

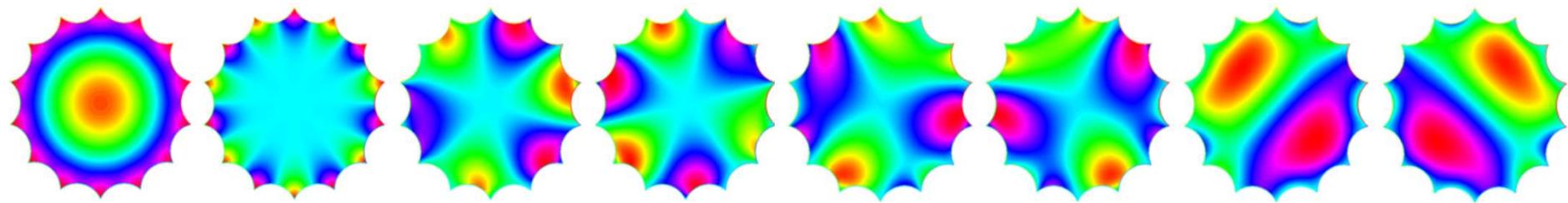
Geometric inequalities from trace formulas

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joint work with Bram Petri

Result

Thm (FB-Petri) Among closed hyperbolic surfaces of genus 3, the multiplicity m_1 of the first positive eigenvalue of the Laplacian is maximized by the Klein quartic Q , where the multiplicity is 8.

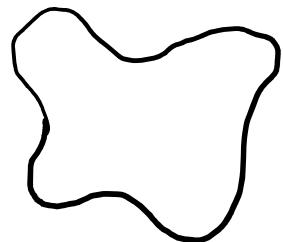


Thm (Sévennec) $m_1(X) \leq 2g+3$ for every Riemannian surface X of genus g .

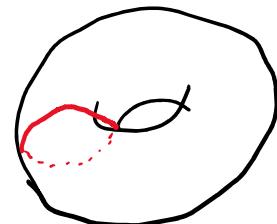
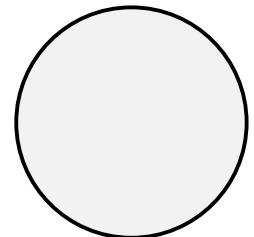
Thm (Cook) $m_1(Q) \in \{6, 7, 8\}$.

Motivation

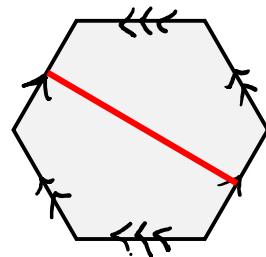
Q: What are the "roundest" shapes?



$$\frac{1}{\text{roundness}} = \text{isoperimetric ratio} = \frac{\text{perimeter}^2}{\text{area}}$$



$$\text{roundness} = \text{(iso)systolic ratio} = \frac{\text{systole}^2}{\text{area}}$$



systole = length of shortest non-contractible closed curve

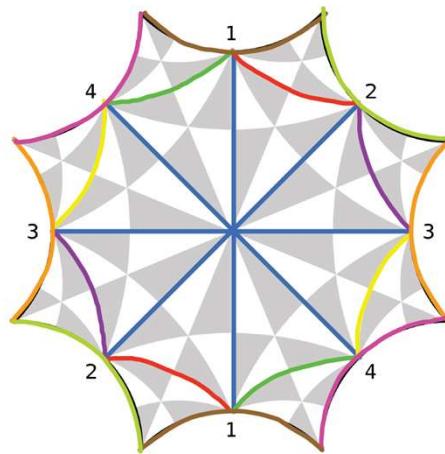
The optimal shape in genus 2 is not known!



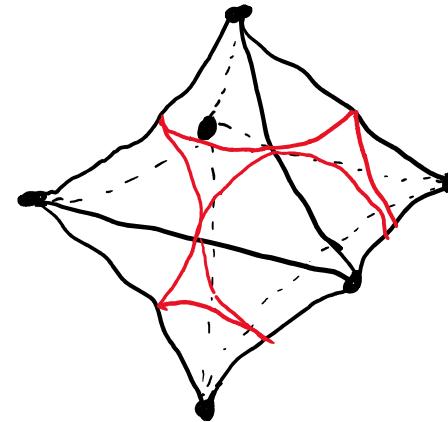
Hyperbolic surfaces

A hyperbolic surface of genus g has area $= 4\pi(g-1)$ so
maximizing systolic ratio \Leftrightarrow maximizing systole

Thm (Jenni) The Bolza surface maximizes the systole for $g=2$.



hyperelliptic
involution



maximizing systole \Leftrightarrow maximizing the area occupied by the
disks of radius $\frac{\text{sys}}{2}$ centered at the cone pts

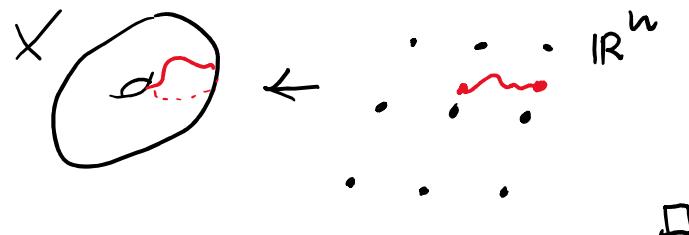
The optimal hyperbolic surface in genus 3 is not known!

Flat tori

$X = \mathbb{R}^n / \Gamma$ where $\Gamma < \mathbb{R}^n$ is a lattice

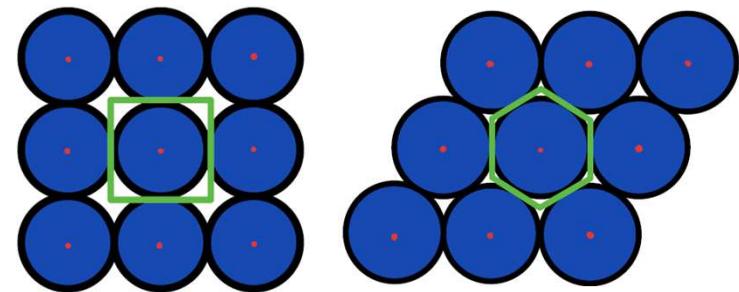
Obs 1 $\text{sys}(X) = \min_{x \in \Gamma \setminus \{0\}} \|x\|$.

Pf



Obs 2 The balls of radius $\frac{\text{sys}(X)}{2}$ centered on Γ form a sphere packing.

Pf Triangle inequality. \square



$$\text{packing density} = \frac{\text{vol}(B_{\frac{\text{sys}(X)}{2}}(0))}{\text{vol}(\text{Voronoi cell})} = \left(\frac{\text{sys}(X)}{2}\right)^n \frac{\text{vol}(B_1(0))}{\text{vol}(X)} \propto \frac{\text{sys}(X)^n}{\text{vol}(X)}$$

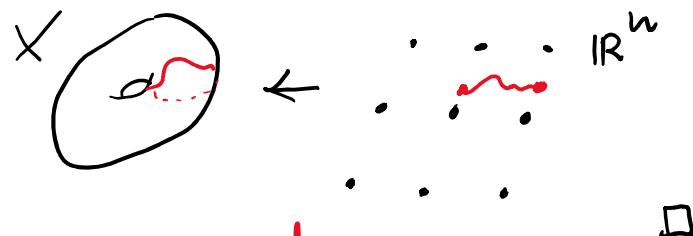
maximizing Systolic ratio \Leftrightarrow maximizing packing density

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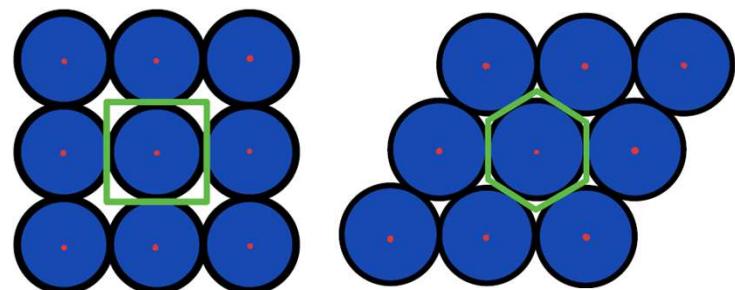
Kissing number = # spheres tangent to any given sphere = # homotopy classes of shortest oriented curves

$$\frac{\text{packing density}}{\text{Vol(Voronoi cell)}} = \frac{\text{vol}(B_{\frac{\text{sys}(X)}{2}}(0))}{\text{vol}(\text{Voronoi cell})} = \left(\frac{\text{sys}(X)}{2}\right)^n \frac{\text{vol}(B_1(0))}{\text{vol}(X)} \propto \frac{\text{sys}(X)^n}{\text{vol}(X)}$$

maximizing Systolic ratio \Leftrightarrow maximizing packing density

Obs 2 The balls of radius $\frac{\text{sys}(X)}{2}$ centered on Γ form a sphere packing.

Pf Triangle inequality. \square



Sphere packings

The densest lattice packing is known in dimensions 1-8 and 24.

(The densest sphere packings are only known for $n=1, 2, 3, 8, 24$)

→ There is an algorithm to enumerate all local maximizers of the systolic ratio. Only carried out up to $n=8$ (there are 2408).

$$\boxed{n=24}$$

Cohn-Kumar (2009): Leech lattice optimal among lattices.

Cohn-Kumar-Miller-Radchenko-Viazovska (2017): among all packings.

Both use a bound of Cohn-Elkies (2003)

Thm (Cohn-Elkies) Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous, radial & fast decay such that:

- $f(x) \leq 0$ if $\|x\| \geq 2$
- $\hat{f}(\xi) \geq 0 \quad \forall \xi \in \mathbb{R}^n$
- $\hat{f}(0) > 0$

Then the packing density in \mathbb{R}^n is bounded by $\text{vol}(B_1(0)) \frac{\hat{f}(0)}{\hat{f}(0)}$.

PF for lattice packings

Rescale Γ so that $\min_{x \in \Gamma \setminus \{0\}} \|x\| = 2$ and hence the balls

in the packing have radius 1. Then

$$f(0) \geq \sum_{x \in \Gamma} f(x) \stackrel{\text{Poisson Summation Formula}}{=} \frac{1}{\text{vol}(\mathbb{R}^n/\Gamma)} \sum_{y \in \Gamma^*} \hat{f}(y) \geq \frac{\hat{f}(0)}{\text{vol}(\mathbb{R}^n/\Gamma)}$$

$$\Rightarrow \text{packing density} = \frac{\text{vol}(B_1(0))}{\text{vol}(\mathbb{R}^n/\Gamma)} \leq \text{vol}(B_1(0)) \cdot \frac{f(0)}{\hat{f}(0)}.$$

□

Tricky bit: Finding good functions f

Solution: Take $f = \text{polynomial} \cdot \text{Gaussian}$

$\rightarrow \hat{f}$ is easy to compute

+ Numerical optimization

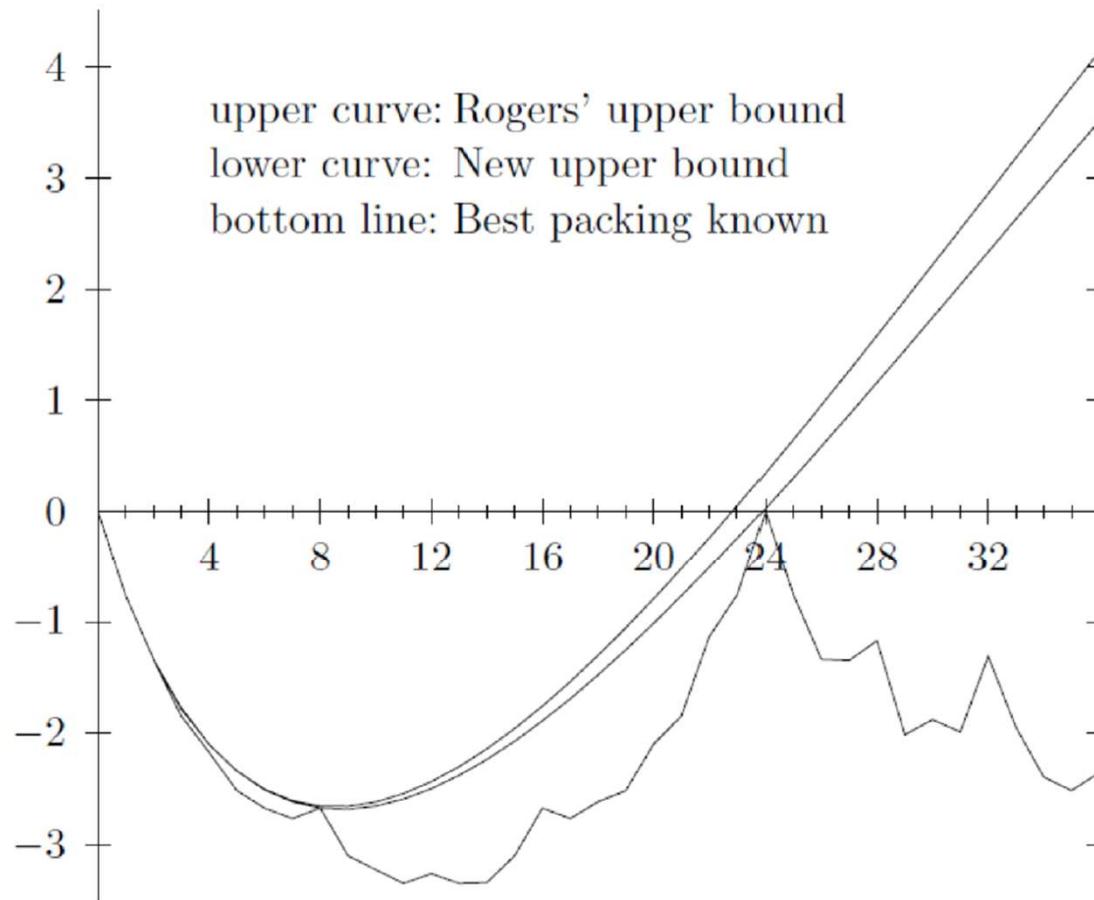


Figure 1. Plot of $\log_2 \delta + n(24 - n)/96$ vs. dimension n .

Trace formulas

X = flat torus, hyperbolic surface, finite regular graph

$\mathcal{C}(X)$ = homotopy classes of oriented closed geodesics on X (\leftrightarrow Conjugacy classes in $\widetilde{\pi}_1(X) \setminus \{\text{id}\}$)

$l(\gamma)$ = length of γ

Δ = Laplacian on X = $-\text{div} \circ \text{grad}$

$\sigma(X)$ = { eigenvalues of Δ listed with multiplicity }

length
spectrum



eigenspectrum

$\text{sys}(X)$

$\lambda_1(X)$

$\text{kiss}(X)$

$m_1(X)$

Poisson Summation Formula $X = \mathbb{R}^n / \Gamma$, $\Gamma^* = \{y \in \mathbb{R}^n : x \cdot y \in \mathbb{Z} \text{ } \forall x \in \Gamma\}$

$f: \mathbb{R}^n \rightarrow \mathbb{R}$ radial, fast decay, $\hat{f} = FT(f)$

$$\sum_{x \in \Gamma} f(\|x\|) = \frac{1}{\text{vol}(X)} \sum_{y \in \Gamma^*} \hat{f}(\|y\|)$$

Poisson Summation Formula $X = \mathbb{R}^n / \Gamma$, $\Gamma^* = \{y \in \mathbb{R}^n : x \cdot y \in \mathbb{Z} \ \forall x \in \Gamma\}$

$f: \mathbb{R}^n \rightarrow \mathbb{R}$ radial, fast decay, $\hat{f} = FT(f)$

$$f(0) + \sum_{\gamma \in C(X)} f(\ell(\gamma)) = \frac{1}{vol(X)} \sum_{\lambda \in \sigma(X)} \hat{f}\left(\frac{\lambda}{2\pi}\right) \quad \sigma(X) = \left\{ (2\pi \|y\|)^2 : y \in \Gamma^* \right\}$$

Poisson Summation Formula $X = \mathbb{R}^n / \Gamma$, $\Gamma^* = \{y \in \mathbb{R}^n : x \cdot y \in \mathbb{Z} \text{ } \forall x \in \Gamma\}$

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$$\sum_{\gamma \in C(X)} f(l(\gamma)) = \frac{1}{vol(X)} \sum_{\lambda \in \sigma(X)} \hat{f}\left(\frac{\lambda}{2\pi}\right) - \int_0^\infty \hat{f}(r) \frac{2\pi^{n/2} r^{n-1}}{\Gamma(n/2)} dr$$

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Selberg Trace Formula $X = \text{closed hyperbolic surface}$, $f: \mathbb{R} \rightarrow \mathbb{R}$ even, fast decay

$$\frac{1}{\sqrt{2\pi}} \sum_{\gamma \in C(X)} \frac{\Lambda(\gamma)}{2\sinh(l(\gamma)/2)} f(l(\gamma)) = \sum_{\lambda \in O(X)} \hat{f}\left(\sqrt{\lambda - \frac{1}{4}}\right) - \frac{\text{vol}(X)}{2\pi} \int_0^\infty \hat{f}(r) \tanh(\pi r) r dr$$

Poisson Summation Formula $X = \mathbb{R}^n / \Gamma$, $\Gamma^* = \{y \in \mathbb{R}^n : x \cdot y \in \mathbb{Z} \text{ } \forall x \in \Gamma\}$

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Selberg Trace Formula $X = \text{closed hyperbolic surface}$, $f: \mathbb{R} \rightarrow \mathbb{R}$ even, fast decay

$$\frac{1}{\sqrt{2\pi}} \sum_{r \in C(X)} \frac{\lambda(r)}{2\sinh(l(r)/2)} f(l(r)) = \sum_{\lambda \in O(X)} \hat{f}\left(\sqrt{\lambda - \frac{1}{4}}\right) - \frac{vol(X)}{2\pi} \int_0^\infty \hat{f}(r) \tanh(\pi r) r dr$$

Ahmad Trace Formula $X = d\text{-regular graph}$, $f: \mathbb{Z} \rightarrow \mathbb{R}$ Fourier coeffs of \hat{f}

$$\sum_{\gamma \in C(X)} \frac{\lambda(\gamma)}{(d-1)^{e(\gamma)/2}} f(l(\gamma)) = \sum_{\lambda \in O(X)} \hat{f}\left(\arccos\left(\frac{d-1}{2\sqrt{d-1}}\right)\right) - vol(X) \int_0^\pi \hat{f}(\theta) \psi_d(\theta) d\theta$$

Cohn-Elkies for hyperbolic surfaces

X -closed hyp. surface of genus $g \geq 2$.

Theorem (FB-Petri). Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is an admissible function and $R > 0$ is such that

- $f(x) \leq 0$ if $x \geq R$;
- $\hat{f}(\xi) \geq 0$ for every $\xi \in \mathbb{R} \cup i[-\frac{1}{2}, \frac{1}{2}]$;
- $\hat{f}(i/2) > 2(g-1) \int_0^\infty \hat{f}(r) \tanh(\pi r) r dr$.

Then $\text{sys}(X) \leq R$ for every closed hyperbolic surface X of genus g .

Theorem (FB-Petri). Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is an admissible function and $L > 0$ is such that

- $f(x) \geq 0$ for all $x \in \mathbb{R}$;
- $\hat{f}(\sqrt{\lambda - 1/4}) \leq 0$ if $\lambda \geq L$;
- $\hat{f}(i/2) < 2(g-1) \int_0^\infty \hat{f}(r) \tanh(\pi r) r dr$.

Then $\lambda_1(X) \leq L$ for every closed hyperbolic surface X of genus g .

Theorem (FB-Petri). Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is an admissible function such that

- $\hat{f}(\xi) \geq 0$ for all $\xi \in \mathbb{R} \cup i[-\frac{1}{2}, \frac{1}{2}]$;
- $f(x) \leq 0$ if $x \geq \text{sys}(X)$;
- $f(\text{sys}(X)) < 0$.

Then

$$\text{kiss}(X) \leq \frac{2\sqrt{2\pi} \sinh(\text{sys}(X)/2)}{-\text{sys}(X)f(\text{sys}(X))} \left(2(g-1) \int_0^\infty \hat{f}(r) \tanh(\pi r) r dr - \hat{f}(i/2) \right).$$

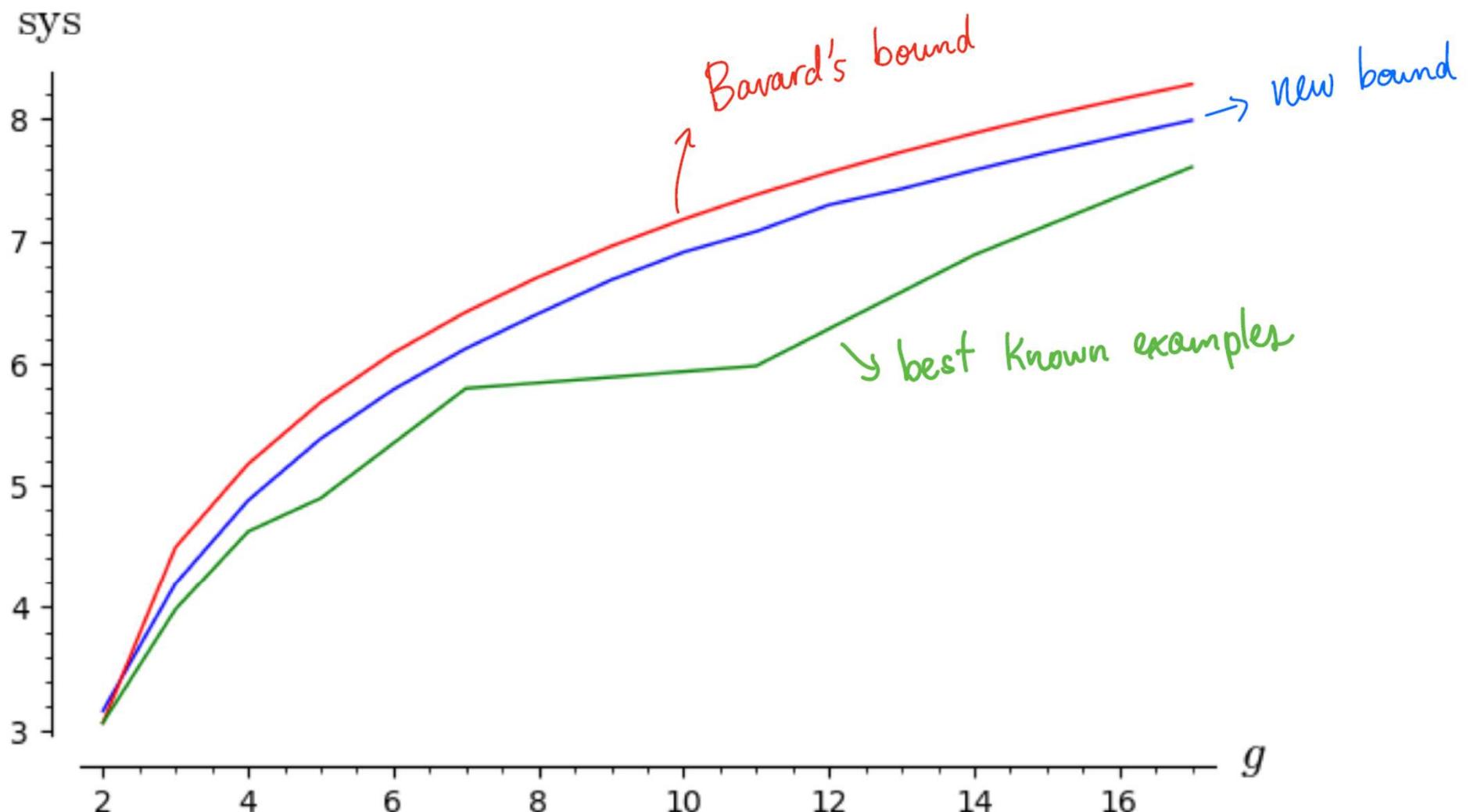
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- $\hat{f}(\sqrt{\lambda_1(X) - 1/4}) < 0$.

Then

$$m_1(X) \leq \frac{\hat{f}(i/2) - 2(g-1) \int_0^\infty \hat{f}(r) \tanh(\pi r) r dr}{-\hat{f}(\sqrt{\lambda_1(X) - 1/4})}.$$

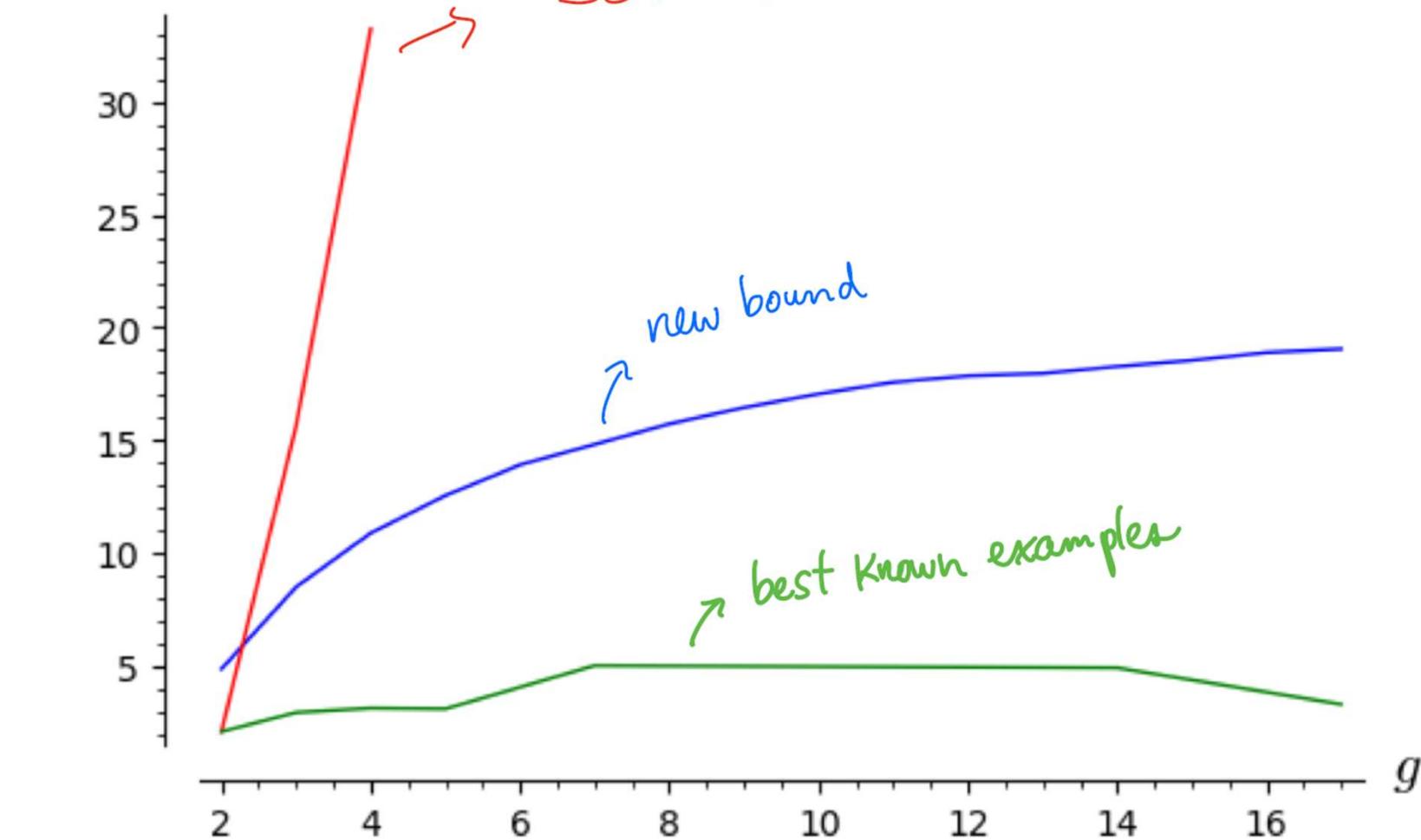
Bounds on Sys



Bounds on kiss

$$\text{kiss} \cdot \log(g)/g^2$$

Schnitz, Malestein-Rivin-Theran, Przytycki

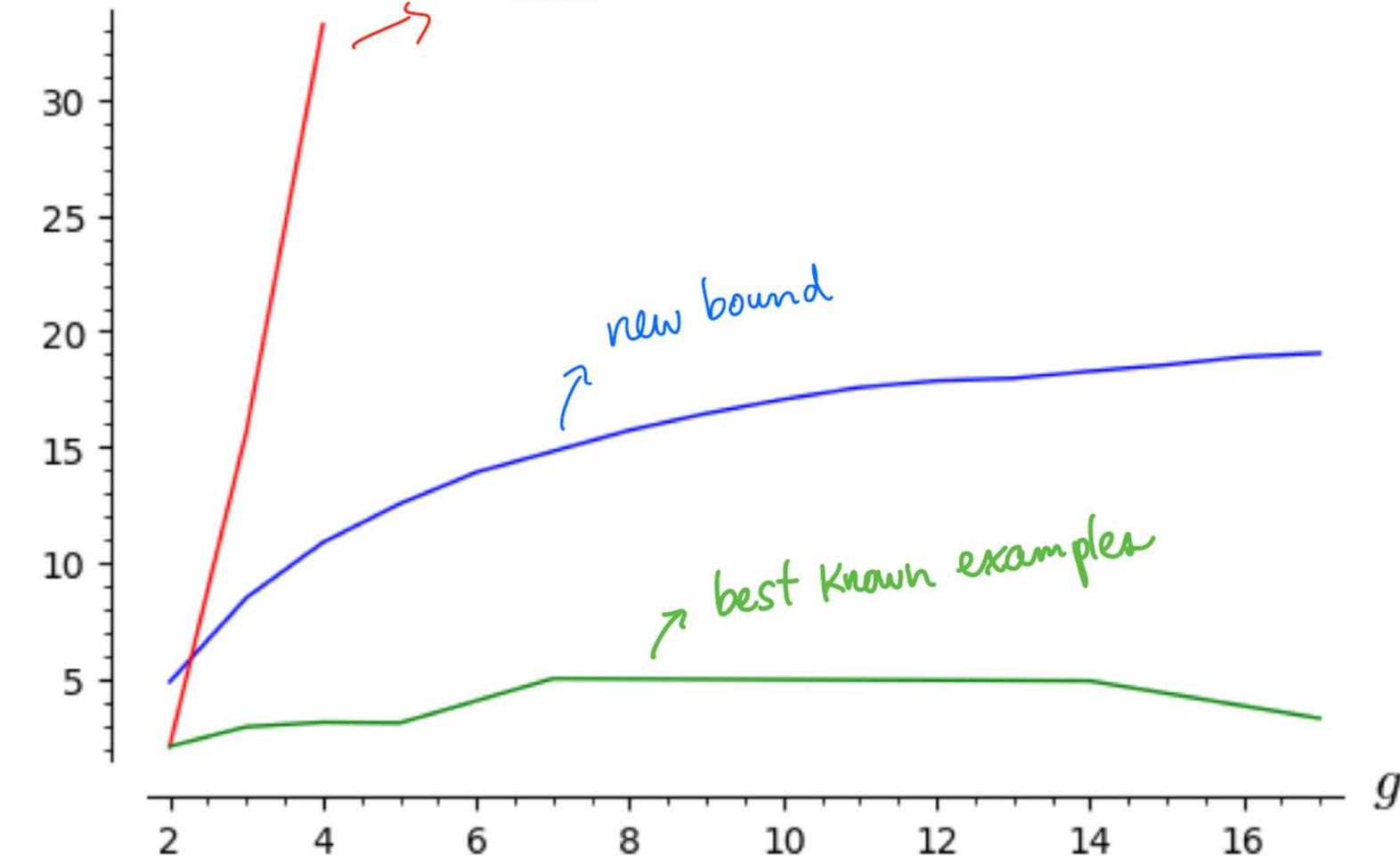


Bounds on kiss

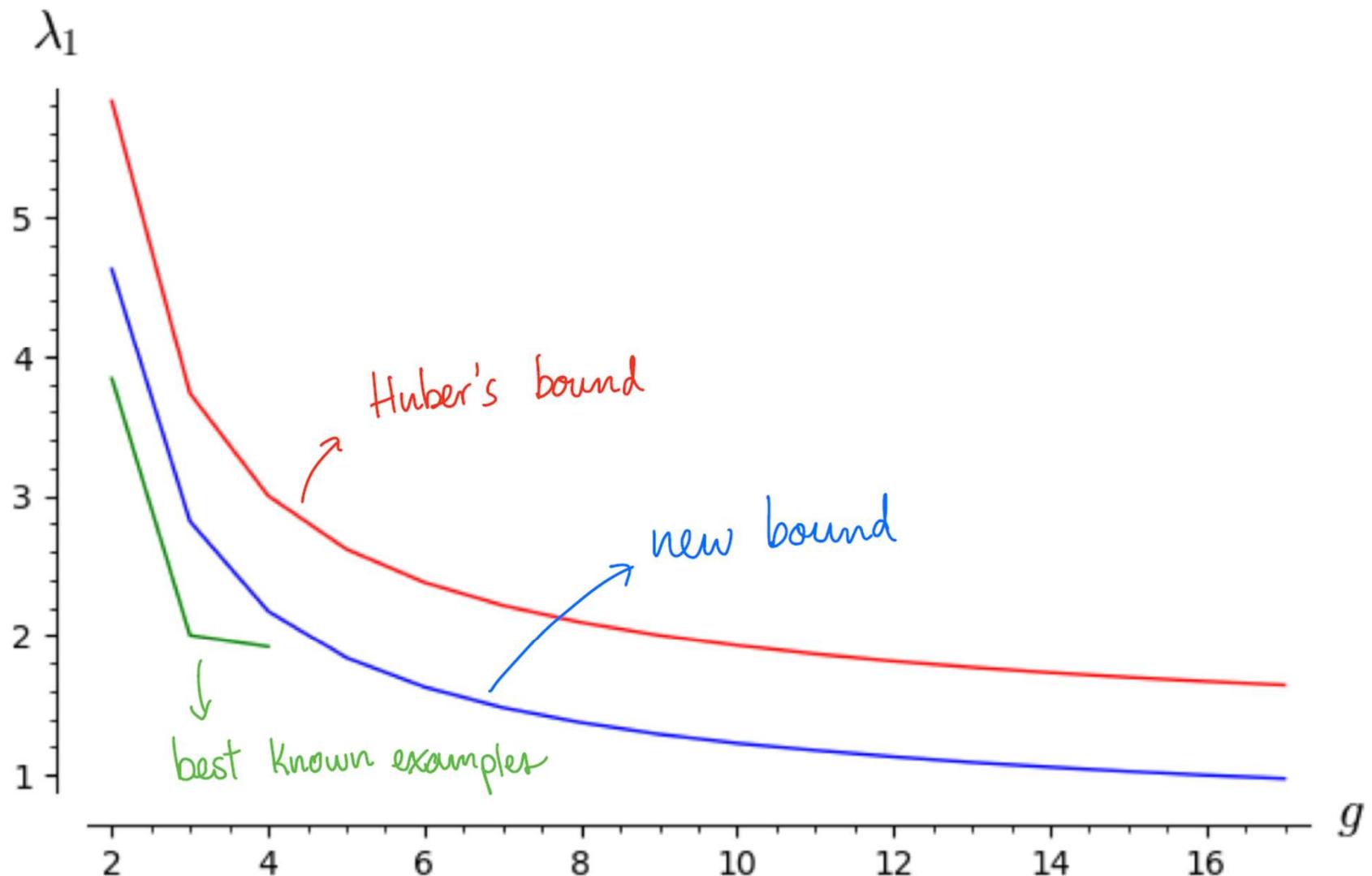
Thm (Parlier) $\exists C > 0$ st. $\text{kiss}(X) \leq \frac{Cg^2}{\log(g)}$ $\forall X \in \mathcal{M}_g$.

$\text{kiss} \cdot \log(g)/g^2$

Schnitz, Malestein-Rivin-Theran, Przytycki



Bounds on λ_1



Bounds on m_1

Problem: Our bound for m_1 depends on λ_1 , and it explodes as $\lambda_1 \rightarrow \frac{1}{4}$ or $\lambda_1 \rightarrow \infty$.

Solution: • Use upper bound on λ_1

• Use different method for $\lambda_1 \in (0, \frac{1}{4} + \varepsilon]$

Thm (Otal-Rosas) There are at most $2g-3$ eigenvalues in $(0, \frac{1}{4}]$ counting multiplicity.

Thm (FB-Petri) $\exists \varepsilon_g > 0$ s.t. $m_1(X) \leq 2g-1$ if $\lambda_1(X) \in (\frac{1}{4}, \frac{1}{4} + \varepsilon_g]$.

$g=3$

Thm (FB-Petri) $m_1(X) \leq 8$ for every closed hyp. surface X of genus 3, with equality if $X = \text{Klein quartic } Q$.

Rem It suffices to prove $m_1(X) < 9 \quad \forall X \in \mathcal{M}_3$
and $7 < m_1(Q) < 9$.

Pf uses

- Previous bounds on $(0, 1.04]$
- Upper bound $\lambda_1 \leq 2(4 - \sqrt{7}) \approx 2.708497\dots$ by Ros
- Trace formula methods on $[1.04, 2.71]$
- Representation theory of $\text{Isom}(Q)$ + trace formula
to show $m_1(Q) = 8$.

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Thank you for watching!