COHERENT STATES AND QUANTIZATION OF THE PARTICLE MOTION ON THE LINE, ON THE CIRCLE, ON $1+1$-DE SITTER SPACE-TIME, AND OF MORE GENERAL SYSTEMS

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Renc. Physique Théorique, Tanger, 2004

- General framework :
  - measure space and coherent states,
  - coherent states quantization.
- Motion of a particle on the line.
- Motion of a particle on the circle.
- Motion of a particle on 1+1 de Sitter space.
- Other less standard systems :
  - quantizations of finite sets,
  - quantizations of the unit interval,
  - quantization of Grassmann algebras.
- Polymer quantization of the motion on the line as a toy model for quantum gravity.
- The fuzzy sphere as a CS quantization of the sphere.

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General setting : "Quantum" processing of a measure space

Quantum Physics and Signal Analysis have similar formalism :

- *raw* set $X = \{x\}$ of basic parameters or data (*observation* set).

- X is equipped with a measure $\mu(dx)$

- statistical reading of the set of measurable real or complex valued functions $f(x)$ on X

- usual framework of studies : real (Euclidean) or complex (Hilbert) spaces, $L^2(X, \mu)$ of square integrable functions $f(x)$ on X: *finite-energy signal*

"Quantum processing" of X differs from signal processing on at least three points:

1. not all square integrable functions are eligible as quantum states,

2. a quantum state is defined up to a nonzero factor,

3. $f(x)$ eligible as quantum states with unit norm, $\int_X |f(x)|^2 \, \mu(dx) = 1$, give rise to a probability interpretation:

$$X \supset \Delta \to \int_\Delta |f(x)|^2 \mu(dx)$$

   is a probability measure interpretable in terms of localisation in the measurable Δ.

Quantization problem

How to select quantum states among simple signals? In other words, how to select the right (projective) Hilbert space $\mathcal{H}$, a closed subspace of $L^2(X, \mu)$, or equivalently the corresponding orthogonal projecteur $\mathbb{I}_{\mathcal{H}}$?

Coherent state approach

Select, among elements of $L^2(X, \mu)$, an orthonormal set $\mathcal{S}_N = \{\phi_n(x)\}_{n=1}^{N}$, N being finite or infinite, which spans, by definition, the separable Hilbert subspace $\mathcal{H} \equiv \mathcal{H}_N$.

Crucial assumption :

$$\mathcal{N}(x) \equiv \sum_n |\phi_n(x)|^2 < \infty \text{ almost everywhere.}$$

Who selects?

Suppose that some experimental device allows one to measure all possible and exclusive issues

$$\{a_n \in \mathbb{R}\}$$

of a physical quantity. The latter is encoded under the form of an operator in $L^2(X, \mu)$, the spectral resolution of which reads

$$A = \sum_n a_n |\phi_n\rangle\langle\phi_n|,$$

in which the ket $|\phi_n\rangle$ designates (or through some Hilbert isomorphism corresponds to) the element $\phi_n(x)$ in a "Dirac-Fock" notation.

Then consider the family of states $\{|x\rangle\}_{x\in X}$ through the following linear superpositions:

$$|x\rangle \equiv \frac{1}{\sqrt{\mathcal{N}(x)}} \sum_n \phi_n^\star(x)|\phi_n\rangle.$$

This defines an injective map

$$X \ni x \to |x\rangle \in \mathcal{H}_N,$$

These *coherent* states obey

- **Normalisation**

$$\langle\, x\,|x\rangle = 1,$$

- **Resolution of the unity in $\mathcal{H}_N$**

$$\int_X |x\rangle\langle x|\ \nu(dx) = \mathbb{I}_{\mathcal{H}_N}, \quad \nu(dx) = \mathcal{N}(x)\,\mu(dx)$$

Comments

- States $|x\rangle$ form in general an overcomplete basis of $\mathcal{H}$.

- To any vector $|\phi\rangle$ in $\mathcal{H}_N$ is isometrically associated the function in $L^2(X, \mu)$:

$$\phi(x) \equiv \sqrt{\mathcal{N}(x)} \langle x \,|\phi\rangle$$

- This function obeys

$$\phi(x) = \int_X \sqrt{\mathcal{N}(x)\mathcal{N}(x')} \langle x|x'\rangle \phi(x')\, \mu(dx').$$

- So $\mathcal{H}_N$ is isometric to a reproducing Hilbert space with kernel

$$\mathcal{K}(x, x') = \sqrt{\mathcal{N}(x)\mathcal{N}(x')} \langle x \,|x'\rangle.$$

Berezin-Toeplitz quantization

A *classical* observable is a function $f(x)$ on X having specific properties in relationship with some supplementary structure allocated to X, topology, geometry or something else. Its quantization consists in associating to $f(x)$ the operator

$$A_f := \int_X f(x)|x\rangle\langle x|\,\nu(dx).$$

- Function $f(x) \equiv \widehat{A}_f(x)$: upper (or contravariant) symbol of the operator A_f (nonunique in general).

- Mean value $\langle x|A_f|x\rangle \equiv \check{A}_f(x)$: lower (or covariant) symbol of A_f.

Comments

Such a quantization of the observation set is in one-to-one correspondence with the choice of the frame

$$\int_X |x\rangle\langle x| \; \nu(dx) = \mathbb{I}_{\mathcal{H}_N}.$$

To a certain extent, a quantization scheme consists in adopting a certain point of view in dealing with X*. This frame can be discrete, continuous, depending on the topology furthermore allocated to the set* X*, and it can be overcomplete, of course. Compare with Fourier or wavelet analysis in signal processing. Here, the validity of a precise frame choice is asserted by comparing spectral characteristics of quantum observables* A_f *with data provided by specific protocole in the observation of* X*.*

Standard example : motion of a particle on the line

- Observation set X is the classical phase space $\mathbb{R}^2 \simeq \mathbb{C} = \{x \equiv z = \frac{1}{\sqrt{2}}(q + ip)\}$ of a particle with one degree of freedom

- Measure on X is Gaussian, $\mu(dx) = \frac{1}{\pi} e^{-|z|^2} d^2 z$ where $d^2 z$ is the Lebesgue measure of the plane

- Functions $\phi_n(x)$ are the normalised powers of the complex variable z, $\phi_n(x) \equiv \frac{z^{\star n}}{\sqrt{n!}} \equiv \frac{\bar{z}^n}{\sqrt{n!}}$

- So that the Hilbert subspace $\mathcal{H}$ is the so-called Fock-Bargmann space of all entire functions that are square integrable with respect to the Gaussian measure

- Coherent states read

$$|z\rangle = e^{-\frac{|z|^2}{2}} \sum_n \frac{z^n}{\sqrt{n!}} |n\rangle,$$

where $|n\rangle \equiv \phi_n$.

The quantization of the observation set is hence achieved by selecting in the original Hilbert space $L^2(\mathbb{C}, \frac{1}{\pi} e^{-|z|^2} \, d^2 z)$ all holomorphic entire functions, which geometric quantization specialists would call a choice of polarization

Van Hove canonical quantization rules

Given a phase space with canonical coordinates $(\boldsymbol{q}, \boldsymbol{p})$

- To the classical observable $f(\boldsymbol{q}, \boldsymbol{p}) = 1$ corresponds the identity operator in the (projective) Hilbert space $\mathcal{H}$ of quantum states

- The correspondence that assigns to a classical observable $f(\boldsymbol{q}, \boldsymbol{p})$ a (essentially) self-adjoint operator on $\mathcal{H}$ is a linear map

- To the classical Poisson bracket corresponds, <u>at least at the order $\hbar$</u>, the quantum commutator, multiplied by $i\hbar$:

$$f_j(\boldsymbol{q}, \boldsymbol{p}) \to A_{f_j} \;\; j = 1, 2, 3$$
$$\{f_1, f_2\} = f_3 \to [A_{f_1}, A_{f_2}] = i\hbar A_{f_3} + o(\hbar)$$

- Some conditions of minimality on the resulting observable algebra

Standard coherent states yield canonical quantization

- Quantum operators acting on $\mathcal{H}$ are yielded by using

$$A_f := \int_X f(x)|x\rangle\langle x| \, \nu(dx).$$

- For the most basic one,

$$\frac{1}{\pi}\int_{\mathbb{C}} z \, |z\rangle\langle z| \, d^2z = \sum_n \sqrt{n+1}|n\rangle\langle n+1| \equiv a,$$

  which is the lowering operator,

$$a|n\rangle = \sqrt{n}|n-1\rangle$$

- Adjoint $a^\dagger$ is obtained by replacing z by $\bar{z}$ in the integral

- We get the factorisation $N = a^\dagger a$ for the number operator, together with the commutation rule $[a, a^\dagger] = \mathbb{I}_{\mathcal{H}}$

- $a^\dagger$ and a realize on $\mathcal{H}$ as multiplication operator and derivation operator respectively, $a^\dagger f(z) = zf(z),\ af(z) = df(z)/dz$

- From $q = \frac{1}{\sqrt{2}}(z+\bar{z})$ et $p = \frac{1}{\sqrt{2}i}(z-\bar{z})$, q and p are upper symbols for $\frac{1}{\sqrt{2}}(a+a^\dagger) \equiv Q$ and $\frac{1}{\sqrt{2}i}(a-a^\dagger) \equiv P$ respectively

- So the self-adjoint operators Q and P obey the canonical commutation rule $[Q,P] = i\mathbb{I}_{\mathcal{H}}$

Q and P fully deserve the name of position and momentum operators of the usual (galilean) quantum mechanics, together with all localisation properties specific to the latter, in particular the famous $\Delta Q\, \Delta P \geq \frac{1}{2}$

Heisenberg and Weyl

Standard coherent states have many interesting properties. Among them:

- They are eigenvectors of the lowering operator, $a|z\rangle = z|z\rangle$, and they saturate the Heisenberg inequalities : $\Delta Q \, \Delta P = \frac{1}{2}$
- they pertain to the group theoretical construction since they are obtained from unitary Weyl-Heisenberg transport of the ground state: $|z\rangle = \exp(za^{\dagger} - \bar{z}a)|0\rangle$.

Less standard example : motion of a particle on the circle

- Observation set X is the cylinder $S^1 \times \mathbb{R} = \{x \equiv (\beta, J), | 0 \leq \beta < 2\pi, J \in \mathbb{R}\}$, phase space of a particle moving on the circle

- J, β are canonically conjugate variables and $dJ\, d\beta$ is the invariant measure on the phase space

- Measure on X is partly Gaussian, $\mu(dx) = \sqrt{\frac{\epsilon}{\pi}} \frac{1}{2\pi} e^{-\epsilon J^2} dJ\, d\beta$ where $\epsilon > 0$ can be arbitrarily small

- Functions $\phi_n(x)$ are suitably weighted Fourier exponentials:

$$\phi_n(x) = e^{(-\epsilon n^2/2)}\, e^{n(\epsilon J + i\beta)}$$

- $\mathcal{N}(x) \equiv \mathcal{N}(J) = \sum_n e^{(-\epsilon n^2)}\, e^{2n\epsilon J} < \infty$
  (Theta function)

- Coherent states read
$$|J,\beta\rangle = \frac{1}{\sqrt{\mathcal{N}(J)}} \sum_n e^{(-\epsilon n^2/2)}\, e^{n(\epsilon J - i\beta)} |\phi_n\rangle,$$

(De Bièvre-González (92-93), González-Del Olmo (1998), Kowalski-Rembieliński-Papaloucas (1996))

The quantization of the observation set is hence achieved by selecting in the original Hilbert space $L^2(S^1 \times \mathbb{R}, \sqrt{\frac{\epsilon}{\pi}}\frac{1}{2\pi} e^{-\epsilon J^2}\, dJ\, d\beta)$ *all Laurent series in the complex variable* $e^{\epsilon J - i\beta}$ *(choice of polarization)*

Quantization

- Quantum operators acting on $\mathcal{H}$ are yielded by using

$$A_f := \int_X f(x)|x\rangle\langle x|\mathcal{N}(x)\mu(dx)$$

- For the most basic one,

$$\int_X \mu(dx)\mathcal{N}(J)\, J\,|J,\beta\rangle\langle J,\beta| = \sum_n n\,|\phi_n\rangle\langle\phi_n|,$$

- For an arbitrary function of β:

$$\int_X \mu(dx)\mathcal{N}(J)f(\beta)\, J\,|J,\beta\rangle\langle J,\beta|$$
$$= \sum_{n,n'} e^{-\frac{\epsilon}{4}(n-n')^2}\, c_{n-n'}(f)|\phi_n\rangle\langle\phi_{n'}| = A_{f(\beta)}$$

$c_{n-n'}(f)$: Fourier coefficient of f

- Operator “angle” :

$$A_\beta = \pi \mathbb{I}_{\mathcal{H}} + \sum_{n \neq n'} i \frac{e^{-\frac{\epsilon}{4}(n-n')^2}}{n-n'} |\phi_n\rangle\langle\phi_{n'}|$$

- Operator “Fourier”

$$A_{e^{i\beta}} = e^{-\frac{\epsilon}{4}} \sum_n |\phi_{n+1}\rangle\langle\phi_n|$$

- In an isomorphic realisation of $\mathcal{H}$ in which the kets $|\phi_n\rangle$'s are just the Fourier exponentials $e^{i\,n\beta}$:

$$A_J = -i\frac{\partial}{\partial\beta}$$

  and $A_{e^{i\beta}}$ is multiplication operator by $e^{i\beta}$ up to the (arbitrarily small) factor $e^{-\frac{\epsilon}{4}}$

Did you say canonical ?

- "Canonical" commutation rule

$$[A_J, A_{e^{i\beta}}] = A_{e^{i\beta}}$$

canonical in the sense that it is in exact correspondence with the classical Poisson bracket

$$\left\{J, e^{i\beta}\right\} = ie^{i\beta}$$

It is actually the only non trivial one having this exact correspondence

- Interpretative difficulties with others of the type

$$[A_J, A_{f(\beta)}] = \sum_{n,n'} \frac{e^{-\frac{\epsilon}{4}(n-n')^2}}{n-n'} c_{n-n'}(f)\, |\phi_n\rangle\langle\phi_{n'}|$$

- in particular for the angle operator:

$$[A_J, A_\beta] = i \sum_{n\neq n'} e^{-\frac{\epsilon}{4}(n-n')^2}\, |\phi_n\rangle\langle\phi_{n'}|$$

to be compared with the classical $\{J, \beta\} = 1$!!

Problem leading to well known quantum angular localisation problems

Nevertheless ...

Actually, these difficulties are only apparent ones and are due to the discontinuity of the 2π-periodic function $B(\beta)$ which is equal to β on $[0, 2\pi)$. They can be circumvented if we examine, for instance the behaviour of the corresponding lower symbols at the limit $\epsilon \to 0$. For the angle operator,

$$\langle J_0, \beta_0 | A_\beta | J_0, \beta_0 \rangle =$$
$$\pi + \left(1 + e^{\epsilon(J_0 - \frac{1}{4})} \frac{\mathcal{N}(J_0 - \frac{1}{2})}{\mathcal{N}(J_0)}\right) \sum_{n \neq 0} \frac{e^{-\frac{\epsilon}{2}n^2 + in\beta_0}}{n}$$
$$\underset{\epsilon \to 0}{\sim} \pi + \sum_{n \neq 0} \frac{e^{in\beta_0}}{n},$$

where we recognize at the limit the Fourier series of $B(\beta_0)$. For the commutator,

$$\langle J_0, \beta_0 | [A_J, A_\beta] | J_0, \beta_0 \rangle =$$
$$-i \left(1 + e^{\epsilon(J_0 - \frac{1}{4})} \frac{\mathcal{N}(J_0 - \frac{1}{2})}{\mathcal{N}(J_0)}\right) + \sum_{n \in \mathbb{Z}} e^{-\frac{\epsilon}{2}n^2 + in\beta_0}$$
$$\underset{\epsilon \to 0}{\sim} -i + \sum_{n} \delta(\beta_0 - 2\pi n).$$

So we (almost) recover the canonical commutation rule except for the singularity at the origin $\bmod 2\pi$.

From the motion of the circle to the motion on 1+1-de Sitter space-time

- Phase space X is also a one-sheeted hyperboloid:

$$J_1^2 + J_2^2 - J_0^2 = \kappa^2 > 0,$$

with (local) canonical coordinates (J, β), as for the motion on the circle.

- Phase space coordinates are now viewed as basic classical observables,

$$J_0 = J,\ J_1 = J \cos\beta - \kappa \sin\beta,\ J_2 = J \sin\beta + \kappa \cos\beta,$$

and obey the Poisson bracket relations

$$\{J_0, J_2\} = -J_1,\ \{J_0, J_1\} = -J_1,\ \{J_1, J_2\} = J_0.$$

They are, as expected, the commutation relations of $so(1,2) \simeq sl(2,\mathbb{R})$, which is the kinematical symmetry algebra of the system.

Toward the principal series of $SO_0(1,2)$

- Applying the coherent states quantization at $\epsilon \neq 0$ produces the basic quantum observables:

$$A_{J_0} = \sum_n n|\phi_n\rangle\langle\phi_n|,$$
$$A^{\epsilon}_{J_1} = \frac{1}{2}e^{-\frac{\epsilon}{4}} \sum_n (n + \frac{1}{2} + i\kappa)|\phi_{n+1}\rangle\langle\phi_n| + \text{ cc},$$
$$A^{\epsilon}_{J_2} = \frac{1}{2i}e^{-\frac{\epsilon}{4}} \sum_n (n + \frac{1}{2} + i\kappa)|\phi_{n+1}\rangle\langle\phi_n| - \text{ cc}.$$

- The quantization is asymptotically exact for these basic observables since

$$[A_{J_0}, A^{\epsilon}_{J_1}] = iA^{\epsilon}_{J_2},\ [A_{J_0}, A^{\epsilon}_{J_2}] = -iA^{\epsilon}_{J_1},\ [A^{\epsilon}_{J_1}, A^{\epsilon}_{J_2}] = -ie^{-\frac{\epsilon}{4}}A_{J_0}.$$

- The quadratic operator

$$C^{\epsilon} = (A^{\epsilon}_{J_1})^2 + (A^{\epsilon}_{J_2})^2 - (A_{J_0})^2$$
$$= \sum_n (e^{-\frac{\epsilon}{4}}(n^2 + \kappa^2 + \frac{1}{4}) - n^2)|\phi_n\rangle\langle\phi_n|,$$

admits the limit $C^{\epsilon} \underset{\epsilon\to 0}{\sim} (\kappa^2 + \frac{1}{4})\mathbb{I}$.

Hence we have produced a coherent states quantization which leads asymptotically to the principal series of $SO_0(1,2)$.

Non standard example : 2d quantum processing of a N-element set

- Choose N-element set $X = \{x_i\}$ as Observation Set with non-degenerate measure :

$$\mu(dx) = \sum_{i=1}^{N} a_i \delta_{\{x_i\}}, \;\; a_i > 0.$$

- Hilbert space $L^2(X, \mu)$ is isomorphic to $\mathbb{C}^N$.

- Consider the two-element orthonormal set $\{\phi_1, \phi_2\}$ defined in the most generic way by:

$$\phi_{\substack{1\\2}}(x) = \sum_{i=1}^{N} \left\{ \begin{array}{l} \alpha_i \\ \beta_i \end{array} \right. \frac{1}{\sqrt{a_i}} \chi_{\{x_i\}}(x).$$

- $\boldsymbol{\alpha} = \{\alpha_i\}, \boldsymbol{\beta} = \{\beta_i\}$ are orthonormal vectors $\in \mathbb{C}^N$:

$$\sum_{i=1}^{N} |\alpha_i|^2 = 1 = \sum_{i=1}^{N} |\beta_i|^2, \;\; \sum_{i=1}^{N} \alpha_i \overline{\beta_i} = 0.$$

- Coherent states:

$$|x\rangle = \frac{1}{\sqrt{\mathcal{N}(x)}} \left[\phi_1(x)\; |1\rangle + \phi_2(x)\; |2\rangle\right],$$

with $\mathcal{N}(x) = \sum_{i=1}^{N} \frac{|\alpha_i|^2 + |\beta_i|^2}{a_i} \chi_{\{x_i\}}(x)$

- To any real-valued function $f(x)$ on X, *i.e.* to any vector $\boldsymbol{f} \equiv (f(x_i))$ in $\mathbb{R}^N$, there corresponds the following hermitian operator A_f in $\mathbb{C}^2$:

$$A_f = \int_X \mu(dx)\, \mathcal{N}(x)\, f(x) |x\rangle\langle x| = \begin{pmatrix} \sum_{i=1}^{N} |\alpha_i|^2 f(x_i) & \sum_{i=1}^{N} \alpha_i \overline{\beta_i} f(x_i) \\ \sum_{i=1}^{N} \overline{\alpha_i} \beta_i f(x_i) & \sum_{i=1}^{N} |\beta_i|^2 f(x_i) \end{pmatrix}.$$

- In terms of Pauli matrices (non redundant for generic $N \geq 4$) :

$$A_f = \langle f \rangle_+ \sigma_0 + \langle f \rangle_- \sigma_3 + \Re\left(\langle f \rangle_0\right)\sigma_1 - \Im\left(\langle f \rangle_0\right)\sigma_2,$$

$$\langle f \rangle_\pm = \frac{1}{2}\sum_{i=1}^{N}\left(|\alpha_i|^2 \pm |\beta_i|^2\right) f(x_i),$$

$$\langle f \rangle_0 = \sum_{i=1}^{N}\left(\alpha_i \bar{\beta}_i\right) f(x_i).$$

- Spectral values of the quantum observable A_f:

$$\mathrm{Sp}(f) = \left\{\langle f \rangle_+ \pm \sqrt{(\langle f \rangle_-)^2 + |\langle f \rangle_0|^2}\right\}.$$

- Components of the lower symbol of A_f :

$$\langle x_l | A_f | x_l \rangle == \sum_i \varpi_i(l) f(x_i),$$

This is an interesting averaging of all values assumed by fonction f by the probability distribution

$$\varpi_l = |\alpha_l|^2 + |\beta_l|^2, \ \varpi_i = \frac{|\overline{\alpha_l}\alpha_i + \overline{\beta_l}\beta_i|^2}{|\alpha_l|^2 + |\beta_l|^2}, \ i \neq l.$$

In particular, components of lower symbols of Pauli matrices are given by:

$$\check{\sigma}_0(x_l) = 1, \qquad \check{\sigma}_1(x_l) = \frac{2\Re\left(\overline{\alpha_l}\beta_l\right)}{|\alpha_l|^2 + |\beta_l|^2},$$

$$\check{\sigma}_2(x_l) = \frac{2\Im\left(\overline{\alpha_l}\beta_l\right)}{|\alpha_l|^2 + |\beta_l|^2}, \ \check{\sigma}_3(x_l) = \frac{|\alpha_l|^2 - |\beta_l|^2}{|\alpha_l|^2 + |\beta_l|^2}.$$

Lower-dimensional cases $N=2$ and $N=3$

When $N = 2$ the basis change $\phi_i \to \boldsymbol{\alpha},\boldsymbol{\beta}$ reduces to a $U(2)$ transformation with $SU(2)$ parameters $\alpha = \alpha_1$, $\beta = -\overline{\beta}_1$, $|\alpha|^2 - |\beta|^2 = 1$, and some global phase factor. The operator A_f simplifies as

$$A_f = f_+ \mathbb{I} + f_- \begin{pmatrix} |\alpha|^2 - |\beta|^2 & -2\alpha\beta \\ -2\overline{\alpha\beta} & |\beta|^2 - |\alpha|^2 \end{pmatrix},$$

with $f_\pm := (f(x_1) \pm f(x_2))/2$. We now have a two-dimensional commutative algebra of "observables" A_f, generated by the identity matrix $\mathbb{I} = \sigma_0$ and the $SU(2)$ transform of σ_3: $\sigma_3 \to g\sigma_3 g^\dagger$ with $g = \left(\begin{smallmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{smallmatrix}\right) \in SU(2)$. As is easily expected in this case, lower symbols reduce to components:

$$\langle x_l | A_f | x_l \rangle = \check{A}_f(x_l) = f(x_l), \; l = 1, 2.$$

For $N = 3$ case and if all considered vector spaces are real, the basis change involves four real independent parameters, say $\alpha_1, \alpha_2, \beta_1$, and β_2, all with modulus < 1. One has (generically) uniqueness of upper symbols of Pauli matrices σ_1, σ_3, and $\sigma_0 = \mathbb{I}$ which form a basis of the three-dimensional Jordan algebra of real symmetric 2×2-matrices. These upper symbols read in vector form as

$$\hat{\boldsymbol{\sigma}}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; \hat{\boldsymbol{\sigma}}_1 = \mathcal{C}_3{}^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \; \hat{\boldsymbol{\sigma}}_3 = \mathcal{C}_3{}^{-1} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

A commutative 2d quantization of the unit interval

- Observation set is the unit interval $X = [0, 1]$ of the real line and its associated Hilbert space $L^2[0, 1]$.

- Select the two first elements of the orthonormal Haar basis : the characteristic function $\mathbf{1}(x)$ of the unit interval and the Haar wavelet:

$$\phi_1(x) = \mathbf{1}(x),\ \phi_2(x) = \mathbf{1}(2x) - \mathbf{1}(2x-1).$$

  Then $\mathcal{N}(x) = \sum_{n=1}^{2} |\phi_n(x)|^2 = 2$ *a.e.*.

- Coherent states read as

$$|x\rangle = \frac{1}{\sqrt{2}} \left[\phi_1(x)\ |1\rangle + \phi_2(x)\ |2\rangle\right].$$

- To any integrable function $f(x)$ on the interval there corresponds the linear operator A_f on $\mathbb{R}^2$ or $\mathbb{C}^2$:

$$A_f = \begin{pmatrix} \int_0^1 dx\, f(x) & \int_0^1 dx\, f(x)\phi_2(x) \\ \int_0^1 dx\, f(x)\phi_2(x) & \int_0^1 dx\, f(x) \end{pmatrix}.$$

- For instance, with the choice $f = \phi_1$ we recover the identity whereas for $f = \phi_2$, $A_{\phi_2} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) = \sigma_1$.

- For $f(x) = x^p, \Re e\, p > -1$,

$$A_{x^p} = \frac{1}{p+1} \begin{pmatrix} 1 & 2^{-p} - 1 \\ 2^{-p} - 1 & 1 \end{pmatrix}.$$

- Average values (lower symbols) of A_{x^p}.

$$\langle x_0 | A_{x^p} | x_0 \rangle = \begin{cases} \frac{2^{-p}}{p+1} & 0 \le x_0 \le \frac{1}{2}, \\ \frac{2-2^{-p}}{p+1} & \frac{1}{2} \le x_0 \le 1, \end{cases}$$

the two possible values being precisely the eigenvalues of the above matrix.

- Note the average values of the "position" operator: $\langle x_0 | A_x | x_0 \rangle = 1/4$ if $0 \le x_0 \le \frac{1}{2}$ and $3/4$ if $\frac{1}{2} \le x_0 \le 1$.

  Clearly, like in the $N = 2$ case of the previous example, all operators A_f commute, since they are linear combinations of the identity matrix and the Pauli matrix σ_1.

A non-commutative N-d quantization of the unit interval

- Add to the previous set $\{\phi_1, \phi_2\}$ other elements of the Haar basis, up to "scale" J :

$$\{\phi_1(x), \phi_2(x), \phi_3(x) = \sqrt{2}\phi_2(2x), \phi_4(x) = \sqrt{2}\phi_2(2x-1), \cdots, \phi_s(x) = 2^{j/2}\phi_2(2x-k), \phi_N(x) = 2^{J/2}\phi_2(2x-2^J+1)\},$$

- At given $j = 1, 2, \cdots, J$, $0 \leq k \leq 2^j - 1$.

- Total number of elements of this orthonormal system : $N = 2^{J+1}$.

- $\mathcal{N}(x) = 2^{J+1}$ clearly diverges at the limit $J \to \infty$.

- Spectral values as well as average values of the "position" operator are given by

$$\langle x_0 | A_x | x_0 \rangle = (2k+1)/2^{J+1}$$

  for $k/2^J \leq x_0 \leq (k+1)/2^J$ where $0 \leq k \leq 2^J - 1$.

This quantization scheme in the present case achieves a dyadic discretization of the localization in the unit interval.

A non-commutative 2d quantization of the unit interval

- Now we choose another orthonormal system, in the form of the two first elements of the trigonometric Fourier basis,

$$\phi_1(x) = 1(x), \;\; \phi_2(x) = \sqrt{2}\sin 2\pi x.$$

Then $\mathcal{N}(x) = \sum_{n=1}^{2} |\phi_n(x)|^2 = 1 + 2\sin^2 2\pi x$.

- Corresponding coherent states read as

$$|x\rangle = \frac{1}{\sqrt{1 + 2\;\sin^2 2\pi x}}\left[|1\rangle + \sqrt{2}\;\sin 2\pi x\;|2\rangle\right].$$

- To any integrable function $f(x)$ on the interval corresponds

$$A_f = \begin{pmatrix} \int_0^1 dx\, f(x) & \sqrt{2}\int_0^1 dx\, f(x)\sin 2\pi x \\ \sqrt{2}\int_0^1 dx\, f(x)\sin 2\pi x & 2\int_0^1 dx\, f(x)\sin^2 2\pi x \end{pmatrix}.$$

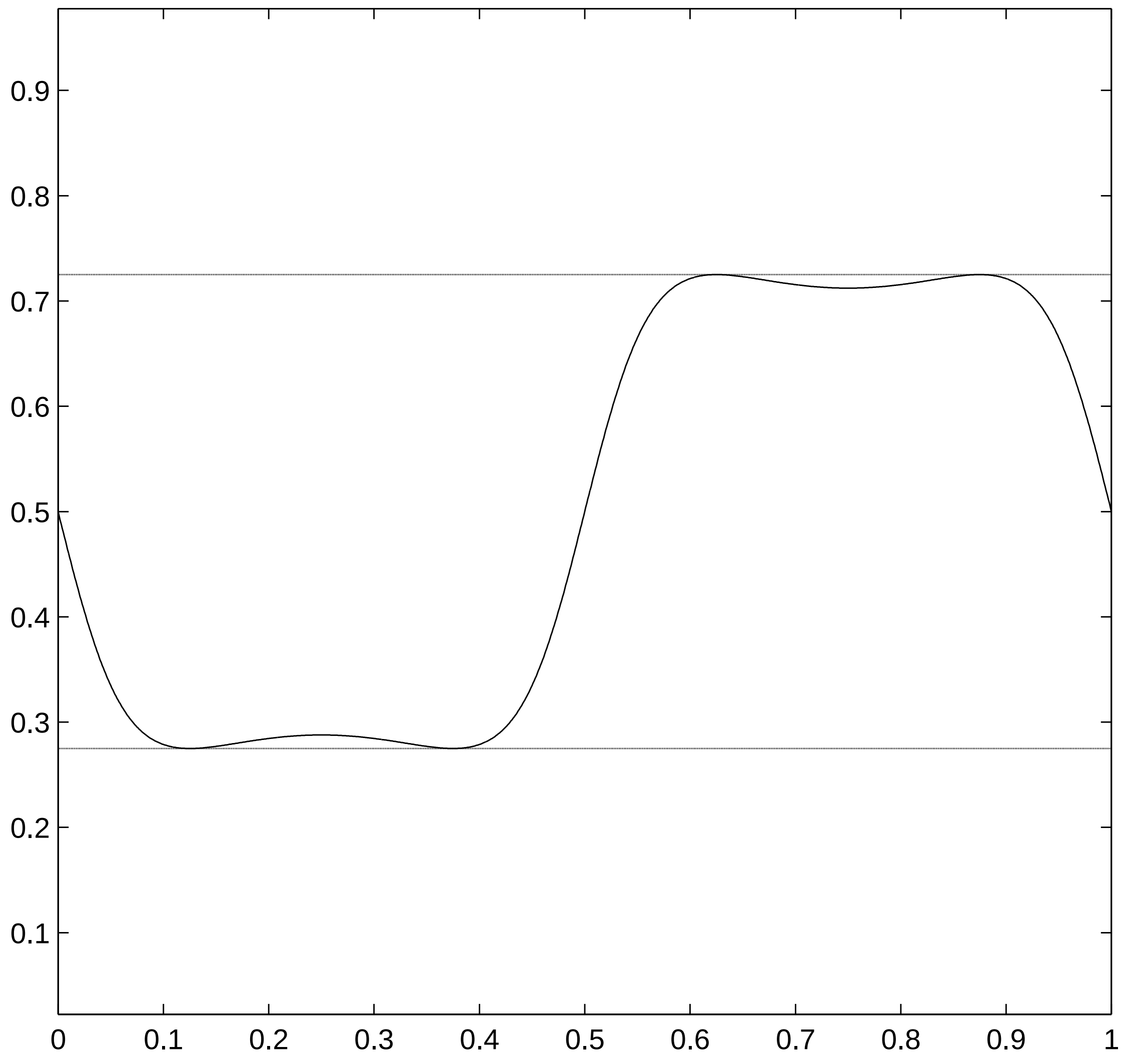
- *Position* operator :

$$A_x = \begin{pmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}\pi} \\ -\frac{1}{\sqrt{2}\pi} & \frac{1}{2} \end{pmatrix},$$

with eigenvalues $\frac{1}{2} \pm \frac{1}{\sqrt{2}\pi}$

- Note its average values :

$$\langle x_0 | A_x | x_0 \rangle = \frac{1}{2} - \frac{2}{\pi} \frac{\sin 2\pi x_0}{1 + 2 \ \sin^2 2\pi x_0}$$



Average value of position operator $\langle x_0 | A_x | x_0 \rangle$ *versus* x_0, *compared with eigenvalues of* A_x

Comments

Like in the previous case, with the choice $f = \phi_1$ we recover the identity whereas for $f = \phi_2$, $A_{\phi_2} = \sigma_1$, the first Pauli matrix.

We now have to deal with a non-commutative Jordan algebra of operators A_f, like in the $N = 3$ real case of the previous section. It is generated by the identity matrix and the two real Pauli matrices σ_1 and σ_3.

Other directions

Daoud M. and Kibler M. : A fractional supersymmetric oscillator and its coherent states *Proceedings of the International Wigner Symposium, Istanbul, Aout 1999*, Eds. M. Arik et al , Bogazici University Press (Istanbul) (2002).

The Grassmannian framework

- The observation set X is the Grassmann algebra Σ_k, *i.e.* linear span of $\left\{1, \theta, \cdots, \theta^{k-1}\right\}$ and their respective conjugates $\bar{\theta}^i$: here θ is a Grassmann variable satisfying $\theta^k = 0$.

- The measure on X is

$$\mu(d\theta d\bar{\theta}) = d\theta\, w(\theta, \bar{\theta})\; d\bar{\theta}.$$

Here, the integral over $d\theta$ and $d\bar{\theta}$ should be understood in the sense of Berezin-Majid-Rodríguez-Plaza integrals:

$$\int d\theta\, \theta^n = 0 = \int d\bar{\theta}\, \bar{\theta}^n, \text{ for } n = 0, 1, \cdots, k-2$$

$$\int d\theta\, \theta^{k-1} = 1 = \int d\bar{\theta}\, \bar{\theta}^{k-1}.$$

- The "weight" $w(\theta,\bar{\theta})$ is given by the q-deformed polynomial

$$w(\theta,\bar{\theta}) = \sum_{n=0}^{k-1} \Big(([n]_q!([n]_{\bar{q}}!\Big)^{\frac{1}{2}} \theta^{k-1-n}\bar{\theta}^{k-1-n},$$

where

$$[n]_q! = [1]_q[2]_q \cdots [n]_q, \text{ with } [x]_q := \frac{1-q^x}{1-q},$$

and $q = e^{\frac{2\pi i}{k}}$ is a root of unity.

Daoud-Kibler k-fermionic coherent states

- Introduce an orthonormal basis $\{|n\rangle\}$ of the Hilbert space $\mathbb{C}^k$.

- The (nonnormalized) DK coherent states should be understood as elements of $\mathbb{C}^k \otimes \Sigma_k$. They read as

$$|\theta) = \sum_{n=0}^{k-1} \frac{\theta^n}{([n]_q!)^{\frac{1}{2}}} |n\rangle.$$

Daoud-Kibler CS quantization

- The Daoud-Kibler CS quantization of the "spinorial" Grassman algebra rests upon the resolution of the unity $\mathbb{I}$ in $\mathbb{C}^k$:

$$\int\int d\theta|\theta)\, w(\theta,\bar{\theta})\,(\theta|\, d\bar{\theta} = \mathbb{I}$$

- The quantization of a Grassmann-valued function $f(\theta,\bar{\theta})$ maps f to the linear operator A_f on $\mathbb{C}^k$

$$A_f = \int\int d\theta|\theta)\, f(\theta,\bar{\theta})w(\theta,\bar{\theta})\,(\theta|\, d\bar{\theta}.$$

We actually recover the $k \times k$-matrix realization of the so-called *k-fermionic algebra* F_k. For instance, we have for the simplest functions:

$$A_\theta = \sum_{n=0}^{k-1} ([n+1]_q)^{\frac{1}{2}}\,|n\rangle\langle n+1|,$$

$$A_{\bar{\theta}} = \sum_{n=0}^{k-1} ([n+1]_{\bar{q}})^{\frac{1}{2}}\,|n+1\rangle\langle n| = A_\theta^\dagger.$$

Their (anti-)commutator reads as

$$[A_\theta, A_{\bar{\theta}}] = \sum_{n=0}^{k-1} \frac{\cos\pi\frac{2n+1}{2k}}{\cos\frac{\pi}{2k}}\,|n\rangle\langle n|,\ \{A_\theta, A_{\bar{\theta}}\} = \sum_{n=0}^{k-1} \frac{\sin\pi\frac{2n+1}{2k}}{\sin\frac{\pi}{2k}}\,|n\rangle\langle n|.$$

In the purely fermionic case, $k=2$, we recover the canonical anticommutation rule $\{A_\theta, A_{\bar{\theta}}\} = \mathbb{I}_2$.

Polymer quantization of the motion on the line

From *Quantum gravity, shadow states, and quantum mechanics*, A. Ashtekar, S. Fairhurst, and J. L. Willis, gr-qc/0207106

The game is to rebuild a "shadow" Schrödinger quantum mechanics on all possible discretisations of the real line. A simple way to do this is to adapt the previous CS quantization of the motion on the circle to an arbitrary discretization of the line. Actually, we shall deal with the phase space of the motion of the particle on the line, *i.e.* the plane, provided with a different measure.

Coherent states for the motion of a particle on a discrete subset of the line

- Observation set X is the plane $\mathbb{R}^2 = \{x \equiv (q,p)\}$,

- Measure on X is partly of the "Bohr type":

$$\mu(f) = \int_{-\infty}^{+\infty} dq \lim_{T\to+\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} dp\, f(q,p).$$

- Functions $\phi_n(x)$ are suitably weighted Fourier exponentials associated to a discrete subset $\gamma = \{a_n\}$ of the real line:

$$\phi_n(x) = \left(\frac{\epsilon}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\epsilon}{2}(q-a_n)^2} e^{-ia_n p},$$

- The “graph” (in the Ashtekar language) γ is supposed to be uniformly discrete (there exists a non zero minimal distance between successive elements) in such a way that the “generalized” theta function

$$\mathcal{N}(x) \equiv \mathcal{N}(q) = \sqrt{\frac{\epsilon}{\pi}} \sum_n e^{-\epsilon(q-a_n)^2}$$

converges.

- Coherent states read

$$|x\rangle = |q,p\rangle = \frac{1}{\sqrt{\mathcal{N}(q)}} \sum_n \phi_n^\star(x) |\phi_n\rangle.$$

Space is quantized

- Quantum operators acting on $\mathcal{H}$ are yielded by using

$$A_f := \int_X f(x)|x\rangle\langle x|\mathcal{N}(x)\mu(dx)$$

- For the most basic one, *i.e.* the "position":

$$\int_X \mu(dx)\mathcal{N}(q)\, q\, |q,p\rangle\langle q,p| = \sum_n a_n\, |\phi_n\rangle\langle\phi_n|.$$

Hence, the graph γ <u>is</u> the "quantization" of space when the latter is viewed from the point of view of coherent states precisely based on γ (...)

Quantum momentum doe's not exist

- Discrete catastrophe for the momentum :

  $$\int_X \mu(dx)\mathcal{N}(q)\, p|q,p\rangle\langle q,p| = \sum_{n,n'}$$

  $$\lim_{T\to+\infty}\left[\frac{\sin\left((a_n - a_{n'})\frac{T}{2}\right)}{(a_n - a_{n'})} - \frac{2}{T}\frac{\sin\left((a_n - a_{n'})\frac{T}{2}\right)}{(a_n - a_{n'})^2}\right]$$

  $$e^{-\frac{\epsilon}{4}(a_n - a_{n'})^2}|\phi_n\rangle\langle\phi_{n'}|,$$

  and the matrix elements are zero for $a_n \neq a_{n'}$ and ∞ for $a_n = a_{n'}$.

Nevertheless ...

- Pick an arbitrary Fourier exponential $e^{i\lambda p}$:
$$\int_X \mu(dx)\mathcal{N}(q)\, e^{i\lambda p}|q,p\rangle\langle q,p|$$
$$= \sum_{n,n'} e^{-\frac{\epsilon}{4}(a_n - a_{n'})^2}\, \delta_{\lambda, a_{n'} - a_n}|\phi_n\rangle\langle\phi_{n'}|$$

- And more generally, for superpositions of Fourier exponentials $e^{i\lambda p}$ (almost-periodic functions).

- In particular, one can define discretized versions of the momentum by considering CS quantized versions of finite differences of Fourier exponentials
$$\frac{1}{i}\frac{e^{i\lambda' p} - e^{i\lambda p}}{\lambda' - \lambda},$$
in which "allowed" frequencies should belong to the set of "interpositions" $\gamma - \gamma'$ in the graph γ.