

Then (FB-MG-VP) In genus 2, the extremal length systole attains a strict local maximum at the Bolza surface, where it takes the value  $\sqrt{2}$ .

Systole If  $M$  is a closed Riemannian manifold,

$$\text{Sys}(M) = \inf \{ \text{length}(\gamma) \mid \gamma: S^1 \rightarrow M \text{ is not contractible} \}$$

Systolic Ratio  $\text{SR}(M) = \frac{\text{Sys}(M)^{\dim(M)}}{\text{Vol}(M)}$  ← scaling invariant

Isosystolic Problem Fix smooth manifold  $\Sigma$ .

What is  $\sup \{ \text{SR}(\Sigma, h) : h \text{ is a Riemannian metric on } \Sigma \}?$

Only known if  $\Sigma = \mathbb{T}^2, \mathbb{RP}^2, K$ .

In all 3 cases, strategy: Fix conformal class of  $h$

Solve restricted problem

Maximize over conformal classes.

$h \overset{\text{conf.}}{\sim} g$  if  $\exists f: \Sigma \rightarrow (0, \infty)$  smooth st.  $h = f \cdot g$ .

$$\sup_h \text{SR}(\Sigma, h) = \sup_{\substack{c \\ \text{comp}}} \sup_{h \in c} \text{SR}(\Sigma, h) = \sup_c \sup_{h \in c} \frac{L(\Gamma_{\text{all}}, h)^2}{\text{area}(h)}$$

where  $\Gamma_{\text{all}} = \{ \text{non-contractible curves in } \Sigma \}$   $\text{EL}(\Gamma_{\text{all}}, c)$

$$L(\Gamma, h) = \inf \{ l(\gamma, h) \mid \gamma \in \Gamma \}$$

### Extremal length

Let  $c = [g]$  be a class of metrics on  $\Sigma$ .

$\Gamma$  any set of curves in  $\Sigma$ .

Then  $\text{EL}(\Gamma, c) = \sup_{h \in c} \frac{L(\Gamma, h)^2}{\text{area}(h)} \left( = \sup_{h \in \bar{c}} \frac{L(\Gamma, h)^2}{\text{area}(h)} \right)$

$$\bar{c} = \left\{ f \cdot g \mid f: \Sigma \rightarrow [0, \infty] \text{ Borel-measurable} \right. \\ \left. 0 < \text{area}(f \cdot g) < \infty \right\}$$

Def<sup>n</sup> Let  $h \in \bar{c}$ .  $(\Sigma, h)$  is plump for  $\Gamma$  if

$\exists \Gamma_0 \subseteq \Gamma$  nonempty s.t.  $l(\gamma, h) = L(\gamma, h)$  for all  $\gamma \in \Gamma_0$

and  $\# \varphi: \Sigma \rightarrow \mathbb{R}$  Borel-meas.

$$\int_{\gamma} \varphi \, ds_h \geq 0 \quad \forall \gamma \in \Gamma_0 \Rightarrow \int_{\Sigma} \varphi \, dA_h \geq 0.$$

Berling's Criterion If  $h \in \bar{c}$  is plump for  $\Gamma$ , then  $h$  is extremal

i.e.,  $\text{EL}(\Gamma, c) = \frac{L(\Gamma, h)^2}{\text{area}(h)}$ .

### Example



Eucl. cyl. is plump for  $\Gamma_{\text{all}}$  or  $\Gamma = \{ \text{curves that wrap once} \}$

$$\text{EL} = \frac{\text{Circumference}^2}{\text{Area}}$$

= Circumference.

①  curv. cys. is plump for  $\Gamma_{all}$  or  $\Gamma' = \{$  curves that wrap once  $\}$

$$EL = \frac{\text{circumference}^2}{\text{area}} = \frac{\text{circumference}}{\text{height}}$$

② Any flat torus is plump for  $\Gamma_{all}$ .

$$\Rightarrow EL(\Gamma_{all}) = SR(\text{flat metric in } c) \\ \hookrightarrow \text{maximized for the hex torus}$$



③  $k \in (0, 1)$ ,  $\Sigma = \hat{\mathbb{C}} \setminus \{\pm 1, \pm i_k\}$ ,  $c = [\text{Eucl metric in } \mathbb{C}]$ .

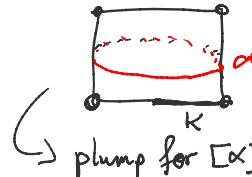
$$\Gamma = [\alpha]$$



$EL([\alpha]) = \frac{4K(k)}{K'(k)}$  where  $K$  is the complete elliptic integral of the first kind.

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad K'(k) = K(k') \text{ where } k' = \sqrt{1-k^2}.$$

Pf The map  $\zeta \mapsto \begin{cases} \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}} \end{cases}$  sends  $\Sigma$  to a pillowcase



$$EL([\alpha]) = \frac{\text{circumference}}{\text{height}} = \frac{4K(k)}{K'(k)} \hookrightarrow \text{plump for } [\alpha] \\ \text{by a change of variable}$$

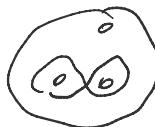
□

Extremal length systole  $(\Sigma, c)$   $\subset$  conf. class.

$$\text{sys}_{EL}(\Sigma, c) = \inf \left\{ EL([\gamma], c) : \gamma \subseteq \Sigma \text{ essential} \right\} \\ (\text{not contractible to a pt or a puncture})$$

$$\geq \sup_{h \in c} SR(\Sigma, h)$$

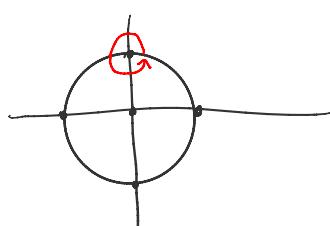
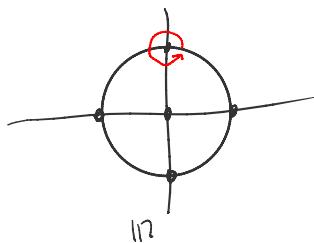
Lemma  $\inf$  is realized by simple closed curves unless  $\Sigma \cong \mathbb{C} \setminus \{0, 1\}$ .



Bolza surface

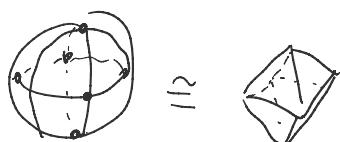
## Bolza surface

$$\mathcal{B} = \{(x, y) \in \mathbb{C}^2 : y^2 = x(x^4 - 1)\}.$$

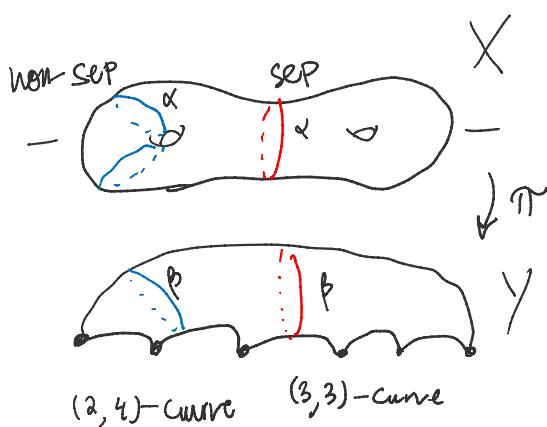


hyperelliptic involution

$$(x, y) \mapsto (x, -y)$$



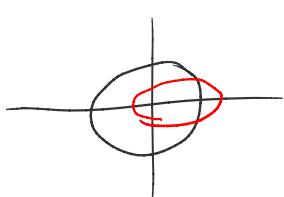
$$\mathcal{O} = \mathbb{C} \setminus \{0, \pm 1, \pm i\}$$



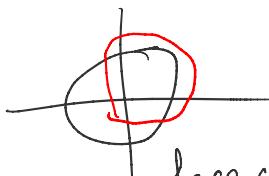
Prop

$$\text{nonsep } EL(\alpha, X) = \frac{1}{2} EL(\beta, Y)$$

$$\text{sep } EL(\alpha, X) = 2 EL(\beta, Y)$$



edge curve



face curve

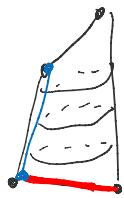
$$EL(\text{edge curve}) = 2\sqrt{2} \approx 2.818\dots$$

$$EL(\text{face curve}) = 6 K(v)/K'(v) \text{ where } v = \frac{1}{\sqrt{27 + 15\sqrt{3}}}$$

$$\approx 2.799\dots$$

Then  $Sys_{EL}(\mathcal{O}) = EL(\text{face curve})$  and the 2nd shortest curves are the edge curves.

Cor  $\text{sys}_{\text{EL}}(\mathcal{B}) = \text{EL}(\text{lifts of edge curves}) = \sqrt{2}$



$\text{sys}_{\text{EL}}(\mathcal{C} \setminus \{0, 1\})$  attained  
by

