A Note On Upper And Lower Solutions For First Order Inclusions Of Upper Semicontinuous Or Lower Semicontinuous Type

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Abstract. Existence results based on fixed point theorems for self maps are used to establish an upper and lower solutions theory for first order inclusions.

1 Introduction

This paper presents an upper and lower solutions theory for the first order inclusions

\[ \begin{cases}
  x'(t) \in F(t, x(t)) & \text{a.e. } t \in [0, T] \\
  x(0) = r
\end{cases} \tag{1.1} \]

where \( F : [0, T] \times \mathbb{R} \to K(\mathbb{R}) \); here \( K(\mathbb{R}) \) denotes the family of nonempty, compact subsets of \( \mathbb{R} \). Throughout the paper our map \( x \mapsto F(t, x) \) is upper semicontinuous or lower semicontinuous for a.e. \( t \in [0, T] \). Recall a map \( G : \mathbb{R} \to K(\mathbb{R}) \) is lower semicontinuous (respectively upper semicontinuous) if the set \( \{ x \in \mathbb{R} : G(x) \cap A \neq \emptyset \} \) is open (respectively closed) for any open (respectively closed) subsets of \( \mathbb{R} \).
closed) set $A$ in $\mathbb{R}$. Also a map $H : [0, T] \rightarrow K(\mathbb{R})$ is measurable if the set 
\{ $t \in [0, T] : H(t) \cap A \neq \emptyset$ \} is measurable for any closed set $A$ in $[0, T]$. The reader is referred to [1-4] for results on multivalued mappings.

We note that part of the theory presented in this paper was inspired by results in [5] for second order problems. Our results rely on the following existence results established in the literature [5, 6]. The first result follows immediately from Ky Fan's fixed point theorem and the second from the Bressan Colombo selection theorem and Schauder's fixed point theorem.

Theorem 1.1. Suppose $F : [0, T] \times \mathbb{R} \rightarrow CK(\mathbb{R})$ (here $CK(\mathbb{R})$ denotes the family of nonempty, convex, compact subsets of $\mathbb{R}$) satisfies the following conditions:

\[ t \mapsto F(t, x) \text{ is measurable for every } x \in \mathbb{R} \quad (1.2) \]

\[ x \mapsto F(t, x) \text{ is upper semicontinuous for a.e. } t \in [0, T] \quad (1.3) \]

and

\[ \exists h \in L^1[0, T] \text{ with } |F(t, x)| \leq h(t) \text{ for a.e. } t \in [0, T] \text{ and } x \in \mathbb{R}. \quad (1.4) \]

Then (1.1) has a solution $y \in W^{1,1}[0, T]$.

Theorem 1.2. Suppose $F : [0, T] \times \mathbb{R} \rightarrow K(\mathbb{R})$ satisfies (1.4) and the following two conditions:

\[ (t, x) \mapsto F(t, x) \text{ is } L \otimes B \text{ measurable} \quad (1.5) \]

and

\[ x \mapsto F(t, x) \text{ is lower semicontinuous for a.e. } t \in [0, T]. \quad (1.6) \]

Then (1.1) has a solution $y \in W^{1,1}[0, T]$.

Remark 1.1. Recall $A \subseteq I \times \mathbb{R}$ is $L \otimes B$ measurable if $A$ belongs to the $\sigma$ algebra generated by all sets of the form $N \times D$ where $N$ is Lebesgue measurable in $[0, T]$ and $D$ is Borel measurable in $\mathbb{R}$.

2 Upper and lower solutions.

In this section we present an upper and lower solutions result for the first order inclusion

\[ \begin{cases} x'(t) \in F(t, x(t)) & \text{a.e. } t \in [0, T] \\ x(0) = r \end{cases} \quad (2.1) \]

where $F : [0, T] \times \mathbb{R} \rightarrow K(\mathbb{R})$.

Definition 2.1. A function $\beta \in W^{1,1}[0, T]$ is said to be an upper solution for (2.1) if for almost every $t \in [0, T]$ there exists $v \in F(t, \beta(t))$ with $v \leq \beta'(t)$ (i.e. $F(t, \beta(t)) \cap (- \infty, \beta'(t)) \neq \emptyset$) and $\beta(0) \geq r$. Similarly a function $\alpha \in W^{1,1}[0, T]$ is said to be a lower solution for (2.1) if for almost every $t \in [0, T]$ there exists $v \in F(t, \alpha(t))$ with $v \geq \alpha'(t)$ (i.e. $F(t, \alpha(t)) \cap [\alpha'(t), \infty) \neq \emptyset$) and $\alpha(0) \leq r$.

We begin with the upper semicontinuous situation.
Theorem 2.1. Suppose \( F : [0, T] \times \mathbb{R} \to C(K) \) satisfies the following conditions:
\[
 t \mapsto F(t, x) \text{ is measurable for every } x \in \mathbb{R} \tag{2.2}
\]
\[
 x \mapsto F(t, x) \text{ is upper semicontinuous for a.e. } t \in [0, T] \tag{2.3}
\]
and
\[
 \left\{ \begin{array}{l}
 \text{for each } r > 0, \exists h_r \in L^1[0, T] \text{ with } |F(t, x)| \leq h_r(t) \\
 \text{for a.e. } t \in [0, T] \text{ and } x \in \mathbb{R} \text{ with } |x| \leq r.
\end{array} \right. \tag{2.4}
\]
Also assume there exists \( \alpha, \beta \in W^{1,1}[0, T] \) respectively lower and upper solutions of (2.1) with \( \alpha(t) \leq \beta(t) \) for \( t \in [0, T] \). Then (2.1) has a solution \( y \in W^{1,1}[0, T] \) with \( \alpha(t) \leq y(t) \leq \beta(t) \) for \( t \in [0, T] \).

PROOF: Let
\[
h(t, x) = \left\{ \begin{array}{ll}
 \alpha(t), & x < \alpha(t), \\
 \beta(t), & x > \beta(t); \\
 x, & \alpha(t) \leq x \leq \beta(t),
\end{array} \right.
\]
\[
\Gamma_+(t, x) = \left\{ \begin{array}{ll}
 \alpha'(t), & x < \alpha(t), \\
 \mathbb{R}, & \alpha(t) \leq x \leq \beta(t), \\
 (-\infty, \beta'(t)), & x > \beta(t);
\end{array} \right.
\]
and let
\[
 F_+(t, x) = F(t, h(t, x)) \cap \Gamma_+(t, x).
\]
Notice \( F_+ : [0, T] \times \mathbb{R} \to C(K) \) (notice the values are nonempty from the definition of upper and lower solutions). Also it is easy to see that \( x \mapsto \Gamma_+(t, x) \) is upper semicontinuous for a.e. \( t \in [0, T] \) so the map \( x \mapsto F_+(t, x) \) is upper semicontinuous for a.e. \( t \in [0, T] \). In addition the map \( t \mapsto F_+(t, x) \) is measurable for each \( x \in \mathbb{R} \). Consider the modified problem
\[
 \left\{ \begin{array}{l}
 x'(t) \in F_+(t, x(t)) \text{ a.e. } t \in [0, T] \\
 x(0) = r.
\end{array} \right. \tag{2.5}
\]
Now Theorem 1.1 guarantees that (2.5) has a solution \( y \in W^{1,1}[0, T] \). To finish the proof it suffices to show \( \alpha(t) \leq y(t) \leq \beta(t) \) for \( t \in [0, T] \). Suppose \( y(t) \not\leq \beta(t) \) for some \( t \in [0, T] \). Since \( y(0) = r \leq \beta(0) \) there exists \( t_1 < t_2 \in [0, T] \) with
\[
y(t_1) = \beta(t_1) \text{ and } y(t) > \beta(t) \text{ for } t \in (t_1, t_2).
\]
For almost every \( t \in (t_1, t_2) \), since \( y(t) > \beta(t) \) we have
\[
y'(t) = w(t) \text{ where } w(t) \in F_+(t, y(t)). \tag{2.6}
\]
In particular \( w(t) \in \Gamma_+(t, y(t)) \) so \( w(t) \in (-\infty, \beta'(t)) \). This together with (2.6) implies \( y'(t) \leq \beta'(t) \). Integration from \( t_1 \) to \( t_2 \) yields \( y(t_2) \leq \beta(t_2) \), a contradiction. Thus \( y(t) \leq \beta(t) \) for \( t \in [0, T] \). A similar argument shows \( y(t) \geq \alpha(t) \) for \( t \in [0, T] \).

Our next result concerns the lower semicontinuous situation.

Theorem 2.2. Suppose \( F : [0, T] \times \mathbb{R} \to K(\mathbb{R}) \) satisfies (2.4) and the following two conditions:
\[
 (t, x) \mapsto F(t, x) \text{ is } \mathcal{L} \otimes \mathcal{B} \text{ measurable} \tag{2.7}
\]
and
\[ x \mapsto F(t, x) \text{ is lower semicontinuous for a.e. } t \in [0, T]. \tag{2.8} \]

Also assume there exists \( \alpha, \beta \in W^{1,1}[0, T] \) respectively lower and upper solutions of (2.1) with \( \alpha(t) \leq \beta(t) \) for \( t \in [0, T] \). Then (2.1) has a solution \( y \in W^{1,1}[0, T] \) with \( \alpha(t) \leq y(t) \leq \beta(t) \) for \( t \in [0, T] \).

**PROOF:** Let
\[ \Gamma_- (t, x) = \begin{cases} [\alpha'(t), \infty), & x \leq \alpha(t) \\ [\alpha(t), \beta(t)), & \alpha(t) < x < \beta(t) \\ (-\infty, \beta'(t)], & x \geq \beta(t) \end{cases} \]

and let
\[ F_-(t, x) = F(t, h(t, x)) \cap \Gamma_-(t, x). \]

Notice \( F_- : [0, T] \times \mathbb{R} \to K(\mathbb{R}) \). We claim \( x \mapsto F_-(t, x) \) is lower semicontinuous for a.e. \( t \in [0, T] \). From (2.8) there exists a null set \( N \) with \( x \mapsto F(t, x) \) is lower semicontinuous for \( t \in [0, T] \setminus N \). Fix \( t \in [0, T] \setminus N \). To show \( F(t, \cdot) \) is lower semicontinuous for \( x \in \mathbb{R} \) take a sequence \( \{x_n\}^\infty_{n=0} \) in \( \mathbb{R} \) with \( x_n \to x \). Take any \( y \in F_-(t, x) \). We must show that there exists a subsequence \( S \) of \( N_0 = \{1, 2, \ldots\} \) and elements \( y_k \in F_-(t, x_k) \), \( k \in S \), with \( y_k \to y \) as \( k \to \infty \) in \( S \). The proof is broken into a number of cases. We discuss three such cases which illustrate the ideas involved. For the first case suppose \( \alpha(t) < x < \beta(t) \) and \( \alpha(t) < x_n < \beta(t) \) for \( n \in N_0 \). Then
\[ y \in F(t, x) \cap \mathbb{R} = F(t, x). \]

Since \( x \mapsto F(t, x) \) is lower semicontinuous then there exists a subsequence \( S \) of \( N_0 \) and elements \( y_k \in F(t, x_k) \), \( k \in S \), with \( y_k \to y \) as \( k \to \infty \) in \( S \). Note \( y_k \in F_-(t, x_k) \), \( k \in S \), since \( \Gamma_-(t, x_k) = \mathbb{R} \) and \( F(t, h(t, x_k)) = F(t, x_k) \). For the second case suppose \( x \geq \beta(t) \) and assume \( x_n \geq \beta(t) \) for \( n \in N_0 \). Then
\[ y \in F(t, \beta(t)) \cap (-\infty, \beta'(t)]. \]

Choose \( S = N_0 \) and \( y_k = y_k \), \( k \in S \). Notice \( y_k \in F_-(t, x_k) \), \( k \in S \), since \( x_n \geq \beta(t) \) for \( n \in N_0 \) implies \( F(t, h(t, x_k)) = F(t, \beta(t)) \) and \( \Gamma_-(t, x_k) = (-\infty, \beta'(t)] \). For the third case suppose \( x = \beta(t) \) and assume \( \alpha(t) < x_n < \beta(t) \) for \( n \in N_0 \). Then
\[ y \in F(t, \beta(t)) \cap (-\infty, \beta'(t)] = F(t, x) \cap (-\infty, \beta'(t)]. \]

Since \( x \mapsto F(t, x) \) is lower semicontinuous then there exists a subsequence \( S \) of \( N_0 \) and elements \( y_k \in F(t, x_k) \), \( k \in S \), with \( y_k \to y \) as \( k \to \infty \) in \( S \). Note \( y_k \in F_-(t, x_k) \), \( k \in S \), since \( \alpha(t) < x_n < \beta(t) \) for \( n \in N_0 \) implies \( F_-(t, x_k) \cap \mathbb{R} = F(t, x_k) \). The other cases are similar, so as a result \( x \mapsto F_-(t, x) \) is lower semicontinuous for a.e. \( t \in [0, T] \). Consider the modified problem
\[ \begin{cases} x'(t) \in F_-(t, x(t)) & \text{a.e. } t \in [0, T] \\ x(0) = r. \end{cases} \tag{2.9} \]

Now Theorem 1.2 guarantees that (2.9) has a solution \( y \in W^{1,1}[0, T] \), and essentially the same argument as in Theorem 2.1 yields \( \alpha(t) \leq y(t) \leq \beta(t) \) for \( t \in [0, T] \).
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