Théorème. *Soit $f\colon[-\pi,\pi]\to\mathbb{R}$ une fonction intégrable. Alors pour tout $k\in\mathbb{N}$, on a*

$$\begin{aligned}\inf\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(f-\sum_{n=-k}^{k}d_ne^{inx}\right)^2\mathrm{d}x \;\middle|\; \begin{matrix} d_{-k},\dots,d_k\in\mathbb{C}\\ d_{-n}=\overline{d_n}\end{matrix}\right\}\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(f-\sum_{n=-k}^{k}c_n(f)e^{inx}\right)^2\mathrm{d}x\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}f^2-\sum_{n=-k}^{k}|c_n(f)|^2\geq 0.\end{aligned}$$

Démonstration. On pose $c_n=c_n(f)$. Pour $k\in\mathbb{N}$, on a

$$\left(f-\sum_{n=-k}^{k}d_ne^{inx}\right)^2=f^2-2f\sum_{n=-k}^{k}d_ne^{inx}+\sum_{n=-k}^{k}\sum_{m=-k}^{k}d_nd_me^{i(m+n)x}.$$

On intègre l'équation :

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(f-\sum_{n=-k}^{k}d_ne^{inx}\Big)^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}f^2-2\sum_{n=-k}^{k}d_nc_{-n}+\sum_{n=-k}^{k}d_nd_{-n}\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}f^2-\sum_{n=-k}^{k}d_nc_{-n}-\sum_{n=-k}^{k}d_{-n}c_n+\sum_{n=-k}^{k}d_nd_{-n}\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}f^2+\sum_{n=-k}^{k}\Big(-d_nc_{-n}-d_{-n}c_n+d_nd_{-n}\Big)\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}f^2+\sum_{n=-k}^{k}\Big((-c_nc_{-n}+c_nc_{-n})-d_nc_{-n}-d_{-n}c_n+d_nd_{-n}\Big)\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}f^2-\sum_{n=-k}^{k}c_nc_{-n}+\sum_{n=-k}^{k}(c_nc_{-n}-d_nc_{-n}-d_nc_{-n}+d_nd_{-n})\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}f^2-\sum_{n=-k}^{k}c_n\overline{c_n}+\sum_{n=-k}^{k}(d_n-c_n)(d_{-n}-c_{-n})\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}f^2-\sum_{n=-k}^{k}|c_n|^2+\sum_{n=-k}^{k}|d_n-c_n|^2\\
&\geq \frac{1}{2\pi}\int_{-\pi}^{\pi}f^2-\sum_{n=-k}^{k}|c_n|^2.\end{aligned}$$

On a égalité ssi $d_n=c_n$, c'est-à-dire $d_n=c_n(f)$.

$\square$

17/3/2026