
\[
B_m^N(\lambda) - A_m^N(\lambda) \to 0 \quad \text{as } \lambda \to +\infty.
\]

Also,
\[
\frac{d}{d\lambda} B_m^N(\lambda) < \frac{d}{d\lambda} A_m^N(\lambda)
\]
for \( \lambda \geq m \) by \([\text{Hor}17, \text{formula following Eq. (58)}]\). As we additionally have
\[
\frac{d}{d\lambda} B_m^D(\lambda) > 0 = \frac{d}{d\lambda} A_m^D(\lambda)
\]
for \( \lambda \in (0, m) \), the function \( B_m^D(\lambda) - A_m^D(\lambda) \) is monotone decreasing on \((0, +\infty)\) and tends to zero at infinity, therefore it is negative on \((0, +\infty)\).\(^8\)

We illustrate Lemma 3.4 in Figure 3.

Theorem 2.1 now immediately follows from Lemma 3.4 with account of formulae \((3.3), (3.7), (3.10)\).

\(^7\)See also \([\text{HBRV15}]\) for all the coefficients of the full asymptotic expansion and some useful remarks.

\(^8\)Surprisingly, despite the simplicity of inequalities \((3.11)\) (re-written in terms of the phase functions) and the fact that all the ingredients of their proof are in \([\text{Hor}17]\), they are not mentioned there.
§ 4. From the lattice point count towards Pólya’s conjecture

We first remark that the validity of Pólya’s conjecture for very small values of $\lambda$ easily follows from known estimates of the zeros of Bessel functions and their derivatives. We have

**Proposition 4.1.** Pólya’s conjecture holds for $\lambda \in [0, \Lambda_0]$ with $\Lambda_0 = \frac{9}{4}$.

*Proof.* Immediately follows from the bound on the first Dirichlet eigenvalue of the disk, $\sqrt{\Lambda_1} = j_{0,1} > \frac{3\pi}{4}$ (see [Het70, Theorem 3]) and the bound on the first non-zero Neumann eigenvalue of the disk, $\sqrt{\mu_2} = j'_{1,1} < 2$ (see [Wat66, page 486, formula (4)]). $\square$

By Theorem 2.1, Pólya’s conjecture would follow immediately from

**Theorem 4.2.** The inequalities

$$\mathcal{D}(\lambda) < \frac{\lambda^2}{4} < \mathcal{N}(\lambda) \quad (4.1)$$

hold for all $\lambda \in (\Lambda_0, +\infty)$.

We split the proof of Theorem 4.2 into two parts, by recalling the definition (1.11) of $\Lambda_1$ and stating

**Theorem 4.3.** The inequalities (4.1) hold for all $\lambda \geq \Lambda_1$.

and

---

The constant $\Lambda_0 = \frac{9}{4}$ is not optimal. Using more sophisticated estimates, the value of $\Lambda_0$ can be pushed a little bit higher; we do not pursue this direction.
**Theorem 4.4.** The inequalities (4.1) hold for all $\lambda \in (\Lambda_0, \Lambda_1)$.

The proof of Theorem 4.3 is purely analytic, and the proof of Theorem 4.4 is computer-assisted.

**Proof of Theorem 4.3.** We will deal with the Dirichlet case in detail, and sketch the modification required in the Neumann case at the end.

We count (and later estimate) the lattice points in (2.8) following the strategy (and often the notation) of Laugesen and Liu [LauLiu18] and Liu’s thesis [Liu17], with a few modifications as our situation is slightly different. Fix some $\alpha_\lambda$ in $(0, \lambda)$ and let

$$\beta_\lambda = f_\lambda(\alpha_\lambda).$$

Let $g_\lambda$ be the inverse function of $f_\lambda$, so that $y = f_\lambda(z)$ if and only if $z = g_\lambda(y)$. Our strategy is to break the region $\mathcal{P}_\lambda \cup \mathcal{P}_\lambda,0$ into three pieces, see Figure 4,

$$\mathcal{P}_\lambda \cup \mathcal{P}_\lambda,0 = \mathcal{P}_\lambda,\text{I} \cup \mathcal{P}_\lambda,\text{II} \cup \mathcal{P}_\lambda,\text{III},$$

where

$$\mathcal{P}_\lambda,\text{I} := \{(z, y) : 0 \leq z < \alpha, \beta < y \leq f(z)\}$$

is the piece above the line $y = \beta$,

$$\mathcal{P}_\lambda,\text{II} := \{(z, y) : \alpha < z \leq \lambda, 0 \leq y \leq f(z)\}$$

is the piece to the right of the line $z = \alpha$, and the rectangle

$$\mathcal{P}_\lambda,\text{III} := [0, \alpha] \times [0, \beta].$$

Laugesen and Liu [LauLiu18] do a vertical count in $\mathcal{P}_\lambda,\text{I}$ and a horizontal count in $\mathcal{P}_\lambda,\text{II}$. We flip this strategy, doing a horizontal count in $\mathcal{P}_\lambda,\text{I}$ and a vertical count in $\mathcal{P}_\lambda,\text{II}$. We also have to take account of the shift by $\frac{1}{4}$, the double-counting of the lattice points $\{m, k - \frac{1}{4}\}$ with $m > 0$, and the single counting of the ones with $m = 0$.

Implementing this strategy, we re-write (2.8) explicitly as

$$\mathcal{P}_\lambda^D(\lambda) = \mathcal{P}_\lambda^D,\text{I}(\lambda) + \mathcal{P}_\lambda^D,\text{II}(\lambda) + \mathcal{P}_\lambda^D,\text{III}(\lambda), \quad (4.2)$$

\[^{10}\text{From now on, we drop the \(\lambda\) subscripts and refer simply to \(\alpha, \beta, f\), and so on.}\]
where

\[ \mathcal{D}_I(\lambda) = 2 \sum_{k \in [\beta + \frac{1}{4}, \beta + \frac{1}{4} + \frac{1}{4}] \cap \mathbb{N}} \left( g \left( k - \frac{1}{4} \right) \right), \]

\[ \mathcal{D}_{II}(\lambda) = 2 \sum_{m \in [\alpha, \lambda] \cap \mathbb{N}} \left( f(m) + \frac{1}{4} \right), \]

\[ \mathcal{D}_{III}(\lambda) = \left( \beta + \frac{1}{4} \right) \cdot (2 \lfloor \alpha \rfloor + 1). \]

We will be evaluating the expressions (4.3) with the help of the sawtooth function

\[ \psi(\zeta) := \zeta - \lfloor \zeta \rfloor - \frac{1}{2}, \]

see Figure 5, substituting

\[ [\zeta] = \zeta - \psi(\zeta) - \frac{1}{2}. \]

We also define, for future use, the antiderivative of \( \psi \)

\[ \Psi(\zeta) := \int_0^\zeta \psi(\eta) \, d\eta = \frac{1}{2} (\zeta - [\zeta] - 1) (\zeta - [\zeta]). \]

In terms of the sawtooth function, the first sum in (4.3) becomes

\[ \mathcal{D}_I(\lambda) = 2 \sum_{k \in [\beta + \frac{1}{4}, \beta + \frac{1}{4} + \frac{1}{4}] \cap \mathbb{N}} \left( g \left( k - \frac{1}{4} \right) - \psi \left( g \left( k - \frac{1}{4} \right) \right) \right), \]

the second one becomes

\[ \mathcal{D}_{II}(\lambda) = 2 \sum_{m \in [\alpha, \lambda] \cap \mathbb{N}} \left( f(m) - \frac{1}{4} - \psi \left( f(m) + \frac{1}{4} \right) \right) \]

\[ = -\frac{1}{2} ([\lambda] - [\alpha]) + 2 \sum_{m \in [\alpha, \lambda] \cap \mathbb{N}} \left( f(m) - \psi \left( f(m) + \frac{1}{4} \right) \right) \]

\[ = -\frac{1}{2} (\lambda - \psi(\lambda)) + \frac{1}{2} (\alpha - \psi(\alpha)) + 2 \sum_{m \in [\alpha, \lambda] \cap \mathbb{N}} \left( f(m) - \psi \left( f(m) + \frac{1}{4} \right) \right), \]

Figure 5: The sawtooth function \( \psi(\zeta) \) (black) and its antiderivative \( \Psi(\zeta) \) (blue).
and the last expression becomes
\[ \mathcal{P}^D_{l\ell}(\lambda) = \left( \beta - \frac{1}{4} - \psi \left( \beta + \frac{1}{4} \right) \right) \left( 2(\alpha - \psi(\alpha)) \right). \]

Plugging these expressions back into (4.2) and simplifying, we obtain\(^1\)
\[ \mathcal{P}^D(\lambda) = 2 \sum_{k \in (\beta + \frac{1}{4} + \frac{1}{2}) \cap \mathbb{N}} g \left( \frac{k - 1}{4} \right) + 2 \sum_{m \in (a, \lambda) \cap \mathbb{N}} f(m) \]
\[ \quad \quad \quad + 2 \left( \beta - \psi \left( \beta + \frac{1}{4} \right) \right) (\alpha - \psi(\alpha)) - \frac{1}{2} (1 - \psi(\lambda)). \]

We now evaluate the sums in the first line of (4.5) using the following version of the Euler–Maclaurin formula, which is easily obtained from the version in [LauLiu18, p. 126]: for any piecewise \( C^1 \)-function \( F(\xi) \) defined on an interval containing \((a, b) \subset (0, +\infty),\)
\[ \sum_{n \in (a, b) \cap \mathbb{N}} F(n) = \int_a^b F(\xi) \, d\xi + \int_a^b F'(\xi) \psi(\xi) \, d\xi + F(a) \psi(a) - F(b) \psi(b). \] \label{eq:4.6}
Applying (4.6) first with \( F(\xi) = g \left( \xi - \frac{1}{4} \right) \) with account of \( g(\beta) = \alpha, g\left( \frac{1}{4} \right) = 0 \), and then with \( F(\xi) = f(\xi) \) with account of \( f(\alpha) = \beta, f(\lambda) = 0 \), we get
\[ \mathcal{P}_{(4.3),1}(\lambda) = 2 \int_{\beta}^{\lambda} g(y) \, dy + 2 \int_{\beta}^{\lambda} g'(y) \psi \left( y + \frac{1}{4} \right) \, dy \]
\[ \quad \quad \quad + 2 \int_{\lambda}^{\lambda} f(z) \, dz + 2 \int_{\alpha}^{\lambda} f'(z) \psi(z) \, dz \]
\[ \quad \quad \quad + 2 \beta \psi(\alpha) + 2 \alpha \psi \left( \beta + \frac{1}{4} \right). \]

We substitute this back into (4.5), simplify using the fact that
\[ \alpha \beta + \int_{\beta}^{\lambda} g(y) \, dy + \int_{\alpha}^{\lambda} f(x) \, dx = \text{Area}(\mathcal{P}_\lambda) = \int_0^{\lambda} f(z) \, dz = \frac{\lambda^2}{8}, \]
and re-arrange, yielding
\[ \mathcal{P}^D(\lambda) = \frac{\lambda^2}{4} - \frac{\lambda}{2} + \frac{1}{2} \psi(\lambda) + 2 \alpha \psi(\alpha) \psi \left( \beta + \frac{1}{4} \right) \]
\[ \quad \quad \quad + 2 \int_{\lambda}^{\lambda} f'(z) \psi(z) \, dz + 2 \int_{\beta}^{\lambda} g'(y) \psi \left( y + \frac{1}{4} \right) \, dy \]
\[ \quad \quad \quad - 2 \sum_{m \in (a, \lambda) \cap \mathbb{N}} \psi \left( f(m) + \frac{1}{4} \right) - 2 \sum_{k \in (\beta + \frac{1}{4}, \alpha) \cap \mathbb{N}} \psi \left( g \left( k - \frac{1}{4} \right) \right). \] \label{eq:4.7}

\(^1\)Here and further on, the various quantities with a formula number in a subscript, like \( \mathcal{P}_{(4.3),1} \), indicate that the corresponding expression is defined in that numbered formula.
We now proceed to estimating the marked terms in (4.7). Firstly, since $|\psi(\zeta)| \leq \frac{1}{2}$ for all $\zeta \geq 0$, we have
\[ |\mathcal{R}_{(4.7),1}(\lambda)| = \frac{1}{2} \psi(\lambda) + 2\psi(\alpha)\psi \left( \beta + \frac{1}{4} \right) \leq \frac{3}{4}. \] (4.8)

Secondly, using the definition (4.4) of the antiderivative $\Psi$ of $\psi$ and integrating by parts, we have
\[ \int_a^\lambda f'(z)\psi(z) \, dz = f'(\lambda)\Psi(\lambda) - f'(\alpha)\Psi(\alpha) - \int_a^\lambda f''(z)\Psi(z) \, dz, \]
which with account of
\[ f'(z) = -\frac{1}{\pi} \arccos \frac{z}{\lambda}, \quad f''(z) = \frac{1}{\pi\sqrt{\lambda^2 - z^2}} \] (4.9)
and
\[-\frac{1}{8} \leq \psi(\zeta) \leq 0 \quad \text{for all} \quad \zeta \geq 0,\]
yields
\[ \left| \int_a^\lambda f'(z)\psi(z) \, dz \right| \leq \frac{1}{8} \left| f'(\alpha) \right| + \frac{1}{8} \int_a^\lambda \left| f''(z) \right| \, dz = \frac{1}{4} \left| f'(\alpha) \right| \leq \frac{1}{8} \] (4.10)
(we note that $f''$ is positive, $f'$ is non-positive, $f'(\lambda) = 0$, and $|f'(\alpha)| = \frac{1}{2}$), and therefore
\[ |\mathcal{R}_{(4.7),2}(\lambda)| \leq \frac{1}{4}. \] (4.11)

Thirdly, in a similar but not identical fashion,
\[ \left| \int_{\beta}^{\lambda/2} g'(y)\psi \left( y + \frac{1}{4} \right) \, dy \right| \leq \left| g' \left( \frac{\lambda}{4} \right) \psi \left( \frac{\lambda}{4} + \frac{1}{4} \right) \right| + \left| g'(\beta)\psi \left( \beta + \frac{1}{4} \right) \right| + \int_{\beta}^{\lambda/2} \left| g''(y)\psi \left( y + \frac{1}{4} \right) \right| \, dy \]
\[ \leq \frac{1}{8} \left| g' \left( \frac{\lambda}{4} \right) \right| + \left| g'(\beta) \right| + \frac{1}{8} \left| g' \left( \frac{\lambda}{4} \right) \right| - \left| g'(\beta) \right| = \frac{1}{4} \left| g'(\beta) \right| = \frac{1}{4} \left| f'(\alpha) \right| \] (4.12)
(since $g''$ is positive, $g'$ is non-positive, and $g'(\beta) = \frac{1}{2} \arccos \frac{c}{\lambda}$ by the inverse function theorem). Therefore, with account of $f'(\alpha) = -\frac{1}{\pi} \arccos \frac{c}{\lambda}$, we have
\[ |\mathcal{R}_{(4.7),3}(\lambda)| \leq \frac{\pi}{2 \arccos \frac{c}{\lambda}}. \] (4.13)

To finish constructing effective estimates of $\mathcal{P}_{(4.7),4}(\lambda)$ we need to estimate the sums $\mathcal{R}_{(4.7),4}(\lambda)$ and $\mathcal{R}_{(4.7),5}(\lambda)$ of sawtooth functions which appear in (4.7). To do so, we utilise

**Proposition 4.5** (following R. S. Laugesen, S. Liu, E. Krätzel, and J. G. van der Corput). Suppose $0 \leq a < b$ and $F \in C^2[a, b]$, with $F''$ monotonic and nonzero on $[a, b]$. Then for any $c > 1$,
\[ \left| \sum_{n(a,b)} \psi(F(n)) \right| \leq \frac{11}{2} c^{2/3} \int_a^b \left| F''(t) \right|^{1/3} \, dt + \frac{11 \sqrt{c}}{\sqrt{c} - 1} \max_{r \in [a,b]} \frac{1}{\sqrt{|F''(t)|}} + \frac{1}{2}. \] (4.14)
Remark 4.6. Proposition 4.5 appears as [LauLiu18, Theorem 18], with the proof in [Liu17, Theorem A.8], but only with the specific $c = \frac{12}{11}^{3/2}$, see also [Kräo, Korollar zu Satz 1.5] and [vdC23, Satz 5]. The proof in [Liu17, Theorem A.8] however leaves some room for manoeuvre allowing us to improve the bound to (4.13). This modification is almost, but not quite, explicitly stated in the proof of [Liu17, Theorem A.8]; the only reason Proposition 4.5 does not appear there is that Liu chooses $c = \frac{12}{11}^{3/2}$ midway through the proof.

For the sake of completeness, we include the proof of Proposition 4.5 in Appendix A3.

Remark 4.7. Using the Laugesen–Liu version of Proposition 4.5 with $c = \frac{12}{11}^{3/2}$ increases the constant $A_1$ in (1.11) (above which we are able to produce a purely analytic proof of Pólya’s conjecture) to about 398300, making an implementation of a computer-assisted proof of Theorem 4.4 significantly more time consuming.

We now want to apply Proposition 4.5 to estimating

$$
|\mathcal{P}(\rho, y)\lambda(\lambda)| = 2 \left| \sum_{m \in (a, A) \cap \mathbb{N}} \psi\left( f(m) + \frac{1}{4} \right) \right|
$$

by taking $F(t) = f(t) + \frac{1}{4}$. We note that $F''(t)$ is monotone increasing and nonzero on $[a, \lambda)$, cf. (4.9), but blows up at $t = \lambda$, so we cannot apply (4.13) directly. We may however apply it after replacing $\lambda$ by $\lambda - \epsilon$ and taking the limit as $\epsilon \to 0^+$. The only minor issue with this arises if $\lambda$ happens to be an integer in which case it may contribute an extra $2|\psi\left( f(\lambda) + \frac{1}{4} \right)| \leq 1$ not accounted by the limit procedure. Taking this and

$$
\max_{t \in [a, A]} \frac{1}{\sqrt{F''(t)}} = \frac{1}{\sqrt{f''(a)}} = \sqrt{\pi} \sqrt{\lambda} \left( 1 - \frac{a}{\lambda} \right)^{1/4}
$$

into account, we obtain

$$
|\mathcal{P}(\rho, y)\lambda(\lambda)| \leq 11 c^{2/3} \frac{2}{\pi^{1/3} \lambda^{1/3}} \int_{a}^{\lambda} \left( 1 - \frac{t}{\lambda} \right)^{-1/6} \frac{1}{\sqrt{\pi} \sqrt{\lambda}} \left( 1 - \frac{a}{\lambda} \right)^{1/4} + 2
$$

$$
= \frac{11 c^{2/3} \lambda^{2/3}}{\pi^{1/3}} \int_{a}^{\lambda} \left( 1 - s^2 \right)^{-1/6} ds + \frac{22 \sqrt{c} \sqrt{\pi} \sqrt{\lambda}}{\sqrt{c - 1}} \frac{1}{\sqrt{\pi} \sqrt{\lambda}} \left( 1 - \frac{a}{\lambda} \right)^{1/4} + 2
$$

(4.14)

after the change of variables $s = \frac{t}{\lambda}$ in the integral.

We now want to apply Proposition 4.5 to estimating

$$
|\mathcal{P}(\rho, y)\lambda(\lambda)| = 2 \left| \sum_{k \in (\beta + \frac{1}{4}, \gamma + \frac{1}{4}]} \psi\left( g\left( k - \frac{1}{4} \right) \right) \right|
$$

by taking $F(t) = g\left( t - \frac{1}{4} \right)$ and $[a, b] = \left[ \beta + \frac{1}{4}, \frac{\alpha}{2} + \frac{1}{4} \right]$. A direct calculation with implicit differentiation yields

$$
g''(y) = \frac{1}{\lambda} \frac{\pi^2}{\arccos g(y)} \left( \frac{\pi}{\sqrt{1 - \left( \frac{g(y)}{\lambda} \right)^2}} \right)^2.
$$

(4.15)

Since $g(y)$ is monotone decreasing in $y$, $F''(t)$ is also monotone decreasing in $t$ (and positive). Moreover $F \in \mathcal{C}^2[a, b]$ as we assumed $g(\beta) = \alpha < \lambda$. Thus, Proposition 4.5 is applicable. We further have

$$
\max_{t \in [\beta + \frac{1}{4}, \gamma + \frac{1}{4}]} \frac{1}{\sqrt{F''(t)}} = \frac{1}{\sqrt{g''(\frac{1}{4})}} = \sqrt{\frac{\pi}{8} \sqrt{\lambda}},
$$

\footnote{We have also noticed that the constant $+1$ appearing in the analogue of (4.13) in [LauLiu18, Theorem 18], [Liu17, Theorem A.8], can be replaced by $+\frac{1}{2}$.}
and (4.13) gives, after the changed of variables \( y = t - 1/4 \) in the integral,

\[
\left| \mathcal{P}_{(4.7),5}(\lambda) \right| \leq 11 \epsilon^{2/3} \int_{\beta}^{\lambda} \frac{g''(y)}{\epsilon^{1/3}} \, dy + \frac{22 \sqrt{\epsilon}}{\sqrt{\epsilon} - 1} \sqrt{\frac{\pi}{8}} \sqrt{\lambda} + 1.
\]

Substituting (4.15) into the integrand and making another change of variables \( s = \frac{1}{\lambda} g(y) \) finally implies

\[
\left| \mathcal{P}_{(4.7),5}(\lambda) \right| \leq 11 \epsilon^{2/3} \lambda^{2/3} \int_{0}^{1/5} \left( 1 - s^2 \right)^{-1/6} \, ds + \frac{22 \sqrt{\epsilon}}{\sqrt{\epsilon} - 1} \sqrt{\frac{\pi}{8}} \sqrt{\lambda} + 1. \tag{4.16}
\]

We now incorporate the five bounds (4.8), (4.10), (4.12), (4.14), and (4.16) back into (4.7), leading to

\[
\mathcal{P}_{D}(\lambda) - \frac{\lambda^2}{4} \leq -\frac{\lambda}{2} + \frac{\pi}{2 \arccos \frac{\alpha}{\lambda}} + \frac{11 \epsilon^{2/3} \lambda^{2/3}}{\pi^{1/3}} \int_{0}^{1} \left( 1 - s^2 \right)^{-1/6} \, ds + \frac{22 \sqrt{\epsilon}}{\sqrt{\epsilon} - 1} \left( 1 - \left( \frac{\alpha}{\lambda} \right)^2 \right)^{1/4} + \frac{1}{2 \sqrt{2}} \right) + 4. \tag{4.17}
\]

The integral

\[
I := \int_{0}^{1} \left( 1 - s^2 \right)^{-1/6} \, ds
\]

can be explicitly evaluated in terms of the \( \Gamma \)-function,\(^{13}\) but it will be simpler for us to work with its approximation in easier terms, which is achieved via the change of variables \( s = \sin \sigma \), giving (using the Taylor expansion)

\[
I = \int_{0}^{\pi/2} (\cos \sigma)^{2/3} \, d\sigma \leq \int_{0}^{\pi/2} \left( 1 - \frac{\sigma^2}{3} - \frac{\sigma^6}{405} \right) \, d\sigma = \frac{\pi}{2} \cdot \frac{\pi^3}{72} - \frac{\pi^7}{362880} =: I. \tag{4.18}
\]

Substituting this into (4.17), we get

\[
\mathcal{P}_{D}(\lambda) - \frac{\lambda^2}{4} \leq -\frac{\lambda}{2} + \frac{\pi}{2 \arccos \frac{\alpha}{\lambda}} + \frac{11 \epsilon^{2/3} \lambda^{2/3}}{\pi^{1/3}} \int_{0}^{1} \left( 1 - s^2 \right)^{-1/6} \, ds + \frac{22 \sqrt{\epsilon}}{\sqrt{\epsilon} - 1} \left( 1 - \left( \frac{\alpha}{\lambda} \right)^2 \right)^{1/4} + \frac{1}{2 \sqrt{2}} \right) + 4. \tag{4.19}
\]

We emphasise that (4.19) holds for any choice of \( \alpha = \alpha_{\lambda} \in (0, \lambda) \) and \( \epsilon > 1 \).

We will now try to optimise the choices of the free parameters \( \alpha_{\lambda} \) and \( \epsilon \). It is convenient to introduce instead the new parameters

\[
\gamma_{\lambda} := \arccos \frac{\alpha}{\lambda} \in \left( 0, \frac{\pi}{2} \right)
\]

and

\[
\kappa := \frac{\sqrt{\epsilon}}{\sqrt{\epsilon} - 1} > 1
\]

so that

\[
\epsilon = \left( \frac{\kappa}{\kappa - 1} \right)^2.
\]

\(^{13}\) We have \( I = \frac{\sqrt{\pi} \Gamma\left( \frac{5}{6} \right)}{2 \Gamma\left( \frac{1}{2} \right)} \).
For brevity, we also denote the numerical constants

\[ a_1 := \frac{11\pi}{1^{1/3}}, \quad a_2 := 22\sqrt{\pi}, \quad a_3 := 11\sqrt{\frac{\pi}{2}}. \]  

(4.20)

Slightly increasing the bound, we estimate

\[ 1 - \left(\frac{a}{\lambda}\right)^2 \right)^{1/4} = \sin \gamma \lambda \leq \sqrt{\gamma \lambda}, \]

so that (4.19) becomes

\[ D^\lambda (\lambda) - \frac{\lambda^2}{4} \leq -\frac{1}{2} \lambda + \frac{\pi}{2\gamma \lambda} + a_2 k \sqrt{\lambda} \gamma + a_1 \left(\frac{k}{k-1}\right)^{4/3} \lambda^{2/3} + a_3 k \sqrt{\lambda} + 4. \]  

(4.21)

We now make a particular choice of \( \gamma \lambda \) in (4.21) in order to minimise the \( \gamma \)-dependent part \( D^\lambda (\gamma) \): to do so, we set

\[ \gamma \lambda := \pi^{2/3} a_2^{-2/3} k^{-2/3} \lambda^{-1/3} \]

(we still have the freedom of choosing \( k > 1 \) later). With this choice of \( \gamma \lambda \) we have

\[ D^\lambda (\gamma) = \frac{3}{2} \pi^{1/3} a_2^{2/3} k^{2/3} \lambda^{1/3}, \]

so that (4.21) becomes

\[ D^\lambda (\lambda) - \frac{\lambda^2}{4} \leq -\frac{1}{2} \lambda + \frac{3}{2} \pi^{1/3} a_2^{2/3} k^{2/3} \lambda^{1/3} + a_1 \left(\frac{k}{k-1}\right)^{4/3} \lambda^{2/3} + a_3 k \sqrt{\lambda} + 4 \leq P_\kappa (\lambda). \]  

(4.22)

Let

\[ W_\kappa (\omega) := -\frac{1}{2} \omega^6 + b_{4,\kappa} \omega^4 + b_{3,\kappa} \omega^3 + b_{2,\kappa} \omega^2 + 4 \]

be the polynomial of degree six in \( \omega \) with \( \kappa \)-dependent coefficients

\[ b_{4,\kappa} := a_1 \left(\frac{k}{k-1}\right)^{4/3}, \quad b_{3,\kappa} := a_3 k, \quad b_{2,\kappa} = \frac{3}{2} \pi^{1/3} a_2^{2/3} k^{2/3}. \]  

(4.23)

Then (4.22) can be re-written as

\[ D^\lambda (\lambda) - \frac{\lambda^2}{4} \leq W_\kappa (\lambda^{1/6}) = P_\kappa (\lambda). \]

It is easily seen from (4.22) that for any \( \kappa > 1 \), \( P_\kappa' (\lambda) \) has a single positive root, which immediately implies that \( P_\kappa (\lambda) \) also has a single positive root \( \Lambda^{*}_\kappa \), and \( W_\kappa (\omega) \) has a single positive root \( \omega^{*}_\kappa = (\Lambda^{*}_\kappa)^{1/6} \), see Figure 6. Thus, with any choice of \( \kappa > 1 \), the Dirichlet inequality in Theorem 4.3 will hold if we choose \( \Lambda_1 > \Lambda^{*}_\kappa \).

Let us set\(^{14}\)

\[ \kappa = \kappa^{*} := \frac{10}{3}, \]

then

\[ W_{\kappa^{*}} (\omega) = -\frac{1}{2} \omega^6 + b_{4,\kappa} \omega^4 + b_{3,\kappa} \omega^3 + b_{2,\kappa} \omega^2 + 4, \]

\(^{14}\)This choice of \( \kappa \) is slightly off the optimal which would be around 3.3356 but the resulting difference in \( \Lambda_1 \) is negligible.
where from (4.20), (4.18), and (4.23) we obtain
\[ b_4 := b_{4,\kappa} = \frac{5^{1/3} 11 \left( 181440 \pi^{2/3} - 5040 \pi^{8/3} - \pi^{20/3} \right)}{28^{1/3} 127008}, \]
\[ b_3 := b_{3,\kappa} = \frac{55 \sqrt{2} \pi}{3}, \quad b_2 := b_{2,\kappa} = 6^{1/3} (55 \pi)^{2/3}. \]
Using the rational approximations
\[ \frac{2818}{957} =: \pi < \frac{2862}{911} , \]
it is easy to check\(^{15}\) that
\[ b_4 < b_4 := \frac{8261}{604}, \quad b_3 < b_3 := \frac{21415}{466}, \quad b_2 < b_2 := \frac{66913}{1187}, \]
and therefore
\[ W_{\kappa,\alpha}(\omega) < W_{\kappa,\alpha}(\omega) := -\frac{1}{2} \omega^6 + b_4 \omega^4 + b_3 \omega^3 + b_2 \omega^2 + 4. \]
Another set of easy computations with \( \omega_0 := \frac{44489}{6721} \) then gives
\[ W_{\kappa,\alpha}(\omega_0) < 0, \]
so that \( \omega_0 > \omega_{\kappa,\alpha}^* \), and choosing
\[ \Lambda_1 = 84123 > \omega_0^6 \]
completes the proof.

The Neumann case is essentially identical. The Neumann counting function may be written as
\[ \mathcal{P}^N(\lambda) = \mathcal{P}^N_1(\lambda) + \mathcal{P}^N_2(\lambda) + \mathcal{P}^N_3(\lambda), \]
where
\[ \mathcal{P}^N_1(\lambda) = 2 \sum_{k \in \mathbb{Z}^+} \left( g \left( k - \frac{3}{4} \right) \right) + \frac{1}{2}, \]
\[ \mathcal{P}^N_2(\lambda) = 2 \sum_{m \in (0, A]} \left( f(m) + \frac{3}{4} \right), \]
\[ \mathcal{P}^N_3(\lambda) = \left( \beta + \frac{3}{4} \right) \cdot (2 \lfloor \alpha \rfloor + 1). \]

\(^{15}\)These checks involve only taking the powers of integers, and are therefore exact. No floating-point approximations have been used in the process.
An identical series of calculations with the sawtooth functions yields the following analogue of (4.7):

\[
P_D(\lambda) = \frac{\lambda^2}{4} + \frac{\lambda}{2} + \frac{1}{2} \psi(\lambda) - 2 \psi(\alpha) \psi\left(\beta + \frac{3}{4}\right) - 2 \int_{\alpha}^{\lambda} f'(z) \psi(z) \, dz + 2 \int_{\beta}^{\gamma} g'(y) \psi\left(y + \frac{3}{4}\right) \, dy - \sum_{m \in [a, \lambda] \cap \mathbb{N}} \psi\left(\frac{f(m)}{\lambda} + \frac{3}{4}\right) - \sum_{k \in [\beta + \frac{3}{4}, \gamma + \frac{3}{4}] \cap \mathbb{N}} \psi\left(\frac{g(k - \frac{3}{4})}{\lambda}\right).
\]

(4.26)

Again by an identical argument, each of the five bounds (4.8), (4.10), (4.12), (4.14), and (4.16) holds for the analogous marked term in (4.26). Therefore, as in (4.17),

\[
P_N(\lambda) - \frac{\lambda^2}{4} \geq \frac{\lambda}{2} - \frac{\pi}{2} \arccos \frac{a}{\lambda} - \frac{11 c^{2/3} \lambda^{2/3}}{\pi^{1/3}} \int_{0}^{1} \left(1 - s^2\right)^{-1/6} \, ds - \frac{22 \sqrt{c} \sqrt{\pi} \sqrt{\lambda}}{\sqrt{\lambda - 1}} \left(\left(1 - \frac{a}{\lambda}\right)^{1/4} + \frac{1}{2 \sqrt{2}}\right) - 4.
\]

(4.27)

The right-hand side of (4.27) is precisely the negative of the right-hand side of (4.17). Thus it is positive whenever \(\lambda > \Lambda_1\), completing the proof of Theorem 4.3 in the Neumann case.

§ 5. Computer-assisted proof of Theorem 4.4

We describe the algorithm (based on the two Principles stated in § 1) of verifying the statements

\[
P_D(\lambda) < \frac{\lambda^2}{4} \quad \text{for all} \quad \lambda \in \left[\frac{9}{4}, 84123\right] = [\Lambda_0, \Lambda_1]
\]

(5.1)

and

\[
P_N(\lambda) > \frac{\lambda^2}{4} \quad \text{for all} \quad \lambda \in [\Lambda_0, \Lambda_1],
\]

(5.2)

with the lattice point counts given explicitly by

\[
P_D(\lambda) = \left\lfloor \frac{\lambda}{\pi} \right\rfloor + 2 \sum_{m \in [1, \lfloor \lambda \rfloor] \cap \mathbb{N}} \left| \lambda h\left(\frac{m}{\lambda}\right) + \frac{1}{4}\right|
\]

(5.3)

and

\[
P_N(\lambda) = \left\lfloor \frac{\lambda}{\pi} \right\rfloor + 2 \sum_{m \in [1, \lfloor \lambda \rfloor] \cap \mathbb{N}} \left| \lambda h\left(\frac{m}{\lambda}\right) + \frac{3}{4}\right|
\]

(5.4)

The first Principle is easily implemented with the help of two simple lemmas.

**Lemma 5.1.** Let the inequality in (5.1) hold for a particular \(\lambda_0\), that is we have

\[
ed^D(\lambda_0) := \frac{\lambda_0^2}{4} - P_D(\lambda_0) > 0.
\]

(5.5)

Then this inequality also holds for

\[
\lambda \in [\lambda_0 - \text{shift}^D(\lambda_0), \lambda_0],
\]

where

\[
\text{shift}^D(\lambda) := \left\lfloor \frac{\lambda}{\pi} \right\rfloor + \sum_{m \in [1, \lfloor \lambda \rfloor] \cap \mathbb{N}} \left| \lambda h\left(\frac{m}{\lambda}\right) + \frac{1}{4}\right|.
\]

(5.6)
where $$\text{shift}^D(\lambda_0) := \frac{2e^D(\lambda_0)}{\lambda_0}$$.

**Proof.** For $$\delta \in [0, \text{shift}^D(\lambda_0)]$$, we have

$$\mathcal{D}^D(\lambda_0 - \delta) \leq \mathcal{D}^D(\lambda_0) = \frac{\lambda_0^2}{4} - e^D(\lambda_0) \leq \frac{\lambda_0^2}{4} - \frac{\delta \lambda_0}{2} = \frac{1}{4} (\lambda_0 - \delta)^2 - \frac{1}{4} \delta^2 \leq \frac{(\lambda_0 - \delta)^2}{4},$$

and the result follows. \(\square\)

**Lemma 5.2.** If the inequality in (5.2) holds for a particular $$\lambda_0$$, that is we have

$$e^N(\lambda_0) := \mathcal{N}^N(\lambda_0) - \frac{\lambda_0^2}{4} > 0,$$ \quad (5.6)

this inequality also holds for $$\lambda \in (\lambda_0, \lambda_0 + \text{shift}^N(\lambda_0)]$$, where

$$\text{shift}^N(\lambda_0) := \frac{2e^N(\lambda_0)}{\lambda_0 + 2e^N(\lambda_0)}.$$

**Proof.** Let $$\delta_0(\lambda_0) := \frac{\lambda_0^2}{4} + 4e^N(\lambda_0) - \lambda_0$$ denote the positive root of the quadratic equation $$\delta^2 + 2\lambda_0\delta - 4e^N(\lambda_0) = 0$$. Then for $$\delta \in [0, \delta_0(\lambda_0)]$$ we have

$$\mathcal{N}^N(\lambda_0 + \delta) \geq \mathcal{N}^N(\lambda_0) = \frac{\lambda_0^2}{4} + e^N(\lambda_0) \geq \frac{(\lambda_0 + \delta)^2}{4},$$

and the result follows since

$$\delta_0(\lambda_0) = \frac{4e^N(\lambda_0)}{\sqrt{\lambda_0^2 + 4e^N(\lambda_0)} + \lambda_0} \geq \frac{4e^N(\lambda_0)}{2\lambda_0 + \frac{2e^N(\lambda_0)}{\lambda_0}} = \text{shift}^N(\lambda_0).$$ \(\square\)

Thus, the basic algorithms work as follows. For proving (5.1), we move downwards: set $$\lambda = \Lambda_1$$ from Theorem 4.3, compute the margin $$e^D(\Lambda_1)$$ from (5.3), set $$\lambda = \Lambda_1 - \text{shift}^D(\Lambda_1)$$, and continue the process. If the differences are positive on each step, the processes will stop successfully if after a finite number $$S^D$$ of steps we reach $$\Lambda_0 = \frac{3}{4}$$. For proving (5.2) we move upwards, starting with $$\lambda = \Lambda_0$$, and stopping once we reached $$\Lambda_1$$ in a finite number of steps $$S^N$$. 

A realisation of Principle 2 is less straightforward since the function $$h$$ does not map rationals into rationals, and therefore cannot be used directly. To bypass this issue we create its lower and upper rational approximating functions $$\overline{h}_Q, \underline{h}_Q : Q \cap [0, 1] \to Q$$, such that $$\overline{h}_Q(x) \leq h(x) \leq \underline{h}_Q(x)$$ for all $$x \in Q \cap [0, 1]$$. Their explicit construction is described below.

Once the functions $$\overline{h}_Q$$ and $$\underline{h}_Q$$ are constructed, we can replace the exact point count $$\mathcal{D}^D(\lambda)$$ by its upper bound $$\mathcal{D}^D(\lambda)$$ obtained by substituting $$\overline{h}_Q$$ for $$h$$ in (5.3), and the exact point count $$\mathcal{N}^D(\lambda)$$ by its lower bound $$\mathcal{N}^D(\lambda)$$ obtained by substituting $$\underline{h}_Q$$ for $$h$$ in (5.4). We then implement the basic algorithm outlined above for $$\mathcal{D}^D(\lambda)$$ and $$\mathcal{N}^D(\lambda)$$.  

We could have taken $$\text{shift}^N(\lambda_0) = \delta_0(\lambda_0)$$ but refrained from doing so in order to keep $$\text{shift}^N(\lambda_0)$$ rational for a rational $$\lambda_0$$. 

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**M. Levitin, I. Polterovich, and D. A. Sher**
Algorithm for proving (5.1)

\[ \lambda \leftarrow \Lambda_1 \]

while \( \lambda > \Lambda_0 \) do

\[ p \leftarrow \mathcal{P}(\lambda) \]

\[ e^D = \lambda^2 / 4 - p \]

if \( e^D > 0 \) then

\[ \text{shift}^D = -2e^D / \lambda \]

\[ \lambda = \lambda - \text{shift}^D \]

else

print “Proof failed \( \mathbb{Q} \)”; exit

end if

end while

print “Dirichlet case proved \( \mathbb{Q} \)”

Algorithm for proving (5.2)

\[ \lambda \leftarrow \Lambda_0 \]

while \( \lambda < \Lambda_1 \) do

\[ p \leftarrow \mathcal{P}(\lambda) \]

\[ e^{N} = p - \lambda^2 / 4 \]

if \( e^{N} > 0 \) then

\[ \text{shift}^N = -2e^N / (\lambda + 2e^N / \lambda) \]

\[ \lambda = \lambda + \text{shift}^N \]

else

print “Proof failed \( \mathbb{Q} \)”; exit

end if

end while

print “Neumann case proved \( \mathbb{Q} \)”

Figure 7: Basic algorithms.

Remark 5.3. Before explaining how to construct rational approximating functions \( h_Q \) and \( \bar{h}_Q \) of \( h \), we want to stress that these approximations should be rather tight. Let us concentrate on \( \bar{h}_Q \). We already know from [KuzFed65] or [CdV10] (or by arguing as in the proof of Theorem 4.3) that we asymptotically have

\[ e^D(\lambda) = \frac{\lambda^2}{4} - \mathcal{P}(\lambda) = \frac{\lambda}{2} + o(\lambda) \quad \text{as} \quad \lambda \to \infty. \]

If we overestimate \( h(x) \) by \( \bar{h}_Q \) with a relative error \( \epsilon \), \( \bar{h}_Q(x) \approx (1 + \epsilon)h(x) \), then we underestimate \( e^D(\lambda) \) with an error asymptotically given by \( \epsilon \frac{\lambda^2}{4} \), so we need to have \( \epsilon \lesssim \frac{2}{\Lambda_1} \approx 2 \cdot 10^{-5} \) to ensure that \( \lambda^2 / 4 - \mathcal{P}(\lambda) \) stays positive. Also, asymptotically each step of the verification algorithm decreases \( \lambda \) by at most \( \text{shift}^D(\lambda) = \frac{2e^D(\lambda)}{\lambda} = 1 + o(\lambda) \); substantially underestimating \( e^D(\lambda) \) would have led to a significant increase in the number of (computationally costly) steps required for the algorithm’s convergence.

It will be simpler to work with the function

\[ v(x) := \sqrt{1 - x^2} - x \arccos x \]

instead of \( h(x) \), and to take care of the factor \( \frac{1}{\pi} \) later.

To create upper and lower rational approximating functions \( \bar{v}_Q(x) \) and \( v_Q(x) \) we proceed in two steps. On the first step, we create upper and lower pre-rational approximating functions \( \bar{v}_{\text{pre}-Q}(x) \) and \( v_{\text{pre}-Q}(x) \) which bound \( v(x) \) from above and below, respectively, but still involve a small number of irrational constants. Finding good rational approximations for these constants and incorporating them into the explicit expressions for \( \frac{1}{\pi} \bar{v}_{\text{pre}-Q}(x) \) and \( \frac{1}{\pi} v_{\text{pre}-Q}(x) \) (see Appendix A2) then completes the job.

We will create both pre-rational approximations as piecewise smooth functions by first partitioning \([0,1]\) into fourteen intervals \( E_i := [x_{i-1}, x_i], \quad i = 1, \ldots, 13, \quad E_{14} := [x_{14}, x_{15}] \), with \( x_i := \frac{i-1}{10} \) for \( i = 1, \ldots, 7 \), and \( x_i := \frac{6}{10} + \frac{i-7}{20} \) for \( i = 8, \ldots, 15 \) (taking finer intervals on the right improves the accuracy of the approximations). We further sub-partition each interval \( E_i \) into \( E_{i, \pm} := [x_{c,i}, x_{c,i+1}] \), where \( x_{c,i} := \frac{1}{2} (x_i + x_{i+1}) \) is the centre of \( E_i \). We will also need an additional subdivision of the right-most interval \( E_{14} := E_{14} \) into ten equal length sub-intervals \( E_{R,k} := [x_{R,k}, x_{R,k+1}] \), where \( x_{R,k} = \frac{9k}{100} + \frac{k-1}{200}, \quad k = 1, \ldots, 11 \), see Figure 8.
Figure 8: The division of the interval $[0,1]$ into fourteen intervals $E_i$ and further sub-intervals.

The construction of $v_{\text{pre-Q}}(x)$ on all fourteen intervals $E_i$, and of $\overline{v}_{\text{pre-Q}}(x)$ on the first thirteen intervals is essentially the same and uses the fourth order Taylor polynomial of $v(x)$ at the centre $x_{c,i}$ of the interval, with explicit remainder estimates done separately for upper and lower bounds, and for $x \in E_{i,-}$ and $x \in E_{i,+}$. As $v'(x) = -\arccos x$, $v''(x) = (1 - x^2)^{-1/2}$, $v'''(x) = x(1 - x^2)^{-3/2}$, $v^{(4)}(x) = (1 + 2x^2)(1 - x^2)^{-5/2}$, $v^{(5)}(x) = 3x(3 + 2x^2)(1 - x^2)^{-7/2}$, then taking into account the signs of $x - x_{c,i}$ on sub-intervals $E_{i,\pm}$, and the fact that the fifth derivative of $v$ is positive and monotone increasing on $[0,1)$, we set

$$v_{\text{pre-Q}}|_{x \in E_{i,\pm}}(x) = v_{\text{pre-Q},i,\pm}(x) = \sum_{j=0}^{4} \alpha_{i,j} (x - x_{c,i})^j + \alpha_{i,5} (x - x_{c,i})^5, \quad i = 1, \ldots, 14, \quad (5.7)$$

and

$$\overline{v}_{\text{pre-Q}}|_{x \in E_{i,\pm}}(x) = \overline{v}_{\text{pre-Q},i,\pm}(x) = \sum_{j=0}^{4} \alpha_{i,j} (x - x_{c,i})^j + \alpha_{i,5,\pm} (x - x_{c,i})^5, \quad i = 1, \ldots, 13, \quad (5.8)$$

where

$$\alpha_{i,j} = \frac{v^{(j)}(x_{c,i})}{j!}, \quad j = 0, \ldots, 5, \quad (5.9)$$

$$\alpha_{i,5,\pm} = \frac{v^{(5)}(x_{i+1})}{5!}, \quad \alpha_{i,5,-} = \alpha_{i-1,5,+} = \frac{v^{(5)}(x_i)}{5!}, \quad \alpha_{1,5,-} = 0. \quad (5.10)$$

The explicit expressions for the coefficients $\alpha_{i,j}$ and $\alpha_{i,5,\pm}$ are collected in Table 1 in Appendix A1. One notices that they still contain some irrational numbers: particular values of the arccosine, and some square roots. They will be taken care of a bit later.

The method of constructing an upper pre-rational approximation described above does not work on the last interval $E_R = E_{14} = \left[\frac{65}{100},1\right]$ as the remainder term blows up at $x = 1$, so we employ a different strategy there. Let $w(x) := (1 - x)^{3/2}$.

We start with a following technical

**Lemma 5.4.** The function

$$C(x) := \frac{v(x)}{w(x)}, \quad x \in [0,1), \quad C(1) := \frac{2\sqrt{2}}{3},$$

is monotone decreasing on the interval $[0,1]$. 


Proof. Since the derivative of $C(x)$ is

$$C'(x) = \frac{3\sqrt{1-x^2} - (2 + x) \arccos x}{2(1-x)^{5/2}},$$

it is enough to show that the numerator in the expression above is negative for $x \in [0, 1)$, or equivalently, switching to the variable $s = \arccos x$, that

$$\eta(s) := 3s - s(2 + \cos s)$$

is negative on $\left[0, \frac{\pi}{2}\right]$. We now note that $\eta(0) = 0$, and

$$\eta'(s) = 2\cos s - 2 + s \sin s = 2\left(\frac{s}{2} - \tan\frac{s}{2}\right)\sin s < 0$$

as required. \qed

Lemma 5.4 immediately implies

**Corollary 5.5.** Let $[a, b] \subseteq [0, 1]$, $a < b$. Then

$$w(x) \leq C(a)w(x) \quad \text{for all } x \in [a, b].$$

We now observe that $w(x)$ is a convex function, therefore its graph on any given interval lies below the chord joining the endpoints of the graph, and $w(x)$ can be bounded above by a linear function. Using the sub-division of the interval $E_R = E_{14}$, we define

$$\overline{\eta}_{pre-Q}(x)_{E_{R,k}}(x) = C_k \left( w_k - \frac{w_k - w_{k+1}}{x_{R,k+1} - x_{R,k}}(x - x_{R,k}) \right), \quad k = 1, \ldots, 10, \quad (5.11)$$

where

$$C_k := C(x_{R,k}), \quad w_k := w(x_{R,k}). \quad (5.12)$$

The explicit expressions for the coefficients $C_k$ and $w_k$ are collected in Table 2 in Appendix A1.

We have now fully described the construction of the lower and upper pre-rational approximating functions $\overline{\eta}_{pre-Q}(x)$ and $\overline{\eta}_{pre-Q}(x)$. We now need to construct a proper lower rational approximation $\overline{\eta}_{Q}(x)$ of a proper upper rational approximation $\overline{\eta}_{Q}(x)$ of $\frac{1}{\pi} \overline{\eta}_{pre-Q}(x)$. This is done by finding and using as appropriate (with account of signs of corresponding expressions on each sub-interval) verified upper and lower rational approximations of some square roots and arccosines (and of course of $\pi$) listed in Table 3 in Appendix A2. As a result, we finally obtain

$$\overline{\eta}_{Q}(x) = \sum_{j=1}^{5} q_{i,j}(x - x_{c,i})^{j/2}, \quad \text{for } x \in E_{i,\pi}, \quad i = 1, \ldots, 14, \quad (5.13)$$

$$\overline{\eta}_{Q}(x) = \left\{ \begin{array}{ll}
\sum_{j=1}^{5} Q_{j,0,1} (x - x_{c,i})^{j/2}, & \text{for } x \in E_{i,\pi}, \quad i = 1, \ldots, 13, \\
Q_{R,R,0} + Q_{R,R,1} (x - x_{R,k}), & \text{for } x \in E_{R,k}, \quad k = 1, \ldots, 10.
\end{array} \right. \quad (5.14)$$

The explicit values of the rational constants $q_{i,j}$ and $Q_{i,j}$ we used in the actual calculations are given in Table 4 in Appendix A2.

For illustration purposes only, we plot the differences $\overline{\eta}_{Q}(x) - \overline{\eta}_{Q}(x)$ and $\overline{\eta}_{Q}(x) - \overline{\eta}_{Q}(x)$ in Figure 9.

We can now describe the results of our implementation\footnote{The algorithms were implemented as a simple parallelised sixteen kernel Mathematica 12.1 script running on an AMD Ryzen 9 5950X 16-core processor. The total processor time slightly exceeded thirteen hours.} of the basic algorithms shown in Figure 7. The algorithms in both Dirichlet and Neumann cases converge successfully.
proving (5.1) (and therefore for fully establishing Pólya’s conjecture for the Dirichlet Laplacian for the disk) reaches \( \Lambda_0 \) from \( \Lambda_1 \) in \( S^D = 92197 \) steps. The algorithm for proving (5.2) (and therefore for fully establishing Pólya’s conjecture for the Neumann Laplacian for the disk) reaches \( \Lambda_1 \) from \( \Lambda_0 \) in \( S^N = 90160 \) steps. The progress of both algorithms as a function of a number of steps is shown in Figure 10.\(^{18}\)

The margins of point counts compared to \( \lambda^2 \) are shown (as percentages) in Figure 11. As we have already mentioned in Remark 5.3, they are quite small, thus confirming \textit{a posteriori} the need for tight approximations \( \overline{h}_Q \) and \( \overline{h}_Q \) of \( h \).

\(^{18}\)The full results of the computations are available for download at michaellevitin.net/polya.html.
The shifts on each step of the iteration process are shown in Figure 12. Note that they roughly match asymptotic estimates of Remark 5.3 due to our careful approximation of $h(x)$ by functions which take rational values at rationals.

Figure 11: The margins $e^D(\lambda)$ and $e^N(\lambda)$ shown as percentages of $\lambda^2/4$. The actual margins very slightly exceed those shown on the plots.

Figure 12: Gained shifts in $\lambda$ during the iteration processes.
Appendix A1. Pre-rational approximations

The tables in this Appendix list the coefficients of the pre-rational approximations of \( v(x) \) on the intervals \( E_{1,\ldots,14} \) (the lower approximation) and \( E_{1,\ldots,13} \) (the upper approximation) and their subintervals \( E_{i,\pm \epsilon} \) in Table 1, and the coefficients of the upper pre-rational approximation on the the interval \( E_R \) and its subintervals \( E_{R,k} \) in Table 2.

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<th>( \alpha_{i,0} )</th>
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Table 1: The coefficients used in the constructions of the upper pre-rational approximating functions \( \mathcal{T}_{\text{pre},-0,i,\pm}(x), \ i = 1,\ldots,13, \) on the interval \( [0, \frac{\pi}{2}) \), and the lower pre-rational approximating functions \( \mathcal{L}_{\text{pre},-0,i,\pm}(x), \ i = 1,\ldots,14, \) on the interval \([0, 1]\), see (1.7)–(1.10). Here \( \beta_i = \arccos \kappa_{E,i}. \) We additionally use \( \alpha_{i,5,-} = \alpha_{i-1,5,+}, \alpha_{1,5,-} = 0. \)
Table 2: The coefficients of the upper pre-rational approximating function \( v_{\text{pre-Q}, k}(x) \) on the interval \([\frac{9}{10}, 1]\), see (5.11), (5.12). Here \( \beta_{R,k} = \arccos \frac{x}{R_k} \). We also use \( w_{11} = 0 \).

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<td>( 2(\sqrt{790} - 39\sqrt{10}\beta_{R,6}) )</td>
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Appendix A2. Approximation of irrational constants

Throughout, we denote by

\[ \mathbb{Q} \ni x < \bar{x} \in \mathbb{Q} \]

some lower and upper approximations of \( x \notin \mathbb{Q} \) (see also Remark A2.1).

The verification of the approximations for the square roots is by taking the squares and comparing rational numbers, and is therefore trivial. To verify our approximations of arccosines (taken at rational points) one may proceed as follows. Define the functions \( \cos, \cos : \mathbb{Q} \to \mathbb{Q} \) as

\[ \cos x := T_{12}[\cos](x), \quad \bar{\cos} x := T_{14}[\cos](x), \]

where \( T_t[\cos](x) \) is the Taylor polynomial of \( \cos x \) at \( x = 0 \) of degree \( t \), so that \( \cos x < \cos x < \bar{\cos} x \). Then

\[ \bar{\cos}(\beta) < x < \cos(\beta) \quad \text{implies} \quad \beta < \beta = \arccos x < \bar{\beta}, \]

and the verification is again reduced to elementary operations.

We give an illustration of constructing \( \bar{\theta}_Q(x) \) from \( \frac{1}{2} \bar{v}_{\text{pre-Q}}(x) \) and the data in Tables 1–3. On the first subinterval \( E_{1,-} = (0, \frac{1}{20}) \), the function \( \bar{v}_{\text{pre-Q},1,-}(x) = \bar{v}_{\text{pre-Q}}(x) \) is given by (see (5.8))

\[
\bar{v}_{\text{pre-Q},1,-}(x) = \sum_{j=0}^{4} a_{1,j} \left( x - \frac{1}{20} \right)^j = \frac{\sqrt{399} - \beta_1}{20} - \beta_1 \left( x - \frac{1}{20} \right) + \frac{10}{\sqrt{399}} \left( x - \frac{1}{20} \right)^2 \\
+ \frac{200}{1197\sqrt{399}} \left( x - \frac{1}{20} \right)^3 + \frac{134000}{159201\sqrt{399}} \left( x - \frac{1}{20} \right)^4, 
\]

where \( \beta_1 = \arccos \frac{1}{20} \). Taking into account the signs of the corresponding terms, and whether the square roots appear in the denominators or the numerators, the expression for \( \bar{\theta}_Q(x) \) on this subin-
terval will be

\[
\tilde{\eta}_Q(x)_{x \in E_{\beta}} = \sum_{j=0}^{5} Q_{1,j,-} \left( x - \frac{1}{20} \right)^j
\]

\[
= \frac{\sqrt{399} - \beta_1}{20\pi} - \frac{\beta_1}{\pi} \left( x - \frac{1}{20} \right) + \frac{10}{\pi \sqrt{399}} \left( x - \frac{1}{20} \right)^2
\]

\[
+ \frac{200}{1197\pi \sqrt{399}} \left( x - \frac{1}{20} \right)^3 + \frac{134000}{159201 \pi \sqrt{399}} \left( x - \frac{1}{20} \right)^4
\].

The constants \(Q_{1,j,-}, j = 0, \ldots, 4\), are then obtained by elementary operations, cf. Table 4, and \(Q_{1,5,-} = 0\).

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Table 3: Rational approximations of irrational constants.
Table 4: The rational constants used in actual calculations: $q_{i,j,k}, i = 1, \ldots, 14, j = 0, \ldots, 5$ appearing in (5.11), and $Q_{i,j,k}, i = 1, \ldots, 13, j = 1, \ldots, 5$, as well as $Q_{k,k,j}, k = 1, \ldots, 10, j = 0, 1$ appearing in (5.14). For even values of $j$, $q_{i,j,k} = q_{i,j,-k}$, and $Q_{i,j,k} = Q_{i,j,-k}$. The data collected in this Table are available for download at michaellevitin.net/polya.html.
Appendix A3. Proof of Proposition 4.5

First, we quote

Lemma A3.1 ([Liu17, Corollary A.7], originally [Krä00, Korollar zu Satz 1.4]). Suppose \( 0 < a < b \), and \( F \in C^2[a, b] \) with \( F'' \) monotone and nonzero, and with

\[
\chi := \min_{t \in [a, b]} |F''(t)| > 0.
\]

Let \( \chi_0 := \min(\chi, 121) \). Then

\[
\left| \sum_{n \in (a, b) \cap \mathbb{N}} \psi(F(n)) \right| \leq \frac{11}{2} |F'(b) - F'(a)| \chi_0^{-2/3} + 11 \chi_0^{-1/2}.
\]

As Liu points out, choosing \( \chi_0 \) rather than \( \chi \) is needed for technical reasons when \( a \) and \( b \) are very close.

We now prove Proposition 4.5.

Pick any \( c > 1 \). Assume without loss of generality that \( |F''| \) is increasing on \( [a, b] \), and that the interval \( (a, b) \) contains at least one integer

\[
K := [a] + 1.
\]

We choose \( \tilde{b} \in (a, b) \) so that \( |F''(\tilde{b})| = 121 \) if such a point exists; if not, choose \( \tilde{b} = b \) if \( |F''| < 121 \) on \( [a, b] \), and choose \( \tilde{b} = a \) if \( |F''| > 121 \) on \( [a, b] \). Decompose

\[
\left| \sum_{n \in (a, b) \cap \mathbb{N}} \psi(F(n)) \right| \leq \left| \sum_{n \in (a, b) \cap \mathbb{N}} \psi(F(n)) \right| + \left| \sum_{n \in (b, b) \cap \mathbb{N}} \psi(F(n)) \right|.
\]

The second term is bounded, since \( |\psi(\cdot)| \leq \frac{1}{2} \), by \( \frac{1}{2} (b - \tilde{b} + 1) \). Since \( |F''| \geq 121 \) on \( [\tilde{b}, b] \), it is immediate that

\[
\frac{1}{2} (b - \tilde{b} + 1) \leq \frac{1}{2} \frac{1}{121^{1/3}} \int_{\tilde{b}}^{b} |F''(t)|^{1/3} \, dt + \frac{1}{2} \leq \frac{11}{2} c^{2/3} \int_{\tilde{b}}^{b} |F''(t)|^{1/3} \, dt + \frac{1}{2}.
\]

From this we see that

\[
\left| \sum_{n \in (a, b) \cap \mathbb{N}} \psi(F(n)) \right| \leq \left| \sum_{n \in (a, b) \cap \mathbb{N}} \psi(F(n)) \right| + \frac{11}{2} c^{2/3} \int_{b}^{\tilde{b}} |F''(t)|^{1/3} \, dt + \frac{1}{2}. \tag{A3.1}
\]

Following the strategy in Case I of Liu’s proof, we now decompose \((a, \tilde{b}]\) into \( N + 1 \) subintervals \((a_v, a_{v+1}], v = 0, \ldots, N\), with \( a_0 = a \) and \( a_{N+1} = \tilde{b} \), such that

\[
c^v \leq \frac{|F''(n)|}{|F''(K)|} \leq c^{v+1} \quad \text{for} \ n \in (a_v, a_{v+1}] \cap \mathbb{N}
\]

\footnote{at least, for aesthetic purposes}
(some of the intervals may be empty). Then
\[ \left| \sum_{n \in (a, b] \cap \mathbb{N}} \psi(F(n)) \right| \leq \left| \sum_{n \in (a, b] \cap \mathbb{N}} \psi(F(n)) \right| \leq \sum_{v=0}^{N} \sum_{n \in [a_v, a_{v+1}) \cap \mathbb{N}} \left| \psi(F(n)) \right|.
\]

Now we apply Lemma A3.1 for each \( v \), with \( \chi = \chi = c^v |F''(K)| \) (we recall that \( |F''(\cdot)| \leq 121 \) on \((a, b]\)). We get
\[ \left| \sum_{n \in (a, b] \cap \mathbb{N}} \psi(F(n)) \right| \leq \frac{11}{2} \sum_{v=0}^{N} |F'(a_{v+1}) - F'(a_v)| \left[ c^v |F''(K)| \right]^{-2/3} + 11 \sum_{v=0}^{N} \left( c^v |F''(K)| \right)^{-1/2}.
\]

By the fundamental theorem of calculus in the first term, and by summing up the geometric series in the second term we have
\[ \left| \sum_{n \in (a, b] \cap \mathbb{N}} \psi(F(n)) \right| \leq \frac{11}{2} \sum_{v=0}^{N} \left( c^v |F''(K)| \right)^{-2/3} \int_{a_v}^{a_{v+1}} |F''(t)| \, dt + 11 \sum_{v=0}^{N} \left( c^v |F''(K)| \right)^{-1/2}. \quad (A3.2)
\]

For each \( v \) we estimate
\[ |F''(t)| = \left| F''(t) \right|^{1/3} \left| F''(t) \right|^{1/3} \leq \left( c^{v+1} |F''(K)| \right)^{2/3} \left| F''(t) \right|^{1/3} \quad \text{for } t \in (a_v, a_{v+1}).
\]

Plugging this into the integral in (A3.2), while also bounding the last term there by the maximum over the interval gives
\[ \left| \sum_{n \in (a, b] \cap \mathbb{N}} \psi(F(n)) \right| \leq \frac{11}{2} c^{2/3} \sum_{v=0}^{N} \int_{a_v}^{a_{v+1}} \left| F''(t) \right|^{1/3} \, dt + 11 \sum_{v=0}^{N} \max_{t \in [a, b]} \frac{1}{\left| F''(t) \right|^{1/2}}.
\]

Substituting this back into (A3.1) and combining all the integrals concludes the proof of Proposition 4.5.
References


N. V. Kuznetsov, Asymptotic distribution of the eigenfrequencies of a plane membrane in the case when the variables can be separated, Differ. Uravn. 2 (1966), 385–1402. Full text (in Russian) on Mathnet.ru.


