Confluence in Antiholomorphic Dynamics

Unfolding of a Parabolic Fixed Point

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Content

Dynamics of antipolynomials

- Study the iterations of $z \mapsto \overline{z}^k + c$
  - The connectedness locus is a *multicorn*

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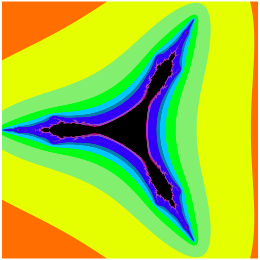
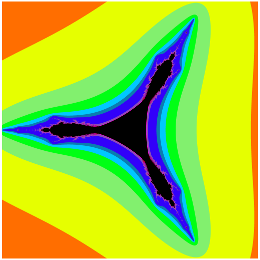


Tricorn (k=2)

Dynamics of antipolynomials

- Study the iterations of $z \mapsto \bar{z}^k + c$
  - The connectedness locus is a *multicorn*
- Key difference from the holomorphic setting :
  - Real analytic dependence of the parameter



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Local dynamics near a parabolic fixed point

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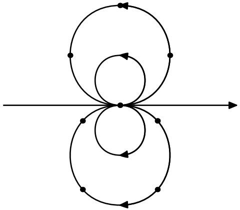
- $g := f \circ f$ is a holomorphic with a parabolic fixed point
- **Model.** $\overline{v^{\frac{1}{2}}}$, where v^t is the time-t of the vector field $\dot{z} = z^2$



Flot of $\dot{z} = z^2$

- an orbit of f jumps up and down

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- b the formal invariant (the residue of $\frac{1}{f \circ f - z}$)
- ψ one of the horn maps of $g = f \circ f$
  - the second horn map is given by $\sigma \circ v^{\frac{1}{2}} \circ \psi \circ \sigma \circ v^{\frac{1}{2}}$

Unfolding of a parabolic fixed point

We consider a family depending on $\varepsilon = \varepsilon_1 + i\varepsilon_2$ (ε_j real)

$$f_\varepsilon(z) = a_0(\varepsilon) + a_1(\varepsilon)\overline{z} + a_2(\varepsilon)\overline{z}^2 + \cdots$$

Two questions :

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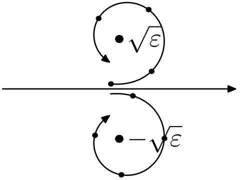
Two questions :

- Can we describe the space of orbits of f_ε for small values of ε ?
- Can we determine when two families are equivalent ?

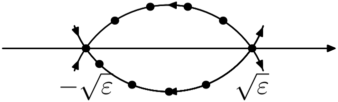
Unfolding of a parabolic fixed point

$$f_\varepsilon(z) = a_0(\varepsilon) + a_1(\varepsilon)\bar{z} + a_2(\varepsilon)\bar{z}^2 + \cdots$$

- This unfolds the parabolic fixed point into either
  - two simple fixed points
  - a periodic orbit of periode 2
  - a parabolic fixed



Periodic orbit



2 fixed points

Unfolding of a parabolic fixed point

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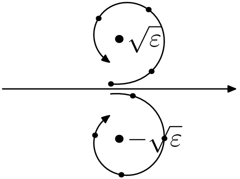
- $g_\varepsilon := f_\varepsilon \circ f_\varepsilon$ unfolds a holomorphic parabolic fixed point



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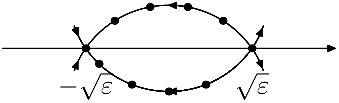
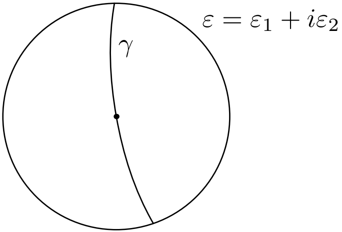


2 fixed points

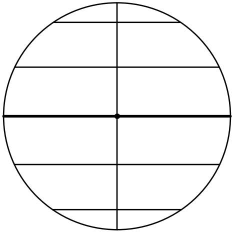
Unfolding of a parabolic fixed point

Lemma

If $\frac{\partial f_\varepsilon}{\partial \varepsilon_1}(0) \neq 0$, there exists a real analytic curve γ in the parameter space on which f_ε is parabolic.



Parameter space



Rectified parameter space

Prepared form

Theorem (2020, Godin, J., Rousseau, C.)

There exists a change of coordinates and parameter that brings $\widetilde{f}_\eta$ to the prepared form

$$f_\varepsilon(z) = \overline{z} + (\overline{z}^2 - \varepsilon_1)R_\varepsilon(\overline{z}),$$

where $\varepsilon_1 := \Re\varepsilon$ is the *unique* parameter given by

$$\varepsilon_1 = \left(\frac{1}{\log(\lambda_+)} - \frac{1}{\log(\lambda_-)}\right)^2, \quad \begin{pmatrix} \widetilde{g}_\eta = \widetilde{f}_\eta \circ \widetilde{f}_\eta \\ \lambda_\pm = \widetilde{g}_\eta'(\pm\sqrt{\eta}) \end{pmatrix}.$$

- ε_1 is the *canonical parameter*

Prepared form

From here, we consider families of one real paramter ε in prepared form

$$f_\varepsilon(z) = \overline{z} + (\overline{z}^2 - \varepsilon)R_\varepsilon(\overline{z}), \qquad \varepsilon \in \mathbb{R} \text{ small.}$$

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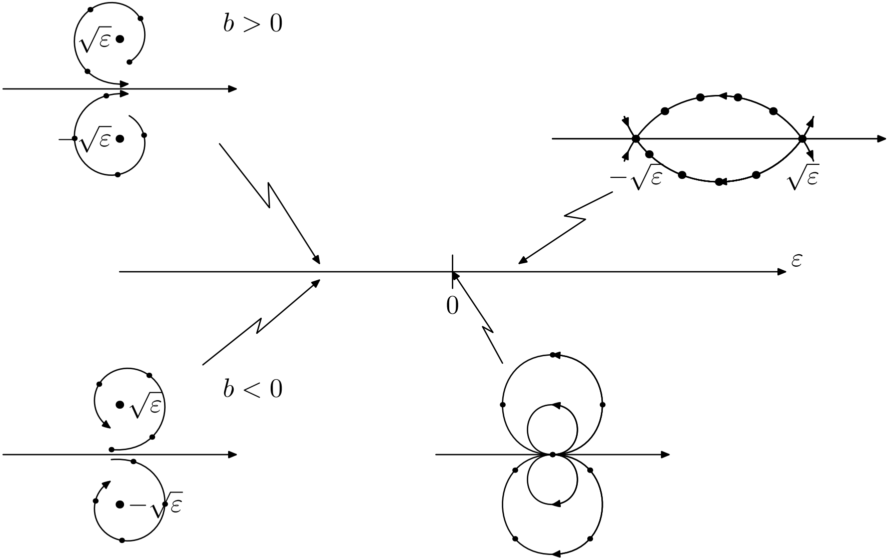
We define

$$b(\varepsilon) = \frac{1}{\log(\lambda_+)} + \frac{1}{\log(\lambda_-)}, \qquad \begin{pmatrix} g_\varepsilon = f_\varepsilon \circ f_\varepsilon, \\ \lambda_\pm = g_\varepsilon'(\pm\sqrt{\varepsilon}), \end{pmatrix}$$

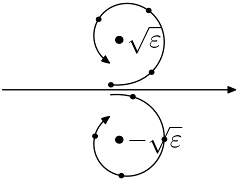
the *formal invariant*.

Bifurcation diagram



Space of orbits for $\varepsilon < 0$

First return map P



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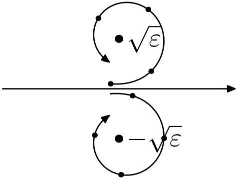

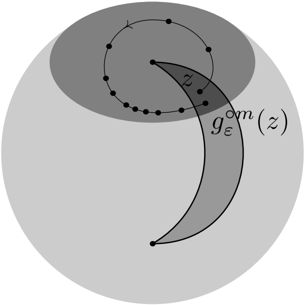

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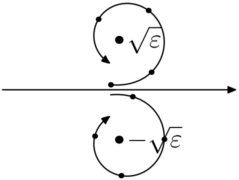
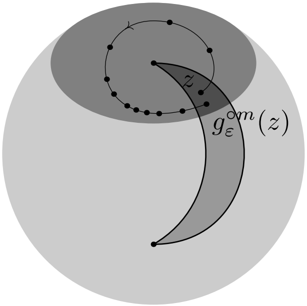
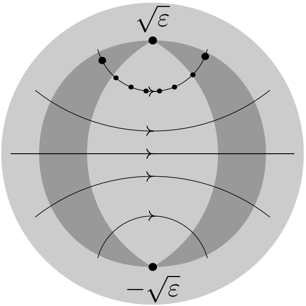


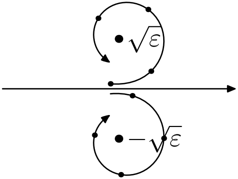


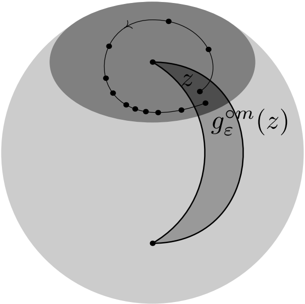


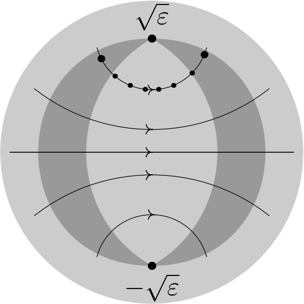
Space of orbits for $\varepsilon < 0$

First return map P decomposes

$$P = L \circ \psi.$$

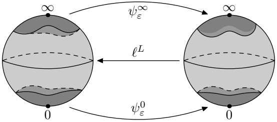






Space of orbits for $\varepsilon < 0$

- Each crescent is identified to a doubly punctured sphere



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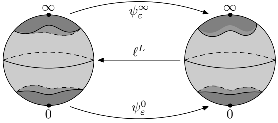
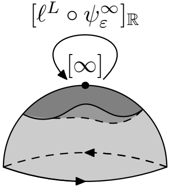


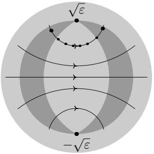
- We quotient by ℓ^L and f_ε to obtain

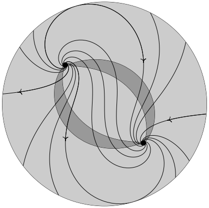


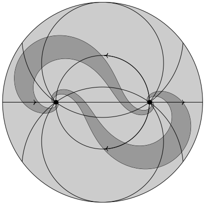
Space of orbits of f_ε : the projective space $\mathbb{R}P^2$

Space of orbits for $\varepsilon > 0$

- We extend the point of view of the previous case
  - We complexifie ε, so that f_ε is antiholomorphic in ε
  - The relation between g_ε and f_ε extends to $g_\varepsilon = f_{\overline{\varepsilon}} \circ f_\varepsilon$







Equivalence Theorem

Theorem (2020, Godin, J., Rousseau, C.)

Two prepared families $f_{j,\varepsilon}$ unfolding a parabolic fixed point are equivalent if and only if they have the same geometric invariant.

Corollary

Two generic families $f_{1,\alpha}$ and $f_{2,\beta}$ each unfolding a parabolic fixed point are equivalent if and only if they have the same modulus of classification $(\varepsilon, b, [\psi])$.