Finite dimensional quantizations of the particle motion: toward new space and momentum inequalities?

Jean-Pierre Gazeau*, François-Xavier Josse-Michaux*, and Pascal Monceau**
* *Fédération de Recherches Astroparticules et Cosmologie and LPTMC*
** *Matière et Systèmes Complexes*
Boite 7020, Université Paris 7 Denis Diderot, F-75251 Paris Cedex 05, France
gazeau@ccr.jussieu.fr, pmo@ccr.jussieu.fr

October 12, 2004

Abstract

We present a N-dimensional quantization *à la* Berezin-Klauder or *frame* quantization of the complex plane based on overcomplete families of states (coherent states) generated by the N first harmonic oscillator eigenstates. The spectra of position and momentum operators are finite and eigenvalues are equal, up to a factor, to the zeros of Hermite polynomials. From numerical and theoretical studies of the large N behavior of the product $\lambda_M(N)\,\lambda_m(N)$ of largest and non null smallest positive eigenvalues, we infer the inequality $\Delta_N(Q)\,\delta_N(Q) = \sigma_N \underset{\mathbb{N}\to\infty}{\overset{<}{\to}} 2\pi$ (resp. $\Delta_N(P)\,\delta_N(P) = \sigma_N \underset{\mathbb{N}\to\infty}{\overset{<}{\to}} 2\pi$) involving, in suitable units, the maximal ($\Delta_N(Q)$) and minimal ($\delta_N(Q)$) sizes of regions of space (resp. momentum) which are accessible to exploration within this finite-dimensional quantum framework. Interesting issues on the measurement process are discussed.

Pacs03.65.Bz, 03.65.Ca

Keywords Quantization, coherent states, quantum localisation

1 General setting: quantum processing of a measure space

In this introductive section, we present the method of quantization we will apply in the sequel to a simple model, for instance the motion of a particle on the line, or more generally a system with one degree of freedom, like the vibration of a linear molecule. The method, which is based on coherent states [1, 2] or *frames* [3] in Hilbert spaces is inspired by previous approaches proposed by Klauder [4, 5] and Berezin [6]. More details and examples concerning the method can be found in the references [7, 8, 9, 10].

Let us start with an arbitrary measure space (X, μ). This set might be a classical phase space, but actually it can be any set of data accessible to observation. The existence of a measure provides us with a statistical reading of the set of measurable real- or complex-valued functions $f(x)$ on X: computing for instance average values on subsets with bounded measure. Actually, both approaches deal with quadratic mean values and correlation/convolution involving signal pairs, and the natural framework of studies is the complex (Hilbert) spaces, $L^2(X, \mu)$ of square integrable functions $f(x)$ on X: $\int_X |f(x)|^2 \, \mu(dx) < \infty$. One will speak of *finite-energy* signal in Signal Analysis and of (pure) quantum state in Quantum Mechanics. However, it is precisely at this stage that "quantum processing" of X differs from signal processing on at least three points:

1. not all square integrable functions are eligible as quantum states,
2. a quantum state is defined up to a nonzero factor,
3. those ones among functions $f(x)$ that are eligible as quantum states with unit norm, $\int_X |f(x)|^2 \, \mu(dx) = 1$, give rise to a probability interpretation : $X \supset \Delta \to \int_\Delta |f(x)|^2 \mu(dx)$ is a probability measure interpretable in terms of localisation in the measurable Δ. This is inherent to the computing of mean values of quantum observables, (essentially) self-adjoint operators with domain included in the set of quantum states.

The first point lies at the heart of the *quantization* problem: what is the more or less canonical procedure allowing to select quantum states among simple signals? In other words, how to select the right (projective) Hilbert space $\mathcal{H}$, a closed subspace of $L^2(X, \mu)$, (resp. some isomorphic copy of it) or equivalently the corresponding orthogonal projecteur $I_\mathcal{H}$ (resp. the identity operator)?

In various circumstances, this question is answered through the selection, among elements of $L^2(X, \mu)$, of an orthonormal set $\mathcal{S}_N = \{\phi_n(x)\}_{n=0}^{N-1}$, N being finite or infinite, which spans, by definition, the separable Hilbert subspace $\mathcal{H} \equiv \mathcal{H}_N$. The crucial point is that these elements have to fulfill the following condition :

$$\mathcal{N}(x) \equiv \sum_n |\phi_n(x)|^2 < \infty \text{ almost everywhere.} \tag{1}$$

Of course, if $N \geq 1$ is finite the above condition is trivially checked.

We now consider the family of states $\{|x\rangle\}_{x \in X}$ <u>in</u> $\mathcal{H}_N$ obtained through the following linear superpositions:

$$|x\rangle \equiv \frac{1}{\sqrt{\mathcal{N}(x)}} \sum_n \overline{\phi_n}(x) |\phi_n\rangle, \tag{2}$$

in which the ket $|\phi_n\rangle$ designates the element $\phi_n(x)$ in a "Fock" notation and $\overline{\phi_n}(x)$ is the complex conjugate of $\phi_n(x)$. This defines an injective map

$$X \ni x \to |x\rangle \in \mathcal{H}_N, \tag{3}$$

and the above Hilbertian superposition makes sense provided that set X is equipped of a mild topological structure for which this map is continuous. It is not difficult to check that states (2) are *coherent* in the sense that they obey the following two conditions:

- **Normalisation**

$$\langle\, x \,|x\rangle = 1, \tag{4}$$

- **Resolution of the unity in $\mathcal{H}_N$**

$$\int_X |x\rangle\langle x|\ \nu(dx) = I_{\mathcal{H}_N}, \tag{5}$$

where $\nu(dx) = \mathcal{N}(x)\,\mu(dx)$ is another measure on X, absolutely continuous with respect to $\mu(dx)$. The coherent states (2) form in general an overcomplete (continuous) basis of $\mathcal{H}_N$. Actually, the term of *frame* [3] is more appropriate for designating the total family $\{|x\rangle\}_{x\in X}$.

The resolution of the unity in $\mathcal{H}_N$ can alternatively be understood in terms of the scalar product $\langle\, x \,|x'\rangle$ of two states of the family. Indeed, (5) implies that, to any vector $|\phi\rangle$ in $\mathcal{H}_N$ one can isometrically associate the function

$$\phi(x) \equiv \sqrt{\mathcal{N}(x)}\langle x\,|\phi\rangle \tag{6}$$

in $L^2(X,\mu)$, and this function obeys

$$\phi(x) = \int_X \sqrt{\mathcal{N}(x)\mathcal{N}(x')}\langle x|x'\rangle\phi(x')\,\mu(dx'). \tag{7}$$

Hence, $\mathcal{H}_N$ is isometric to a reproducing Hilbert space with kernel

$$\mathcal{K}(x,x') = \sqrt{\mathcal{N}(x)\mathcal{N}(x')}\langle x\,|x'\rangle, \tag{8}$$

and the latter assumes finite diagonal values (*a.e.*), $\mathcal{K}(x,x) = \mathcal{N}(x)$, by construction.

A *classical* observable is a function $f(x)$ on X having specific properties in relationship with some supplementary structure allocated to X, topology, geometry or something else. Its quantization simply consists in associating to $f(x)$ the operator

$$A_f := \int_X f(x)|x\rangle\langle x|\,\nu(dx). \tag{9}$$

In this context, $f(x)$ is said upper (or contravariant) symbol of the operator A_f and denoted by $f = \hat{A}_f$, whereas the mean value $\langle x|A|x\rangle$ is said lower (or covariant) symbol of an operator A acting on $\mathcal{H}_N$ [6] and denoted by $\check{A}_f$. Through this approach, one can say that a quantization of the observation set is in one-to-one correspondence with the choice of a frame in the sense of (4) and (5). To a certain extent, a quantization scheme consists in adopting a certain point of view in dealing with X. This frame can be discrete, continuous, depending on the topology furthermore allocated to the set X, and it can be overcomplete, of course. The validity of a precise frame choice with regard to a certain physical context is asserted by comparing spectral characteristics of quantum observables A_f with experimental data.

In order to illustrate the process, we shall first recall the well-known Bargman-Berezin quantization of the complex plane, the latter being viewed as the phase space of a particle moving on the real line or more generally a system with one degree of freedom. The orthonormal set $\mathcal{S}_\infty = \{\phi_n(x)\}_{n=0}^\infty$ is chosen to be the set of eigenstates, in the so-called Fock-Bargmann representation, of the harmonic

oscillator with $N = \infty$. We next study quantizations resulting from the choice, within $\mathcal{S}_\infty$, of increasing subsets $\mathcal{S}_N = \{\phi_n(x)\}_{n=0}^{N-1}, N = 1, 2, 3, \ldots$. After working out the algebras of these finite-dimensional quantizations, we shall explore their respective physical meaning in terms of lower symbols, localisation and momentum range properties. From the existence of a finite spectrum of the position and momentum operators in finite-dimensional quantization, we find that there exists an interesting correlation between the size δ_N of the minimal "forbidden" cell and the width Δ_N of the spectrum ("size of the universe" accessible to measurements from the point of view of the specific system being quantized). This correlation reads in appropriate units $\delta_N \times \Delta_N = \sigma_N$, and numerical explorations, validated by theoretical arguments, indicate that the strictly increasing sequence $\sigma_N \xrightarrow[N\to\infty]{} \sigma \sim 2\pi$. A similar result holds for the spectra of the momentum operators.

2 The standard case

Let us illustrate the above construction with the well-known Klauder-Glauber-Sudarshan coherent states [1]. The observation set X is the classical phase space $\mathbb{R}^2 \simeq \mathbb{C} = \{x \equiv z = \frac{1}{\sqrt{2m\omega u_A}}(m\omega q + ip)\}$ (in complex notations) of a system with one degree of freedom and experiencing a motion with characteristic time ω^{-1} and action u_A. Note that the characteristic length and momentum of this system are $l_c = \sqrt{\frac{u_A}{m\omega}}$ and $p_c = \sqrt{m\omega u_A}$ respectively, whereas the phase-space variable z can be expressed in units of square root of action $\sqrt{u_A}$. Now, we could as well deal with an oscillating system like a biatomic molecule. Of course, in the domain of validity of quantum mechanics, it is natural to choose $u_A = \hbar$. The measure on X is gaussian, $\mu(dx) = \frac{1}{\pi} e^{-\frac{|z|^2}{u_A}} d^2z$ where d^2z is the Lebesgue measure of the plane. In the sequel, we shall work in suitable units, *i.e.* with $m = 1$, $\omega = 1$, and $u_A = 1$.

The functions $\phi_n(x)$ are the normalised powers of the complex variable z, $\phi_n(x) \equiv \frac{\bar{z}^n}{\sqrt{n!}}$, so that the Hilbert subspace $\mathcal{H}$ is the so-called Fock-Bargmann space of all anti-entire functions that are square integrable with respect to the gaussian measure. Those states are eigenvectors of the number operator $\mathfrak{N}$ which is identical to the dilation operator $\mathfrak{N} = z\frac{\partial}{\partial z}$. Since $\sum_n \frac{|z|^{2n}}{n!} = e^{|z|^2}$, the coherent states read

$$|z\rangle = e^{-\frac{|z|^2}{2}} \sum_n \frac{z^n}{\sqrt{n!}} |n\rangle, \tag{10}$$

where we have adopted the usual notation $|n\rangle = |\phi_n\rangle$.

One easily checks the normalisation and unity resolution:

$$\langle z\,|z\rangle = 1, \quad \frac{1}{\pi}\int_{\mathbb{C}} |z\rangle\langle z|\, d^2z = I_{\mathcal{H}}, \tag{11}$$

Note that the reproducing kernel is simply given by $e^{\bar{z}z'}$. The quantization of the observation set is hence achieved by selecting in the original Hilbert space $L^2(\mathbb{C}, \frac{1}{\pi}e^{-|z|^2}\, d^2z)$ all anti-holomorphic entire functions, which geometric quantization specialists would call a choice of polarization. Quantum operators acting on $\mathcal{H}$ are yielded by using (9). We thus have for the most basic one,

$$\frac{1}{\pi}\int_{\mathbb{C}} z\,|z\rangle\langle z|\, d^2z = \sum_n \sqrt{n+1}|n\rangle\langle n+1| \equiv a, \tag{12}$$

which is the lowering operator, $a|n\rangle = \sqrt{n}|n-1\rangle$. Its adjoint $a^\dagger$ is obtained by replacing z by $\bar{z}$ in (12), and we get the factorisation $\mathfrak{N} = a^\dagger a$ together with the commutation rule $[a, a^\dagger] = I_{\mathcal{H}}$. Also note that $a^\dagger$ and a realize on $\mathcal{H}$ as multiplication operator and derivation operator respectively, $a^\dagger f(z) = zf(z),\ af(z) = df(z)/dz$. From $q = \frac{1}{\sqrt{2}}(z + \bar{z})$ et $p = \frac{1}{i\sqrt{2}}(z - \bar{z})$, one easily infers by linearity that q and p are upper symbols for $\frac{1}{\sqrt{2}}(a + a^\dagger) \equiv Q$ and $\frac{1}{i\sqrt{2}}(a - a^\dagger) \equiv P$ respectively. In consequence, the self-adjoint operators Q and P obey the canonical commutation rule $[Q, P] = iI_{\mathcal{H}}$, and for this reason fully deserve the name of position and momentum operators of the usual (galilean) quantum mechanics, together with all localisation properties specific to the latter.

These standard states have many interesting properties. Let us recall two of them: they are eigenvectors of the lowering operator, $a|z\rangle = z|z\rangle$, and they saturate the Heisenberg inequalities : $\Delta Q\, \Delta P = \frac{1}{2}$. It should be noticed that they also pertain to the group theoretical construction since they are obtained from unitary Weyl-Heisenberg transport of the ground state: $|z\rangle = \exp(za^\dagger - \bar{z}a)|0\rangle$.

3 Lowest-dimensional cases

In order to better understand the quantization scheme in the finite-dimensional case, let us examine the simplest situations in which $N = 2$ and $N = 3$ (the case $N = 1$ yields zero values for everything).

3.1 Two-dimensional case

The measure space X is the same as in the previous section : $(X, \mu) = (\mathbb{C}, d\,\mu(z, \bar{z}) = \frac{1}{\pi}\, e^{-|z|^2}\, d^2z)$. Let us now start out by selecting the two first elements of the orthonormal Fock-Bargmann basis , namely

$$\phi_0(x) = 1,\ \phi_1(x) = \bar{z}. \tag{13}$$

Then we have,

$$\mathcal{N}(x) = 1 + |z|^2 \quad a.e.. \tag{14}$$

The corresponding coherent states read as

$$|z\rangle = \frac{1}{\sqrt{1 + |z|^2}}\, [|0\rangle + z\ |1\rangle]\,. \tag{15}$$

To any integrable function $F(z, \bar{z})$ on $\mathbb{C}$ there corresponds the linear operator A_F on $\mathbb{C}^2$ in its matrix form :

$$\begin{aligned} A_F &= \int_{\mathbb{C}} d\,\mu(z, \bar{z})\, (1 + |z|^2)\, F(z, \bar{z})\, |z\rangle\langle z| \\ &= \begin{pmatrix} \frac{1}{\pi}\int_{\mathbb{C}} d^2z\, e^{-|z|^2}\, F(z, \bar{z}) & \frac{1}{\pi}\int_{\mathbb{C}} d^2z\, e^{-|z|^2}\, \bar{z}F(z, \bar{z}) \\ \frac{1}{\pi}\int_{\mathbb{C}} d^2z\, e^{-|z|^2}\, zF(z, \bar{z}) & \frac{1}{\pi}\int_{\mathbb{C}} d^2z\, e^{-|z|^2}\, |z|^2 F(z, \bar{z}) \end{pmatrix} \end{aligned} \tag{16}$$

In particular, with the choice $F(z,\bar{z}) = 1$ we recover the identity, as expected, whereas for $F(z,\bar{z}) = z$ and for $F(z,\bar{z}) = \bar{z}$, we get the projectors

$$A_z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \equiv a, \; A_{\bar{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \equiv a^\dagger. \tag{17}$$

They obey the commutation rule

$$[a, a^\dagger] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3, \tag{18}$$

the third Pauli matrix. For the most general monomial choice $F(z,\bar{z}) = z^u \bar{z}^v$, $u, v \in \mathbb{N}$, one gets

$$A_{z^u \bar{z}^v} = u! \begin{pmatrix} \delta_{u,v} & \delta_{u,v+1} \\ (u+1)\delta_{u,v-1} & (u+1)\delta_{u,v} \end{pmatrix}. \tag{19}$$

Hence, we easily check to what extent this two-dimensional quantization of the complex plane is degenerate since most of the classical observables go to the null operator. We should also compare upper symbol $z^u \bar{z}^v$ of (19) with the lower symbol of the latter :

$$\langle z | A_{z^u \bar{z}^v} | z \rangle = \frac{u!}{1+|z|^2} \left(\left(1 + (u+1)|z|^2\right) \delta_{u,v} + z\delta_{u,v+1} + \bar{z}\delta_{u,v-1} \right). \tag{20}$$

Like in the standard case, from $q = \frac{1}{\sqrt{2}}(z + \bar{z})$ et $p = \frac{1}{i\sqrt{2}}(z - \bar{z})$, we find that q and p are upper symbols for the position and momentum operators:

$$Q \equiv Q_2 \equiv \frac{1}{\sqrt{2}}(a + a^\dagger) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\sigma_1, \tag{21}$$

$$P \equiv P_2 \equiv \frac{1}{i\sqrt{2}}(a - a^\dagger) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\sigma_2, \tag{22}$$

where we note the appearance of the two Pauli matrices σ_1 and σ_2 respectively. In consequence, the self-adjoint operators Q and P obey the spin commutation rule $[Q, P] = i\sigma_3$ and realize a two-dimensional representation of the spin algebra. Their respective lower symbols read

$$\langle z | Q | z \rangle = \frac{q}{1 + \frac{1}{2}(p^2 + q^2)}, \; \langle z | P | z \rangle = \frac{p}{1 + \frac{1}{2}(p^2 + q^2)}. \tag{23}$$

As functions of phase-space variables they give rise to surfaces which are displayed in Fig. 1. In this two-dimensional representation the harmonic oscillator Hamiltonian reduces to a multiple of the identity : $H_2 = \frac{1}{2}\left(P^2 + Q^2\right) = \frac{1}{2}I_2$: all quantum states are stationary. For a general Hamiltonian of the type $H = \frac{1}{2}P^2 + V(Q)$ in which the potential is some (Laurent) series $V(Q) = \sum_i c_i Q^i$ we get the two-level spectrum $\left\{ \frac{1}{4} + \frac{1}{2^n} \sum_n c_{2n} \pm \frac{1}{2^{n+1/2}} \sum_n c_{2n+1} \right\}$. Moreover, the spectra of the position and momentum operators are both equal to $\left\{ \pm\frac{1}{2} \right\}$, which means for instance that two position values only, $\pm\frac{1}{2}$, are accessible to observation within the context of this particular choice of frame, of course.

3.2 Three-dimensional case

We now deal with the orthonormal set :

$$\phi_0(x) = 1, \phi_1(x) = \bar{z}, \phi_2(x) = \frac{\bar{z}^2}{\sqrt{2}}. \tag{24}$$

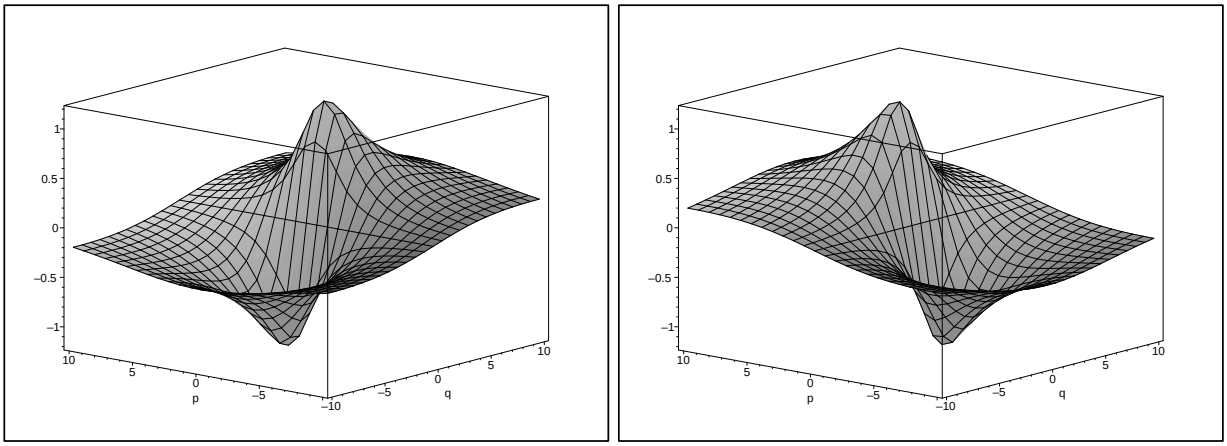

Figure 1: Behavior in function of $z = \frac{1}{\sqrt{2}}(q + ip)$ of the meanvalues $\langle z|Q|z\rangle$ and $\langle z|P|z\rangle$ of position and momentum operators in the two-dimensional quantization case. One should notice the existence of two peaks illustrating two-level system concerning position and momentum.

Accordingly we find

$$\mathcal{N}(x) = 1 + |z|^2 + \frac{|z|^4}{2}. \tag{25}$$

Coherent states read :

$$|z\rangle = \frac{1}{\sqrt{1 + |z|^2 + \frac{|z|^4}{2}}} \left[|0\rangle + z\ |1\rangle + \frac{z^2}{\sqrt{2}}\ |2\rangle \right]. \tag{26}$$

Position and momentum matrix operators read

$$Q \equiv Q_3 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ P \equiv P_3 = -i \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{27}$$

Their commutator is given by:

$$[Q, P] = i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The spectra of Q and P are the same : $\left\{0, \pm\sqrt{\frac{3}{2}}\right\}$.

Their respective lower symbols are given by

$$\langle z|Q|z\rangle = \frac{q + \frac{(q^2+p^2)}{2}\, q}{1 + \frac{(q^2+p^2)}{2} + \frac{(q^2+p^2)^2}{8}},\ \langle z|P|z\rangle = \frac{p + \frac{(q^2+p^2)}{2}\, p}{1 + \frac{(q^2+p^2)}{2} + \frac{(q^2+p^2)^2}{8}}. \tag{28}$$

The Hamiltonian $H_3 = \frac{1}{2}\left(P^2 + Q^2\right)$ is not any more multiple of the identity :

$$H = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{29}$$

and its lower symbol is given by

$$\langle z|H|z\rangle = \frac{\frac{1}{2}+\frac{3}{2}|z|^2+\frac{|z|^4}{2}}{1+|z|^2+\frac{1}{2}|z|^4}. \tag{30}$$

4 The N-dimensional case

Let us now consider the generic orthonormal set with N elements:

$$\phi_0(x) = 1, \phi_1(x) = \bar{z}, \dots \phi_{N-1}(x) = \frac{\bar{z}^{(N-1)}}{\sqrt{(N-1)!}}. \tag{31}$$

The coherent states read :

$$|z\rangle = \frac{1}{\sqrt{\mathcal{N}(x)}} \sum_{n=0}^{N-1} \frac{z^n}{\sqrt{n!}} |n\rangle, \tag{32}$$

with

$$\mathcal{N}(x) = \sum_{n=0}^{N-1} \frac{|z|^{2n}}{n!}. \tag{33}$$

Matrix elements of the position operator $Q \equiv Q_N$ and momentum operator $P \equiv P_N$ are given by

$$Q_N(k,l) = \frac{1}{\sqrt{2}} \left(\sqrt{k}\,\delta_{k,l-1} + \sqrt{k-1}\,\delta_{k,l+1}\right),\ 1 \le k,l \le N. \tag{34}$$

$$P_N(k,l) = -i\,\frac{1}{\sqrt{2}} \left(\sqrt{k}\,\delta_{k,l-1} - \sqrt{k-1}\,\delta_{k,l+1}\right)\ 1 \le k,l \le N. \tag{35}$$

Their commutator is "almost" canonical:

$$[Q_N, P_N] = iI_N - iNE_N, \tag{36}$$

where E_N is the orthogonal projector on the last basis element,

$$E_N = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix}.$$

The appearing of such a projector in (36) is clearly a consequence of the truncation at the Nth level. We shall study the spectra of these operators in the next section.

The corresponding truncated harmonic oscillator hamiltonian is diagonal with matrix elements :

$$H_N(k,l) = \frac{1}{2}\left(2k-1-N\delta_{k,N}\right)\delta_{k,l}. \tag{37}$$

Since H is diagonal, its eigenvalues are trivially $\frac{1}{2}\left(2k-1-N\delta_{N,k}\right)$ and are identical to the lowest eigenenergies of the harmonic oscillator, except for the Nth one which is equal to $\dfrac{N-1}{2}$ instead of

$N - \frac{1}{2}$. One should notice that its nature differs according to the parity of N: it is degenerate if N is even since then $\frac{N}{2} - \frac{1}{2}$ is already present in the spectrum whereas it assumes the intermediate value $\left\lfloor \frac{N}{2} \right\rfloor$ between two expected values if N is odd.

Let us now consider the mean values or lower symbols of the position and momentum operators. We find:

$$\langle z|Q|z\rangle = C(|z|)q, \ \langle z|P|z\rangle = C(|z|)p, \tag{38}$$

where the corrective factor

$$C(|z|) = \frac{1}{\mathcal{N}(z)} \sum_{j=1}^{N-1} \frac{(|z|)^{2(j-1)}}{(j-1)!} \tag{39}$$

goes to 1 at $N \to \infty$.

Lower symbols of the operators Q^2, P^2 and H are given by:

$$\langle z| \left\{ \begin{array}{c} Q^2 \\ P^2 \end{array} \right\} |z\rangle = A(|z|) \pm B(|z|), \ \langle z|H|z\rangle = A(|z|), \tag{40}$$

where

$$A(|z|) = \frac{1}{\mathcal{N}(z)} \sum_{k=1}^{N} \frac{|z|^{2(k-1)}}{(k-1)!} \left(\frac{2k-1-N\delta_{N,k}}{2} \right),$$
$$B(|z|) = \frac{1}{\mathcal{N}(z)} \sum_{k=1}^{N-2} \frac{|z|^{2(k-1)}}{(k-1)!} \frac{z^2+\bar{z}^2}{2}.$$

The behavior of these lower symbols in (40) in function of (q,p), with the particular value $N = 12$, is shown in Fig. 2. One can see that these mean values are identical, albeit the lower symbol of P^2 is obtained from that of Q^2 through a rotation by $\frac{\pi}{2}$ in the complex plane.

From all these meanvalues we can deduce the product $\Delta Q\, \Delta P$, where $\Delta Q = \sqrt{\langle z|Q^2|z\rangle - (\langle z|Q|z\rangle)^2}$. Due to rotational invariance, it is enough to consider its behavior in function of q, at $p = 0$, as is shown in Fig. 3 for different values of N, $N = 2, 5, 10, 15$. One can observe that $\Delta Q\, \Delta P = 1/2$, *i.e.* the product assumes, at the origin of the phase space the minimal value it would have in the infinite-dimensional case (with $\hbar = 1$). Note that, for $N = 2$, the value $1/2$ is a supremum (!), and the latter is reached for almost all values of z except in the range $|z| \lesssim 10$. For higher values of N, there exists around the origin a range of values of $|z|$, where the product is equal to $\frac{1}{2}$. This range increases with N as expected since the Heisenberg inequalities are saturated with standard coherent states ($N = \infty$).

Let us finally consider the behavior in function of $|z|$ of the lower symbol of the harmonic oscillator Hamiltonian given in Eq. (40). From Figs. 4 and 5 in which are shown respectively the meanvalue of H at $N = 5$, and the energy spectrum for different values of N, one can see the influence of truncating the dimension of the space of states.

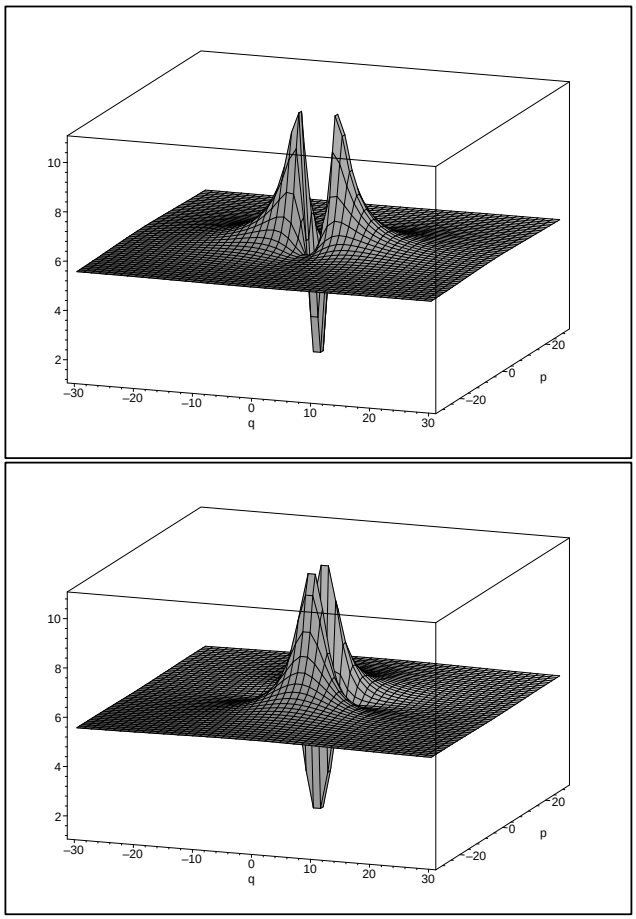

Figure 2: (q, p) behavior of the meanvalues (lower symbols) of the operators Q^2 and P^2 in the coherent state $|z\rangle$ for $N = 12$.

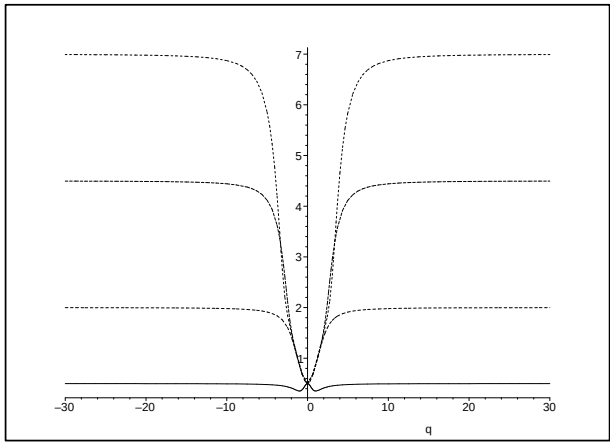

Figure 3: Behavior of $\Delta Q\, \Delta P$ in function of q, at $p = 0$, for different values of N, $N = 2$ (lowest curve), $5, 10, 15$ (upper curve).

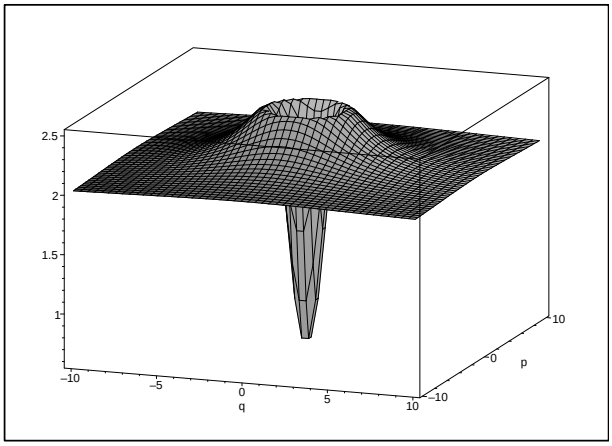

Figure 4: Meanvalue $\langle z|H|z\rangle$ of the harmonic oscillator hamiltonian as a function of $z = \frac{1}{\sqrt{2}}(q+ip)$ for $N = 5$.

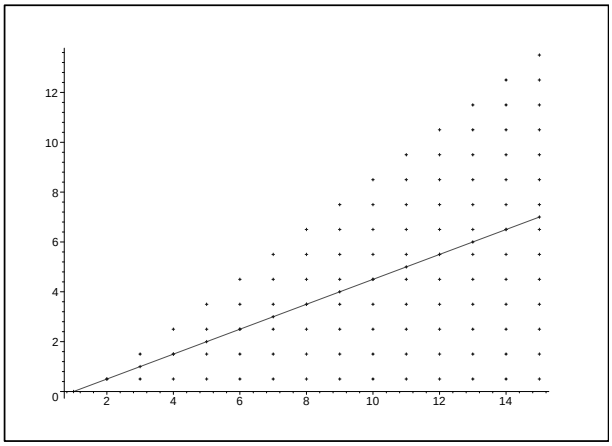

Figure 5: Spectrum of the harmonic oscillator Hamiltonian H in function of N. One clearly sees the appearance of a degeneracy or an intermediate value instead, according to the parity of N.

5 Localization and momentum of the finite-dimensional quantum system

We now examine the spectral features of the position and momentum operators $Q \equiv Q_N$, $P \equiv P_N$ given in the N-dimensional case by Eqs.(34) and (35), *i.e.* in explicit matrix form by:

$$Q_N = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & \dots & 0 \\ \frac{1}{\sqrt{2}} & 0 & 1 & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \dots & \ddots & 0 & \sqrt{\frac{N-1}{2}} \\ 0 & 0 & \dots & \sqrt{\frac{N-1}{2}} & 0 \end{pmatrix}, \; P_N = -i \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & \dots & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 1 & \dots & 0 \\ 0 & -1 & \ddots & \ddots & \vdots \\ \vdots & \dots & \ddots & 0 & \sqrt{\frac{N-1}{2}} \\ 0 & 0 & \dots & -\sqrt{\frac{N-1}{2}} & 0 \end{pmatrix}.$$

Their characteristic equations are the same. Indeed, $p_N(\lambda) = \det(Q_N - \lambda I_N)$ and $\det(P_N - \lambda I_N)$ both obey the same recurrence equation:

$$p_{N+1}(\lambda) = -\lambda p_N(\lambda) - \frac{N}{2} p_{N-1}(\lambda), \tag{41}$$

with $p_0(\lambda) = 1$ and $p_1(\lambda) = -\lambda$. We have just to put $\mathrm{H}_N(\lambda) = (-2)^N p_N(\lambda)$ to ascertain that the H_N's are the Hermite polynomials for obeying the recurrence relation [11]:

$$\mathrm{H}_{N+1}(\lambda) = 2\lambda \mathrm{H}_N(\lambda) - 2N\mathrm{H}_{N-1}(\lambda), \; \mathrm{H}_0(\lambda) = 1, \; \mathrm{H}_1(\lambda) = 2\lambda. \tag{42}$$

Hence the spectral values of the position operator, *i.e.* the allowed or experimentally measurable quantum positions, <u>are</u> just the zeros of the Hermite polynomials! The same result holds for the spectral values of the momentum operator.

The non-null roots of the Hermite polynomial $\mathrm{H}_N(\lambda)$ form the set

$$Z_{\mathrm{H}}(N) = \left\{ -\lambda_{\lfloor \frac{N}{2} \rfloor}(N), -\lambda_{\lfloor \frac{N}{2} \rfloor - 1}(N), \dots, -\lambda_1(N), \lambda_1(N), \dots, \lambda_{\lfloor \frac{N}{2} \rfloor - 1}(N), \lambda_{\lfloor \frac{N}{2} \rfloor}(N) \right\}, \tag{43}$$

symmetrical with respect the origin, where $\lfloor \frac{N}{2} \rfloor = \frac{N}{2}$ if N is even and $\lfloor \frac{N}{2} \rfloor = \frac{N-1}{2}$ if N is odd; moreover $\mathrm{H}_N(0) = 0$ if and only if N is odd. A vast literature exists on the characterization and properties of the zeros of the Hermite polynomials, and many problems concerning their asymptotic behavior at large N are still open. Recent results can be found in [12] with previous references therein. Upper bounds [13] have been provided for $\lambda_m(N)$ and $\lambda_M(N)$ where $\lambda_m(N) = \lambda_1(N)$ and $\lambda_M(N) = \lambda_{\lfloor \frac{N}{2} \rfloor}(N)$ are respectively the smallest and largest positive zeros of H_N.

$$\lambda_M(N) \leq \Lambda_M(N) = \sqrt{2N-2}, \; \forall N > 1 \tag{44}$$

$$\lambda_m(N) \leq \Lambda_m^E(N) = \sqrt{2N-2} \cos\left(\frac{N-2}{2N-2}\pi\right), \; \forall N \text{ even} \tag{45}$$

$$\lambda_m(N) \leq \Lambda_m^O(N) = \sqrt{2N-2} \cos\left(\frac{N-3}{2N-2}\pi\right), \; \forall N \text{ odd} \tag{46}$$

However, it seems that the following observation is not known. We have studied numerically the behavior of the product

$$\varpi_N = \lambda_m(N)\lambda_M(N) \tag{47}$$

The zeros of the Hermite polynomials have been computed by diagonalizing the matrix of the position operator Q_N; since Q_N is tridiagonal symmetric with positive real coefficients, we implemented its diagonalization by using the QR algorithm [14]; such a method enabled us to compute the spectrum of the position operator up to the dimension $N = 10^6$. The respective behaviors of $\lambda_m(N)$ and $\lambda_M(N)$ are shown in Fig. 6 for N even and odd separately.

Now, it is a well-known property of the classical orthogonal polynomials that $\lambda_{i+1}(N) - \lambda_i(N) > \lambda_1(N)$ for all $i \geq 1$ if N is odd, whereas $\lambda_{i+1}(N) - \lambda_i(N) > 2\lambda_1(N)$ for all $i \geq 1$ if N is even, and that the zeros of the Hermite polynomials H_N and H_{N+1} intertwine, as is shown in Fig. 7 in the case of $\lambda_m(N)$ for small values of N.

Furthermore, the asymptotical behaviors of the products of the upper bounds provided by the equations (44, 45, and 46) read respectively:

$$\Lambda_M(N)\,\Lambda_m^E(N) \underset{N\to\infty}{\sim} \pi - \frac{1}{24}\frac{\pi^3}{N^2} + O\left(\frac{1}{N^3}\right),\ N \text{ even} \tag{48}$$

$$\Lambda_M(N)\,\Lambda_m^O(N) \underset{N\to\infty}{\sim} 2\pi - \frac{1}{3}\frac{\pi^3}{N^2} + O\left(\frac{1}{N^3}\right),\ N \text{ odd} \tag{49}$$

Hence, we see that ϖ_N goes asymptotically to $\varpi_e \leq \pi$ for large even N and to $\varpi_o \leq 2\varpi_e$ for large odd N. Therefore, if, at a given N, we define by $\Delta_N(Q) = 2\lambda_M(N)$ the "size" of the "universe" accessible to exploration by the quantum system, and by $\delta_N(Q) = \lambda_m(N)$ (resp. $\delta_N(Q) = 2\lambda_m(N)$) for odd (resp. even) N, the "size" of the smallest "cell" forbidden to exploration by the same system, we find the following upper bound for the product of these two quantities:

$$\delta_N(Q)\Delta_N(Q) \equiv \sigma_N = \begin{cases} 4\lambda_m(N)\lambda_M(N) & \text{for } N \text{ even,} \\ 2\lambda_m(N)\lambda_M(N) & \text{for } N \text{ odd,} \end{cases} \leq 4\pi. \tag{50}$$

However numerical explorations show that this upper bound is not the lowest one. Indeed, consider the behavior of the product σ_N, as a function of N, as is given in Fig. 8.

This strictly increasing function clearly goes asymptotically to a limit which we shall name σ and this limit has a value very close to 2π, as is shown in Table 1 where some values of σ_N up to $N = 10^6$ are given.

Hence, we can assert the interesting inequality for the product (50):

$$\delta_N(Q)\Delta_N(Q) \leq \sigma \approx 2\pi\ \forall N. \tag{51}$$

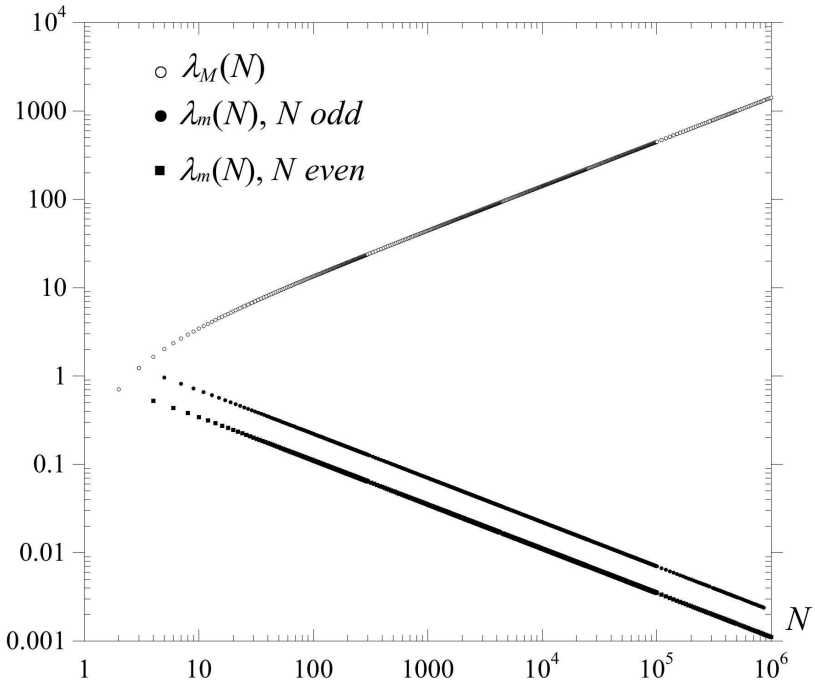

Figure 6: Bottom : behaviors of the lowest positive zero $\lambda_m(N)$ of the Hermite polynomial of degree N for N even and odd separately. Top : behavior of the largest positive zero $\lambda_M(N)$.

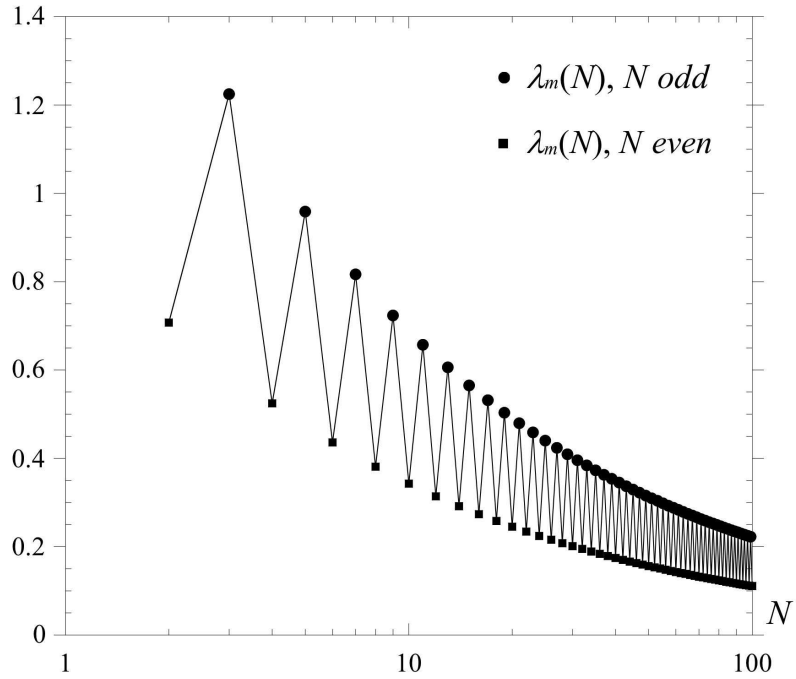

Figure 7: Intertwining of the lowest positive zeros $\lambda_m(N)$ of the Hermite polynomials H_N and H_{N+1} for small values of N.

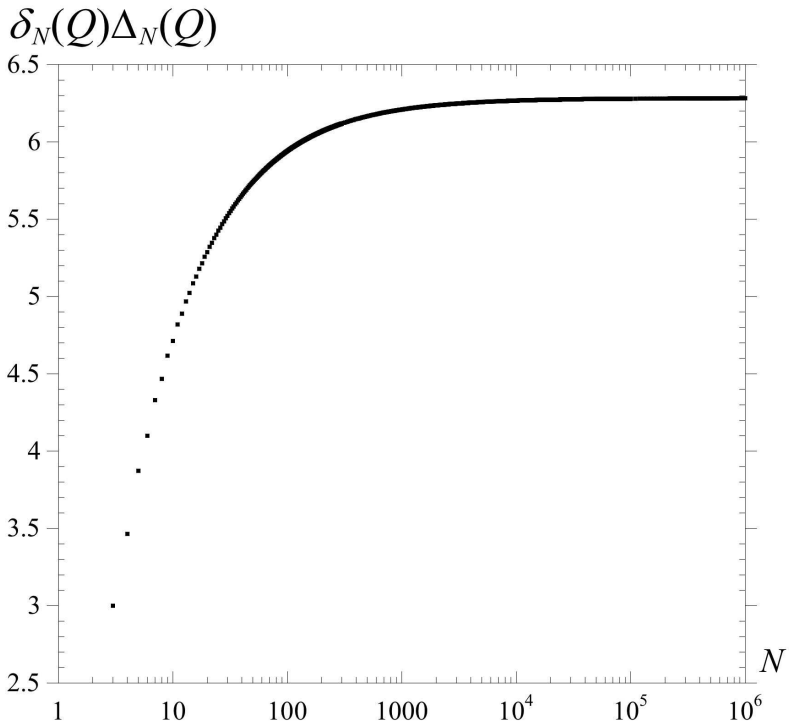

Figure 8: Behavior of the product $\sigma_N = \delta_N(Q)\Delta_N(Q)$, as a function of N.

N	$\delta_N(Q)\Delta_N(Q)$	2π
10	4.713054	
55	5.774856	
100	5.941534	
551	6.173778	
1 000	6.209670	
5 555	6.259760	
10 000	6.267356	
55 255	6.278122	
100 000	6.279776	
500 555	6.282020	
1 000 000	6.282450	6.2831853

Table 1: Values of $\sigma_N = \delta_N(Q)\Delta_N(Q)$ up to $N = 10^6$. Compare with the value of 2π.

Identical result holds for the momentum, of course :

$$\delta_N(P)\Delta_N(P) \leq \sigma \approx 2\pi \ \forall N. \tag{52}$$

In order to fully perceive the physical meaning of such inequalities, it is necessary to reintegrate into them physical constants or scales proper to the considered physical system, *i.e.* characteristic length l_c and momentum p_c as was done at the beginning of Section 2:

$$\delta_N(Q)\Delta_N(Q) \leq \sigma l_c^2, \ \delta_N(P)\Delta_N(P) \leq \sigma p_c^2 \ \forall N, \tag{53}$$

where $\delta_N(Q)$ and $\Delta_N(Q)$ are now expressed in unit l_c. Realistically, in any physical situation, N cannot be infinite: there is an obvious limitation on frequencies or energies accessible to observation/experimentation. So it is natural to work with a finite although large value of N, which need not be determinate. In consequence, there exists irreducible limitations, namely $\delta_N(Q)$ and $\Delta_N(Q)$ in the exploration of small and large distances, and both limitations have the correlation (53).

Let us now suppose there exists, for theoretical reasons, a fundamental or "universal" minimal length, say l_m, something like the Planck length, or equivalently an universal ratio $\rho_u = l_c/l_m \geq 1$. Then, from $\delta_N(Q) \geq l_m$ and (53) we infer that there exists a universal maximal length l_M given by

$$l_M \approx \sigma \rho_u l_c. \tag{54}$$

Of course, if we choose $l_m = l_c$, then the size of the "universe" is $l_M \approx 2\pi l_m$. Now, if we choose a characteristic length proper to Atomic Physics, like the Bohr radius, $l_c \approx 10^{-10}$m, and for the minimal length the Planck length, $l_m \approx 10^{-35}$m, we find for the maximal size the astronomical quantity $l_M \approx 10^{16}$m. On the other hand, if we consider the (controversial) estimate size of our present universe $L_u = cT_u$, with $T_u \approx 13\,10^9$ years [15], we get from $l_p\, L_u \approx 2\pi l_c^2$ a characteristic length $l_c \approx 10^{-5}$m, *i.e.* a wavelength in the infrared electromagnetic spectrum...

6 Conclusion

Of course, we should be very cautious about drawing sound physical consequences from the existence of the inequalities (53). Indeed, one can argue that our scheme of quantization leading to such inequalities is strongly dependent on the choice of orthonormal states used in constructing the "quantizer" frame. For that matter, it would be interesting to consider other simple systems which can be quantized through the same procedure, for instance the motion on a circle like in Ref.[10]. Moreover, the physical interpretation of the inequalities (53) appears to be rather enigmatic : is it a matter of length standard ? Is it instead related to some universal constraint in dealing with spatial degrees of freedom ? At the moment, we cannot provide any reasonable answer to these open questions which certainly deserve a deeper investigation.

References

[1] Feng, D. H., Klauder, J. R., and Strayer, M. (eds.) : *Coherent States: Past, Present and Future (Proc. Oak Ridge 1993)*, World Scientific, Singapore, 1994.

[2] Ali, S. T., Antoine, J.-P., and Gazeau, J.-P. : *Coherent states, wavelets and their generalizations*, Graduate Texts in Contemporary Physics, Springer-Verlag, New York, 2000.

[3] Ali, S. T., Antoine, J.-P., and Gazeau, J.-P. : Continuous Frames in Hilbert Spaces *Ann. of Phys.* **222** (1993), 1–37.

[4] Klauder, J. R. : Continuous-Representation Theory I. Postulates of continuous-representation theory, *J. Math. Phys.* **4** (1963), 1055–1058; Continuous-Representation Theory II. Generalized relation between quantum and classical dynamics, *J. Math. Phys.* **4** (1963), 1058–1073.

[5] Klauder, J. R. : Quantization without Quantization, *Ann. of Phys.* **237** (1995), 147–160.

[6] Berezin, F. A. : General concept of quantization, *Comm. Math. Phys.* **40** (1975), 153–174.

[7] Garidi, T., Gazeau, J-P., Huguet, E., Lachièze Rey, M., and Renaud, J. : Examples of Berezin-Toeplitz Quantization: Finite sets and Unit Interval *Symmetry in Physics. In memory of Robert T. Sharp 2002* eds. P. Winternitz et al (Montréal: CRM Proceedings and Lecture Notes) 2004; Quantization of the sphere with coherent states in *Classical, Stochastic and Quantum gravity, String and Brane Cosmology, Peyresq 2002*, *Int. J. Theor. Phys.* **42** (2003), 1301–1310.(http://arXiv.org/abs/math-ph/0302056).

[8] Gazeau, J-P., Coherent states and quantization of the particle motion on the line, on the circle, on $1+1$-de Sitter space-time and of more general systems, Proceedings of the 2003 Symposium on Contemporary Problems in Mathematical physics (COPROMAPH3), Porto-Novo (Bénin), Eds. J. Govaerts *et al*, World Scientific, Singapore, 2004, to appear.

[9] Ali, S. T., Englis, M., and Gazeau, J-P. : Vector Coherent States from Plancherel's Theorem, Clifford Algebras and Matrix Domains, *J. Phys. A : Math. Gen.* **37** (2004), 6067–6089.

[10] Gazeau, J-P., and Piechocki, W. : Coherent states quantization of a particle in de Sitter space, *J. Phys. A : Math. Gen.* **3**7,(2004) 6977–6986.

[11] Magnus, W., Oberhettinger, F., and Soni, R. P. : *Formulas and Theorems for the Special Functions of Mathematical Physics*, 3rd ed., Springer-Verlag, Berlin, Heidelberg and New York, 1966.

[12] Elbert, A. : Some recent results on the zeros of Bessel functions and orthogonal polynomials, *J. Comput. Appl. Math.* **133** (2001), 65–83.

[13] Area, I., Dimitrov, D. K., Godoy, E., and Ronveaux, A. : Zeros of Gegenbauer and Hermite polynomials and connection coefficients, *Math. Comp.* **73**(2004), 1937–1951.

[14] Press, W.H., Flannery, B.P., Teukolsky, S.A., and Vetterling, W.T. : *Numerical Recipes, the Art of Scientific Computing*, Cambridge University Press, 1986.

[15] Pasquini, L., Bonifacio, P., Randich, S., Galli, D., and Gratton, R.G. : Beryllium in turnoff stars of NGC6397: early Galaxy spallation, cosmochronology and cluster formation, comments, to appear in A& A (2004).