

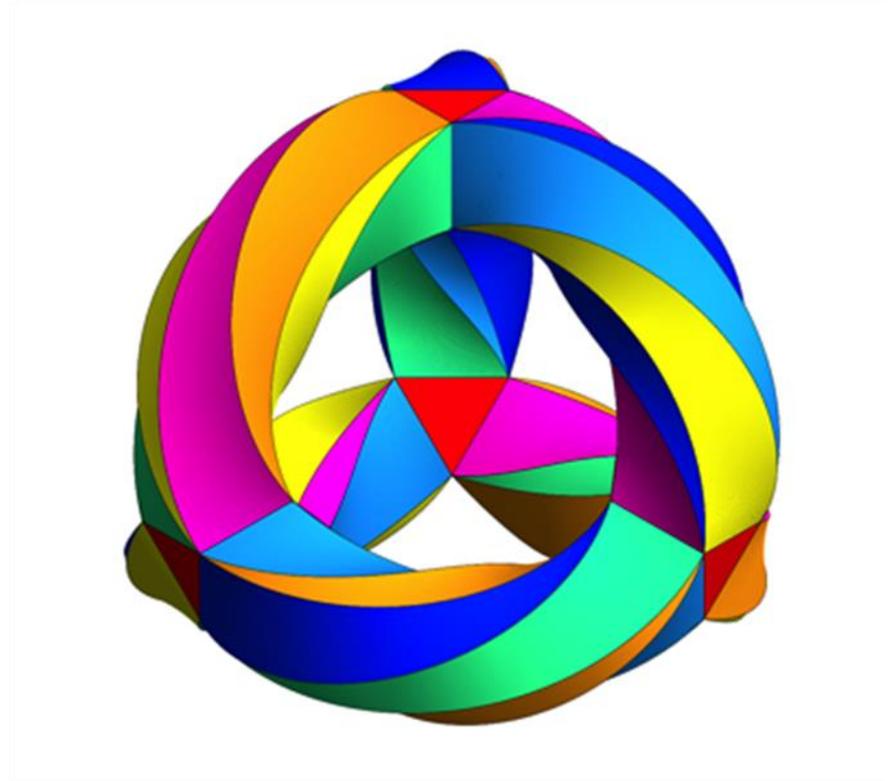
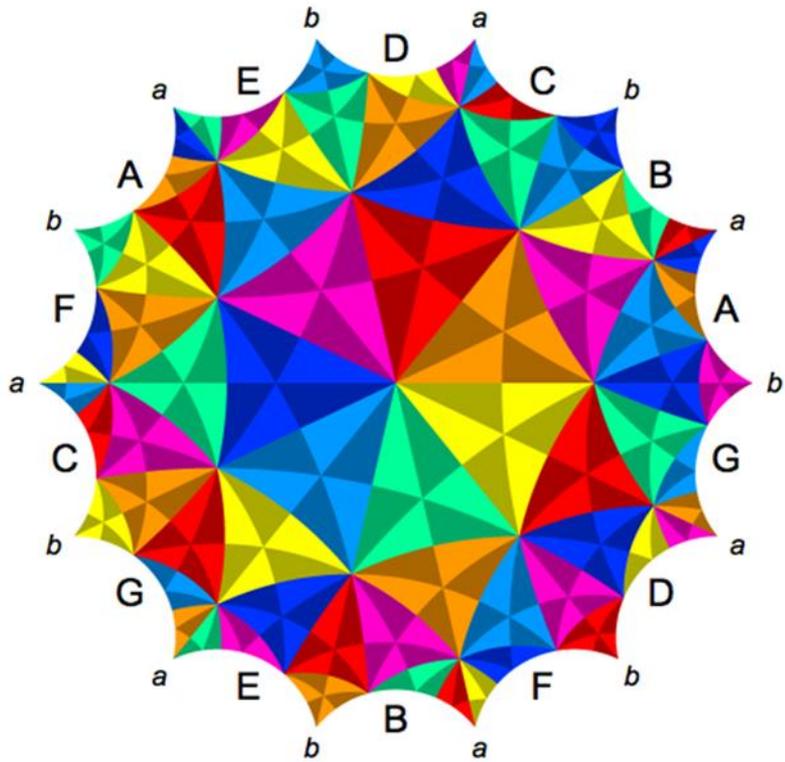
The multiplicity of λ_1
in genus 3

Maxime Fortier Bourque
Université de Montréal

joint work with Bram Petri

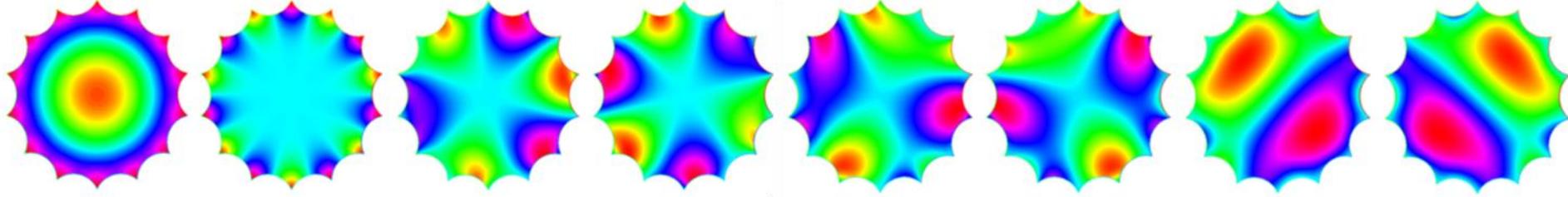
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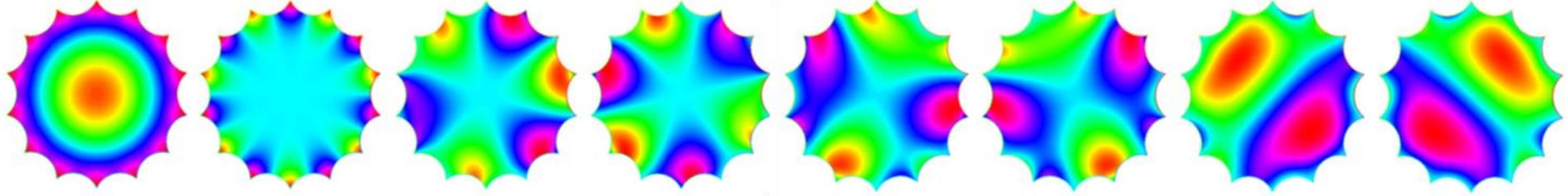


It's the smallest Hurwitz surface, with $\#\text{Isom}(K) = 84(g(K) - 1)$

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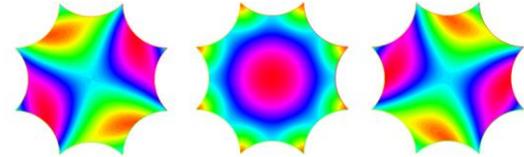


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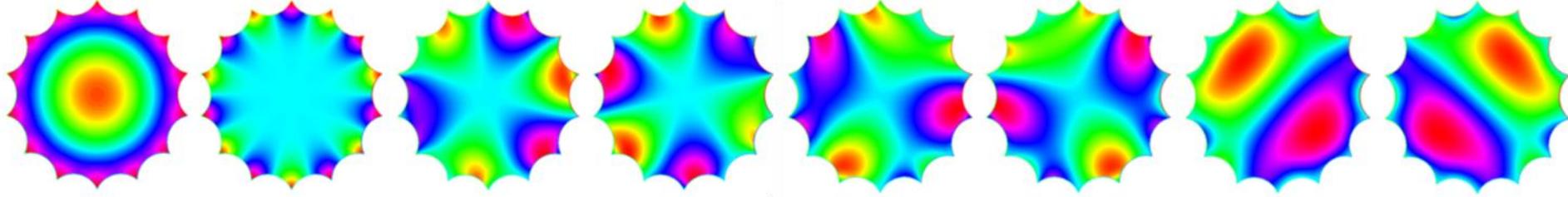


Remarks

- ① Analogous result not known in genus 2.
Conjectured maximizer is the Bolza surface with $m_1=3$.



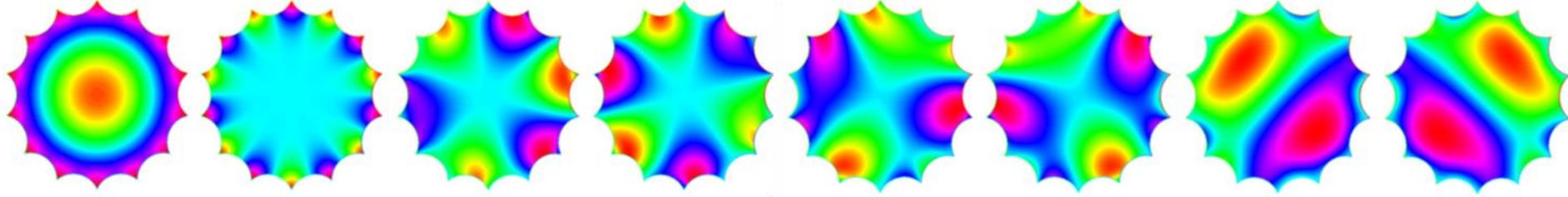
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$$m_1(S) \leq ch(S) - 1 = \left\lfloor \frac{1}{2} (5 + \sqrt{48g + 1}) \right\rfloor = 8 \text{ when } g=3$$
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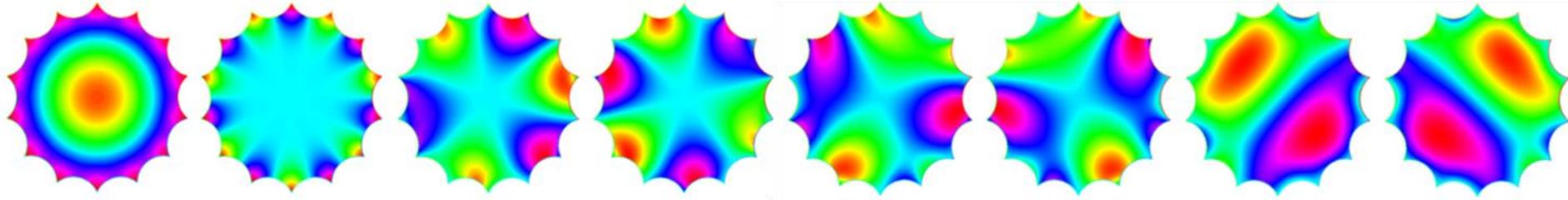
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$$\text{with } m_1(S_g) = \left\lfloor \frac{1}{2} (1 + \sqrt{8g + 1}) \right\rfloor = 3 \text{ when } g=3$$

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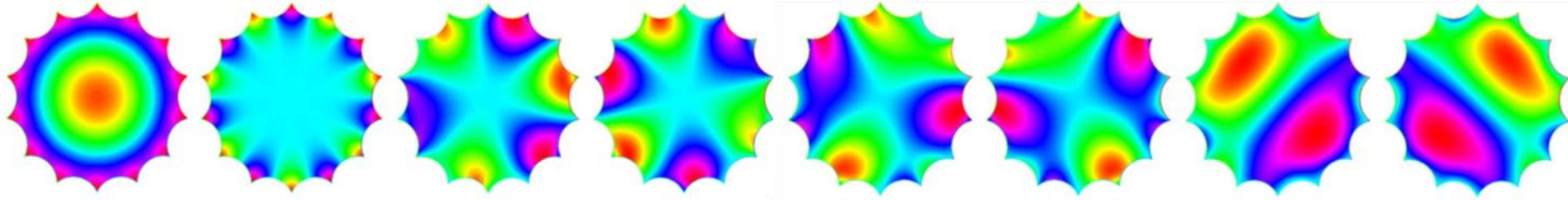
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⑤ Cook had shown $6 \leq m_1(K) \leq 8$ + numerical evidence for $m_1(K) = 8$.

Spectra $S =$ closed hyperbolic surface

$\mathcal{E}(S) =$ eigenspectrum of Δ_S

$$= \{ \lambda_j(S) : j \geq 0 \}$$

where $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$

are repeated according to multiplicity

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λ_1



Systole

m_1



Kissing number

The kissing number bound

Thm (Parlier, FB-Petri) \exists constants $A, B > 0$ s.t. $\forall g \geq 2$ and every closed hyperbolic surface S of genus g ,

$$\text{kiss}(S) \leq A \cdot \frac{e^{\text{sys}(S)/2}}{\text{sys}(S)} \cdot g \leq B \cdot \frac{g^2}{\log(g)}$$

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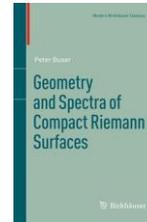
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Huber



Buser



Parlier



Petri



Colbois



The Selberg Trace Formula

S - closed hyperbolic surface of genus g

λ_j - eigenvalues of the Laplacian on S

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Fourier transform If $f: \mathbb{R} \rightarrow \mathbb{C}$ is integrable, then $\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\xi x} dx$.

Defⁿ $f: \mathbb{R} \rightarrow \mathbb{C}$ is **admissible** if

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- integrable
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Thm (Selberg) $\forall S$ hyperbolic of genus g , $\forall f$ admissible,

$$\sum_{j=0}^{\infty} \hat{f}(\sqrt{\lambda_j - \frac{1}{4}}) = 2(g-1) \int_0^{\infty} \hat{f}(r) \tanh(\pi r) r dr + \frac{1}{\sqrt{2\pi}} \sum_{\gamma \in \mathcal{P}(S)} l(\gamma) \sum_{k=1}^{\infty} \frac{f(kl(\gamma))}{2 \sinh(kl(\gamma)/2)}$$

$\mathcal{S} = \mathcal{I} + \mathcal{G}$

Bounds from the STF

$$(0 < a \leq b)$$

Lemma Suppose $\lambda, (s) \in [a, b]$, f admissible st. $\hat{f}(\sqrt{\lambda - 1/4}) \geq c > 0 \quad \forall \lambda \in [a, b]$,

$\hat{f}(\sqrt{\lambda - 1/4}) \geq 0 \quad \forall \lambda \geq b$, and $f(x) \leq 0 \quad \forall x \in \mathbb{R}$. Then

$$m_1(s) \leq \frac{1}{c} \left(2(q-1) \int_0^{\infty} \hat{f}(r) \tanh(\pi r) r dr - \hat{f}(i/2) \right).$$

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How to find good functions? Take $f(x) = p(x^2) e^{-x^2/2}$ where p is a polynomial.
Then $\hat{f}(\xi) = q(\xi^2) e^{-\xi^2/2}$ for some polynomial q .

The map $p \mapsto q$ is linear and easy to compute.

To prevent sign changes, impose some double zeros \leadsto linear system of eqn's

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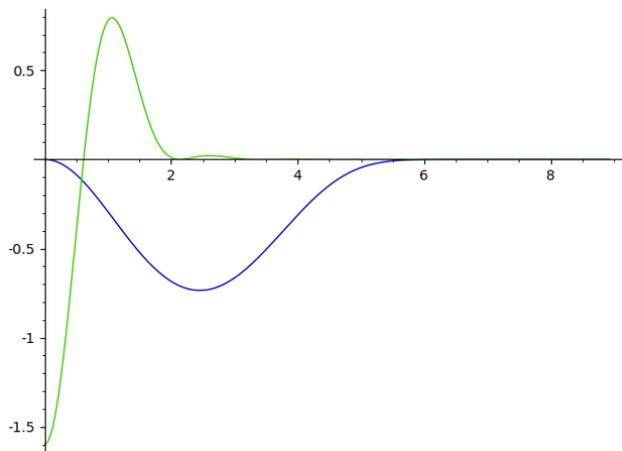
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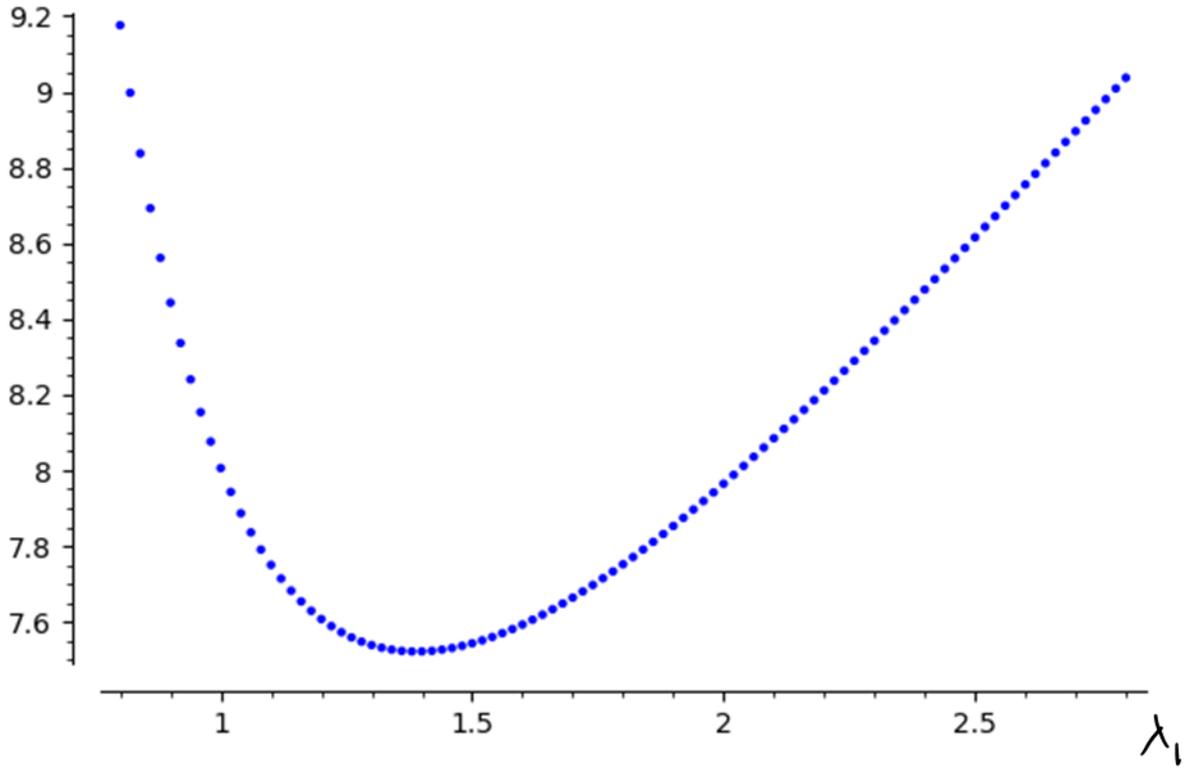
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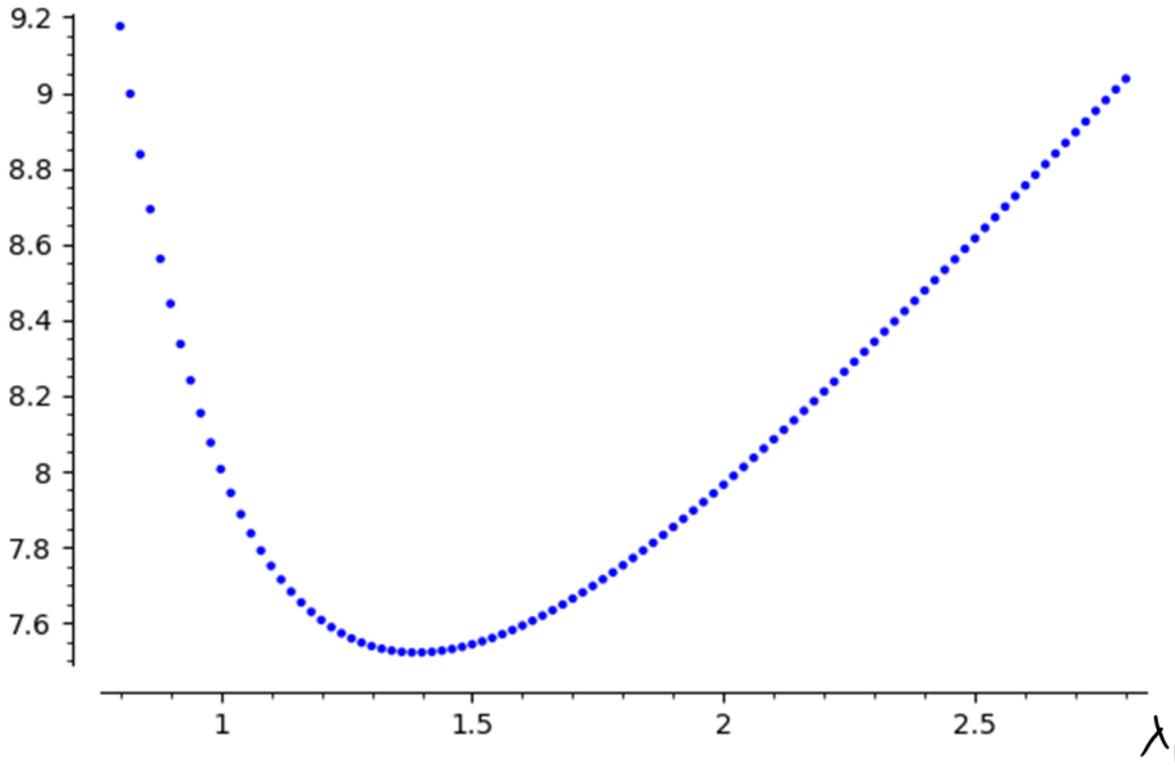
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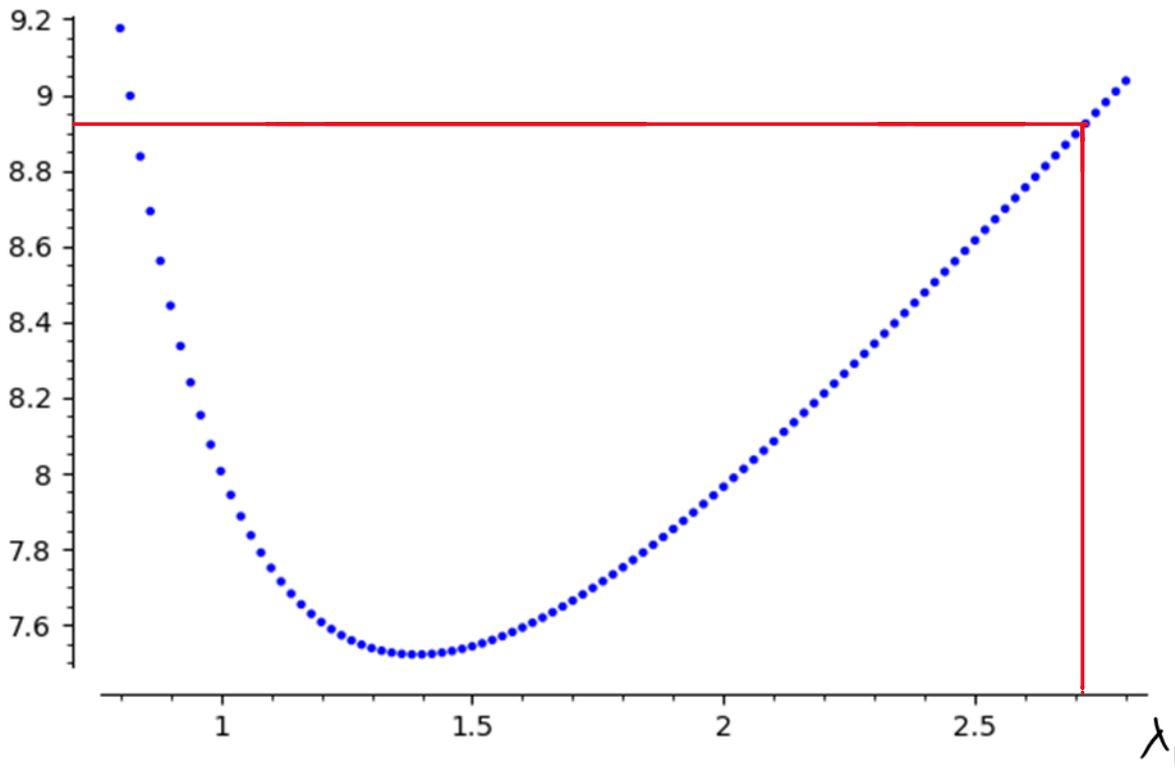
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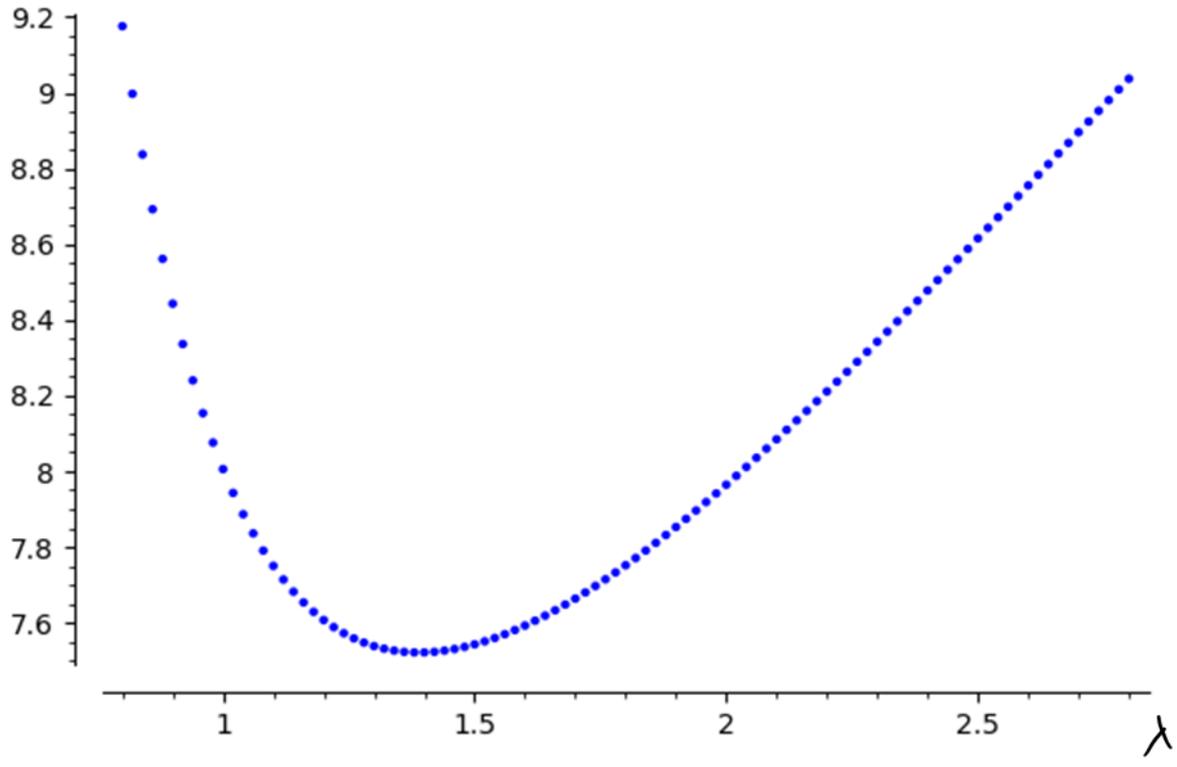
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or small \rightarrow Have to use a different Strategy!

The problem with $\frac{1}{4}$

Thm (Buser) $\forall \varepsilon > 0, \forall N \geq 1$, if S has a short enough geodesic, then

$$\# \mathcal{E}(S) \cap \left(\frac{1}{4}, \frac{1}{4} + \varepsilon\right] \geq N.$$

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How to deal with λ_1 between $\frac{1}{4}$ and $\frac{1}{4} + \varepsilon$?

A: Use Sévenec's proof again, but with a weaker improvement than Otal.

Sévenec's proof

Let $E_1 = \{\text{eigenfunctions for } \lambda_1\}$

$\forall f \in E_1$, $u^+(f) = \{f > 0\}$ and $u^-(f) = \{f < 0\}$ are connected
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$S(E_1) \longrightarrow$ ordered pairs of disjoint domains in S

\downarrow 2-1 covering

$P(E_1) \longrightarrow$ unordered pairs of disjoint domains in S

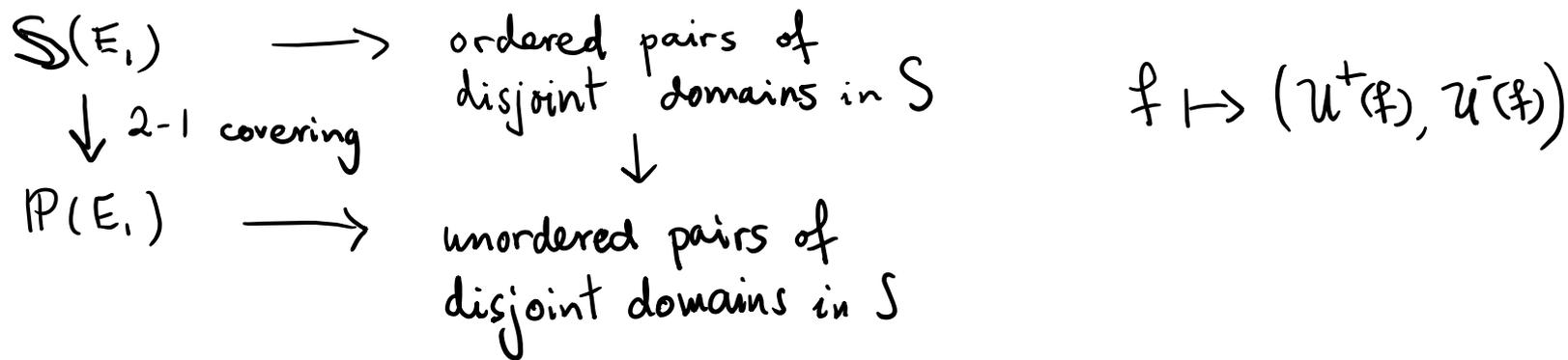
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\hookrightarrow Partition $S(E_1)$ and $P(E_1)$ according to $b_1(U^+) + b_1(U^-) = i$

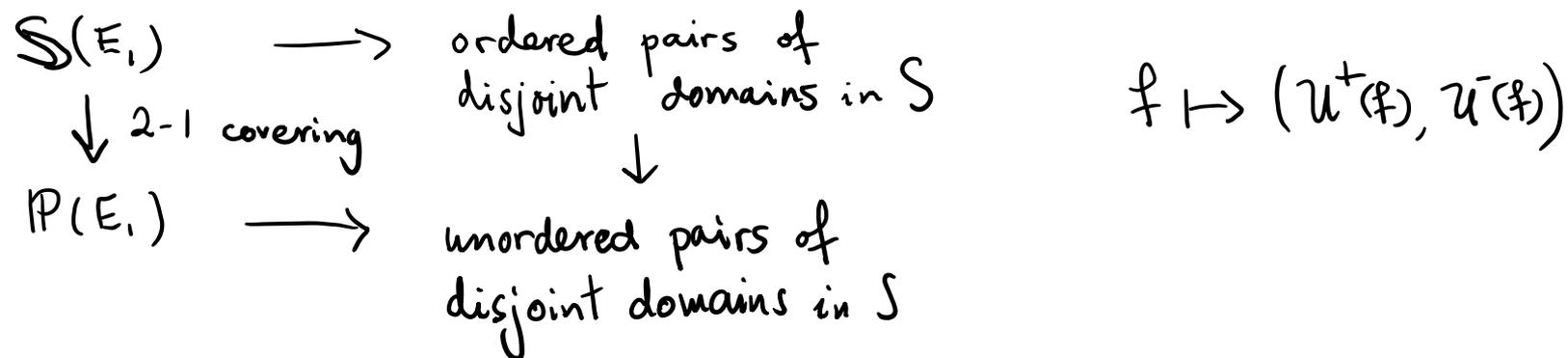
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Prop (Sévenec) $\forall i \geq 1$, the covering $\tilde{X}_i \rightarrow X_i$ is trivial, which contributes 1 to $\dim(E_1)$

The improvement

The problematic case in Sévenec's proof is $i=0$, which he proves contributes at most 3 to $\dim(E_i)$.

$i=0 \Rightarrow U^\pm$ are both contractible

$\Rightarrow U^\pm$ lift to H^2

At least one of them has area $\leq \frac{1}{2} \text{area}(S) = 2\pi(g-1)$

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Faber-Krahn
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Round disk minimizes λ_0 for a given area.

$$\Rightarrow \lambda_1(S) = \lambda_0(U^\pm) \geq \lambda_0(\text{disk of area } 2\pi(g-1))$$

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Antamoshin if $2 \leq g \leq 103$
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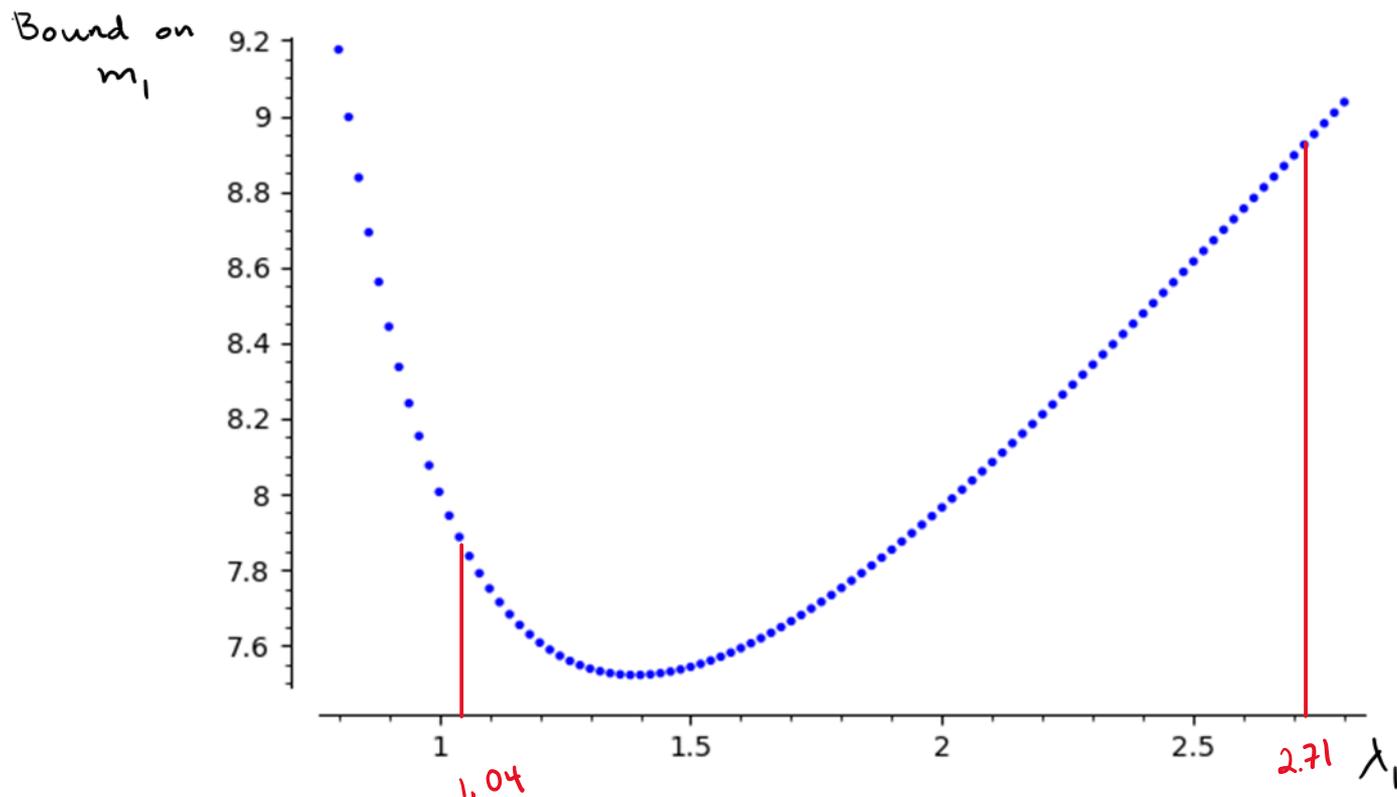
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For $g=3$, this gives $\lambda_1(S) \geq 1.044071\dots$

\Rightarrow If $\lambda_1(S) < 1.044071$, then $i=0$ cannot occur, so $m_1(S) \leq 2g = 6$.

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The Klein quartic | It remains to show that $m_1(K) = 8$.

Cook proved that $6 \leq m_1(K) \leq 8$ using representation theory.

(The idea goes back to Jenni for the Bolza surface)

$$\mathrm{PSL}(2, 7) \cong \mathrm{Isom}(K) \leadsto \text{Eigenspaces of } \Delta_K$$

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If we rule out 1-dim reps in E_1 , then $\dim E_1 \geq 6$.

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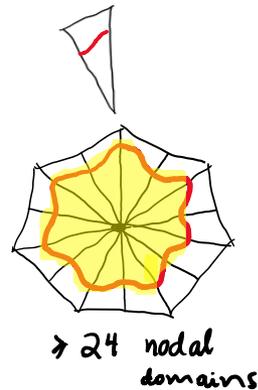
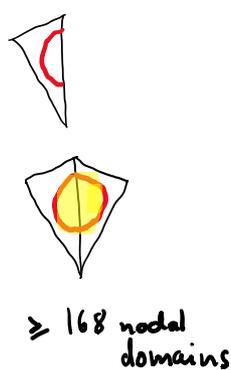
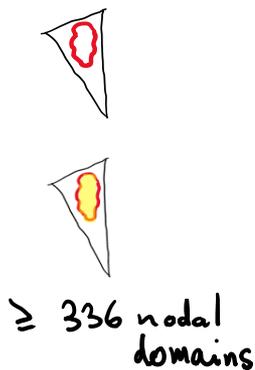
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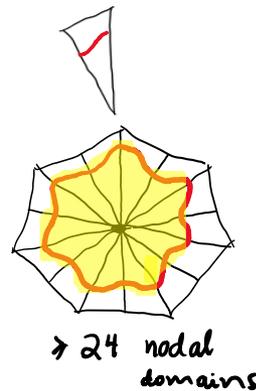
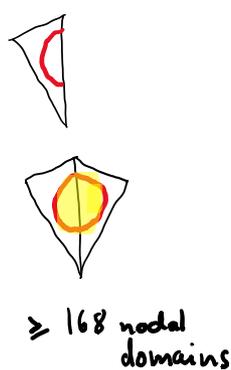
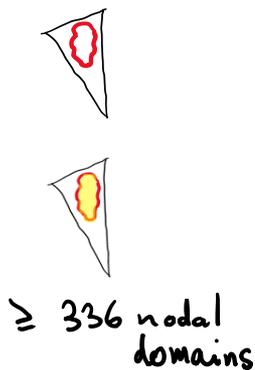
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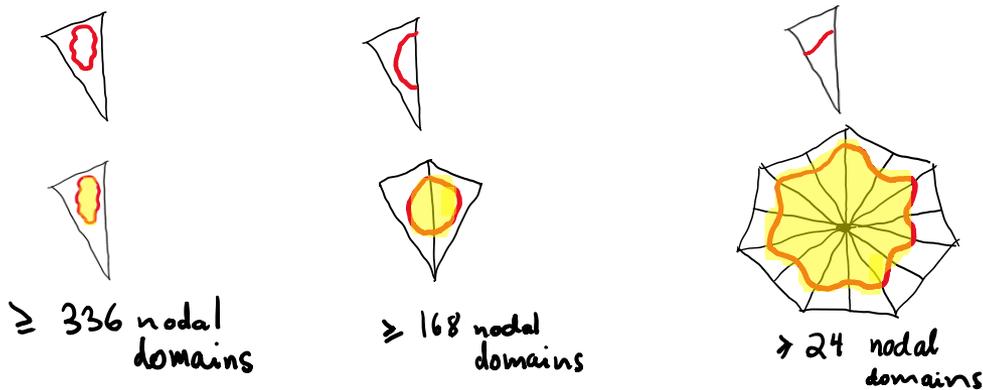
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We can also use Faber-Krahn + Artamoshin to show $\lambda > 7.85$. □

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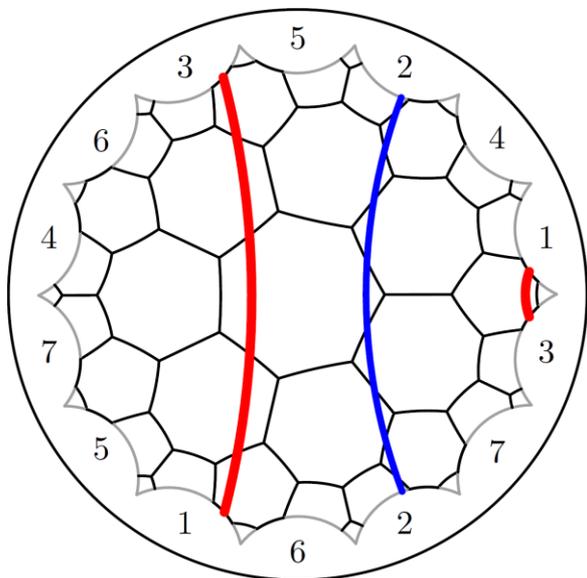
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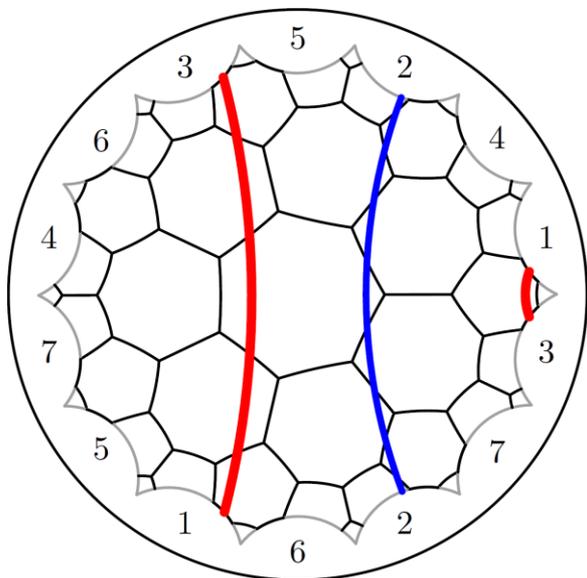
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The End