

Linear programming bounds for hyperbolic surfaces

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February 20, 2023

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joint with
Bram Petri

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Hyperbolic surfaces

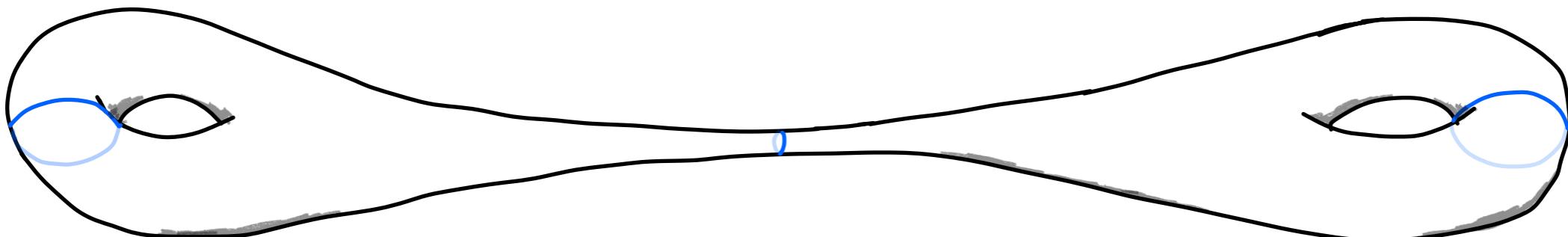
$g \geq 2,$

Moduli space

$\mathcal{M}_g = \left\{ \begin{array}{l} \text{closed, oriented, connected, surfaces of} \\ \text{genus } g \text{ with constant curvature } -1 \end{array} \right\}$

positive
isometries

↪ non-compact $(6g - 6)$ -dimensional orbifold



Hyperbolic surfaces

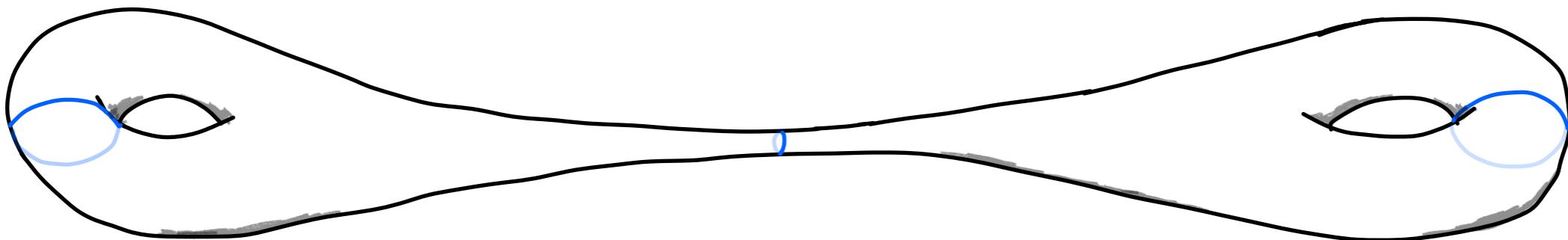
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$$X \in \mathcal{M}_g \Rightarrow \text{area}(X) = 4\pi(g-1)$$

Invariants

Classical mechanics	Quantum mechanics
closed geodesic γ	eigenfunction
length $\ell(\gamma)$	eigenvalue λ

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length spectrum ↵

↪ eigenspectrum

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	first entry > 0

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↪ eigenspectrum

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Counting function	

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Prime geodesic theorem	Weyl's law

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	N_{small}

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Defⁿ An eigenvalue λ is small if $\lambda \in [0, \lambda_4]$. Note: $\sigma(\Delta_{H^2}) = [\lambda_4, \infty)$

$N_{\text{small}}(X) = \# \text{ eigenvalues of } \Delta_X \text{ in } [0, \lambda_4]$
counting multiplicity

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# primitive $\gamma : \ell(\gamma) \leq 2\operatorname{arcsinh}(1)$	N_{small}

length spectrum ↵

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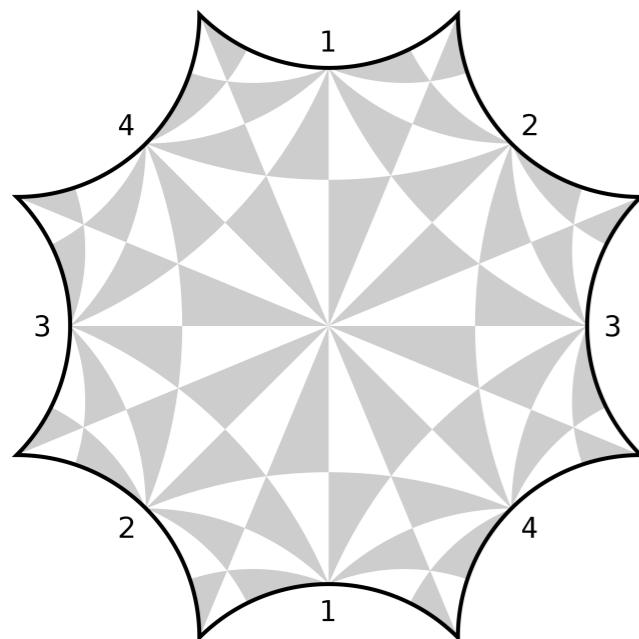
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Known maximizers

Jenni '84

sys



$$g=2$$

Bolza
Surface

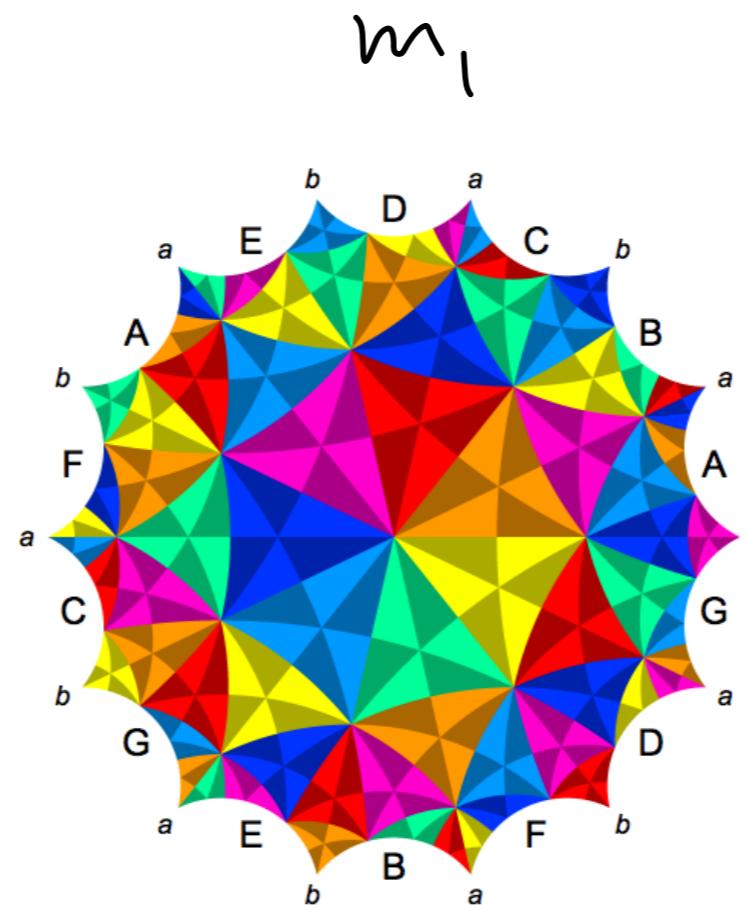
Kiss

Schmutz '94

FB - Petri '21

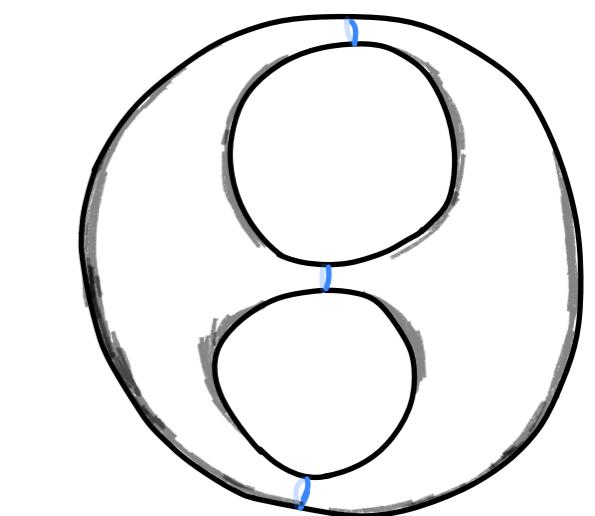
Otal - Rosas '04 (+ Buser '77)

$$\max N_{\text{small}} = 2g - 2$$



$$g = 3$$

Klein
quartic



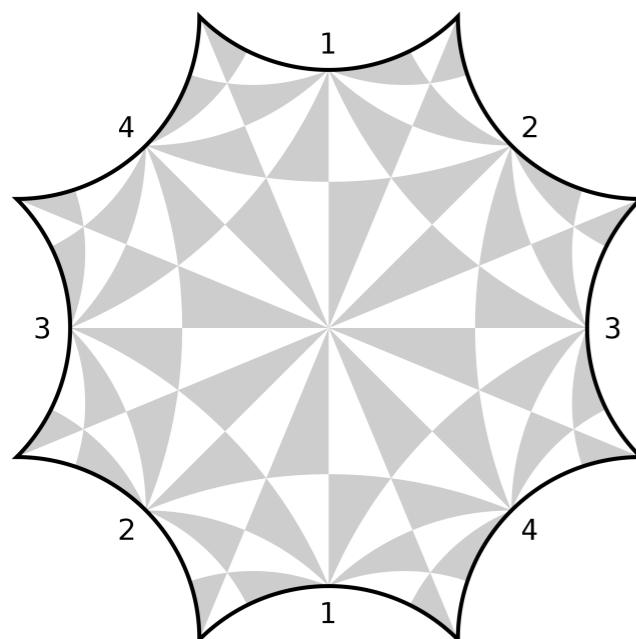
every $g \geq 2$

any surface with
a short pants
decomposition

Known maximizers

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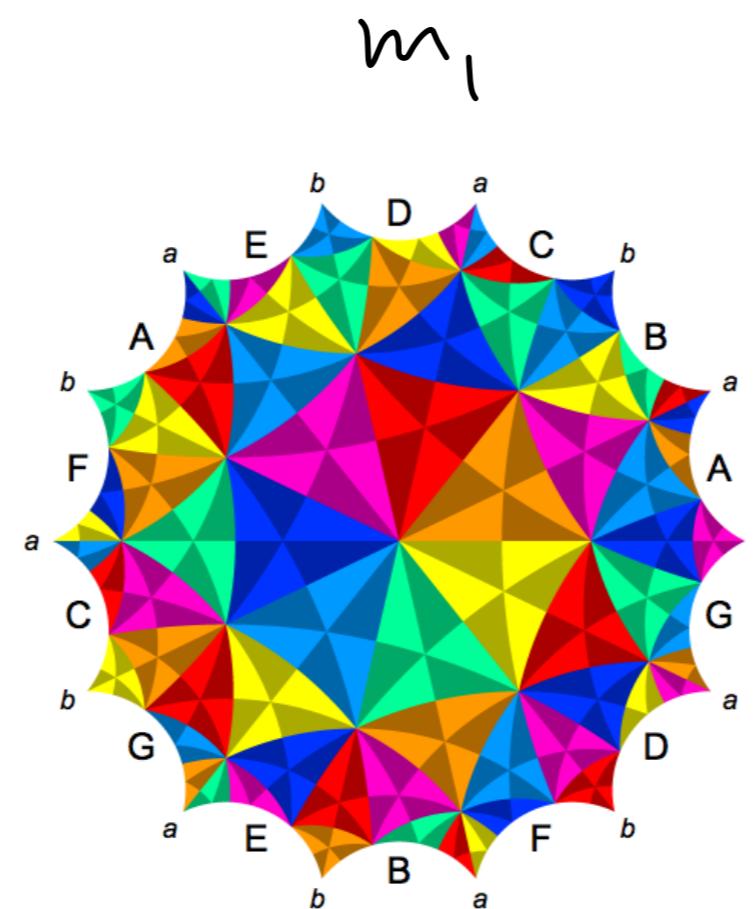


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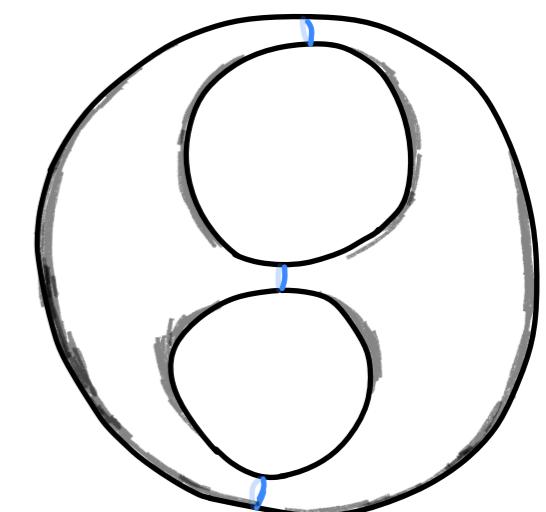
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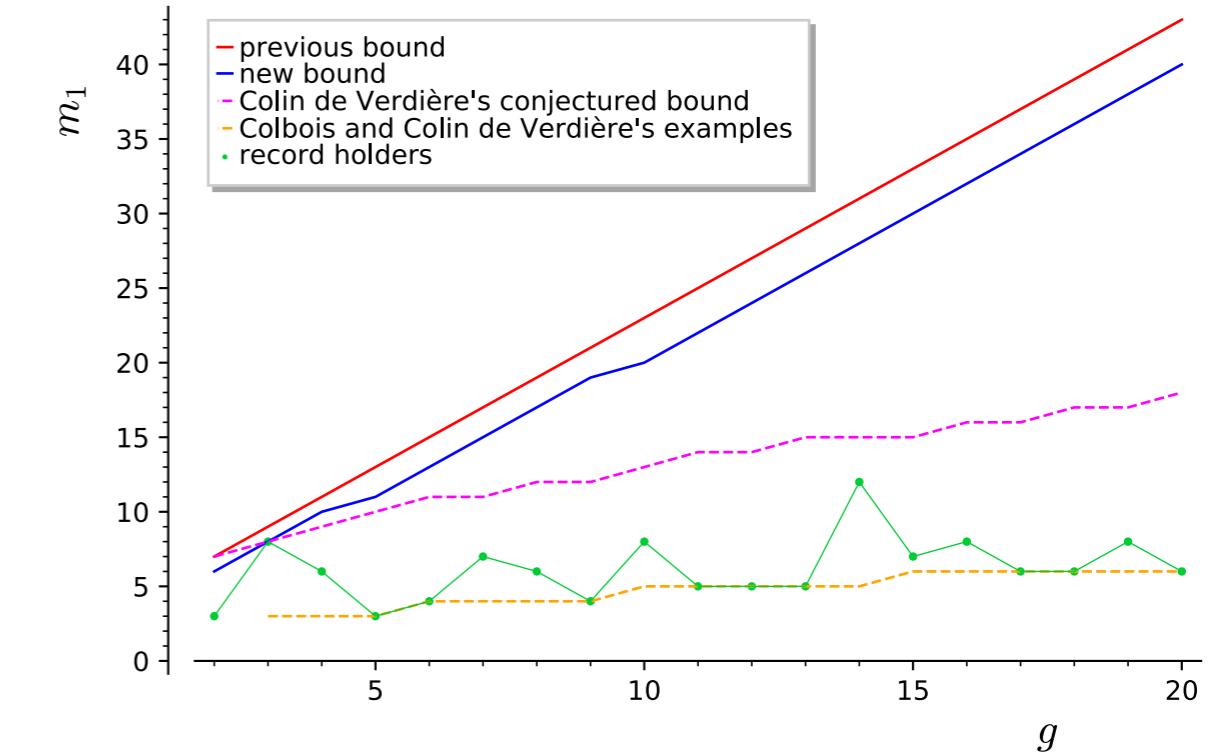
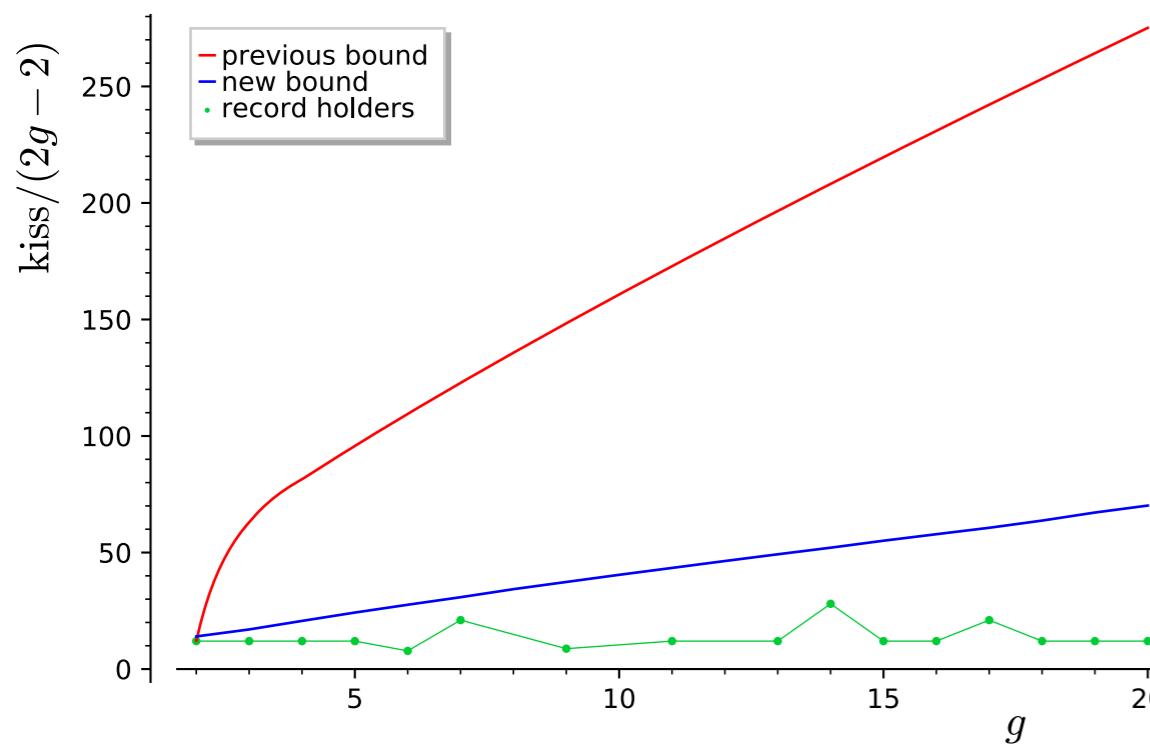
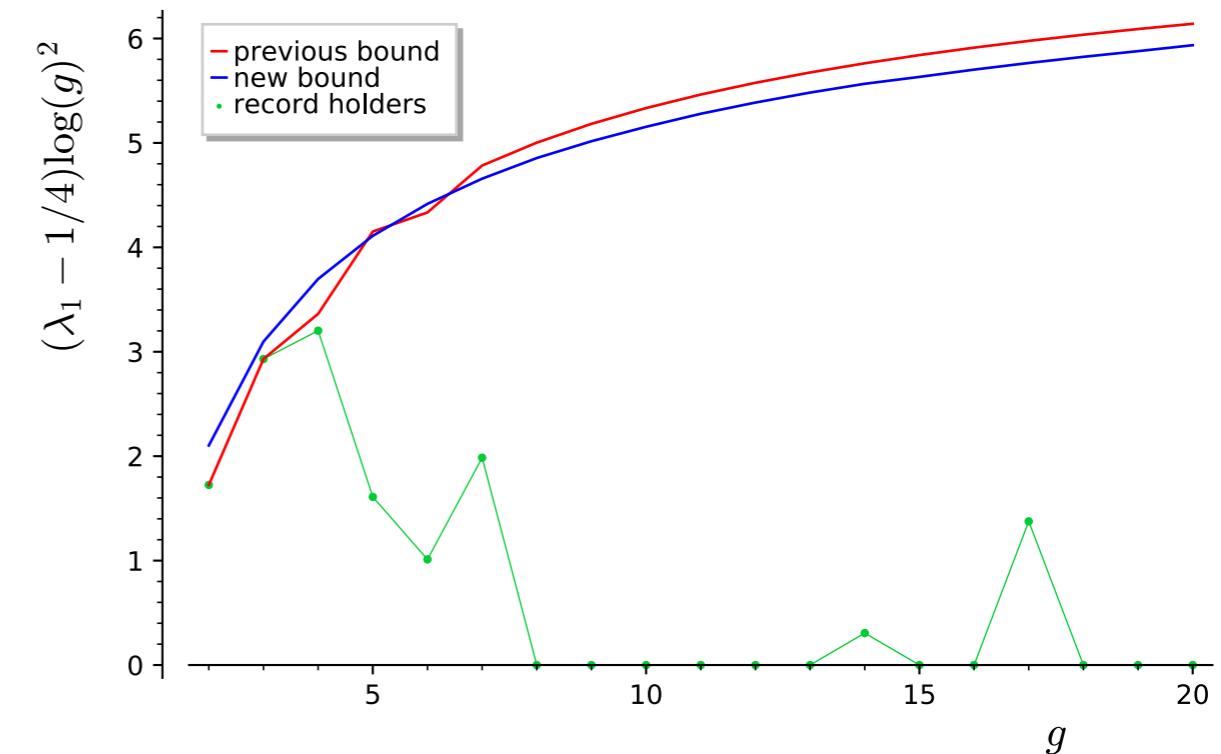
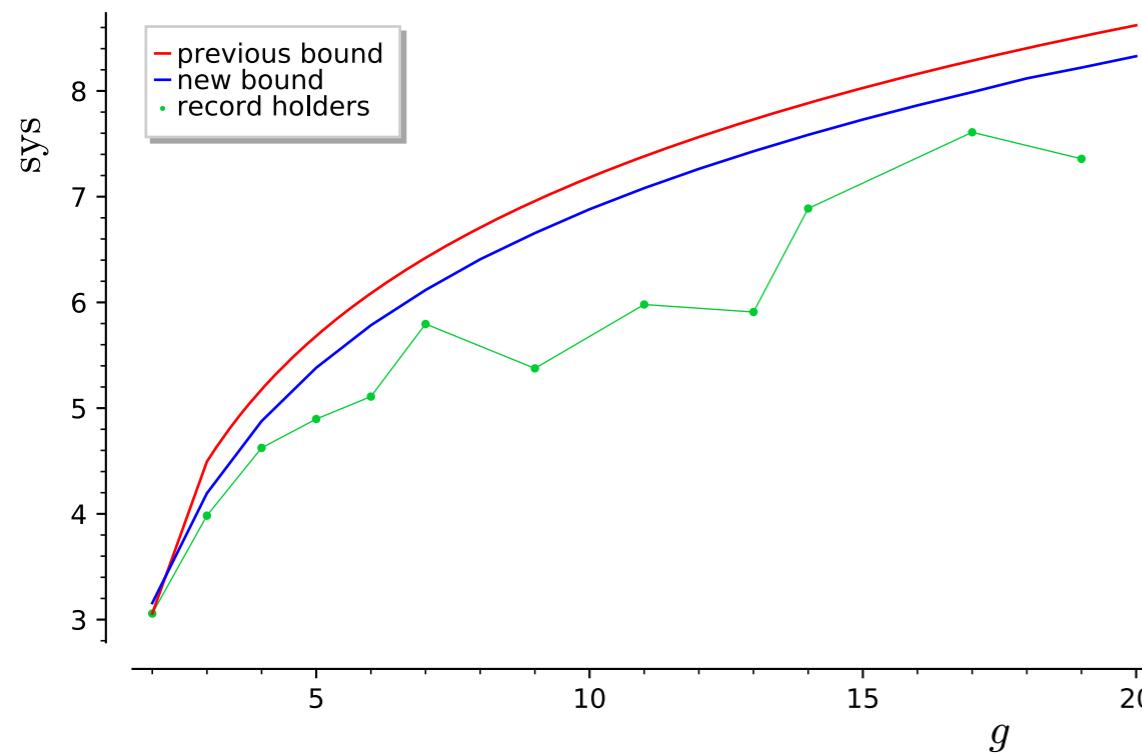
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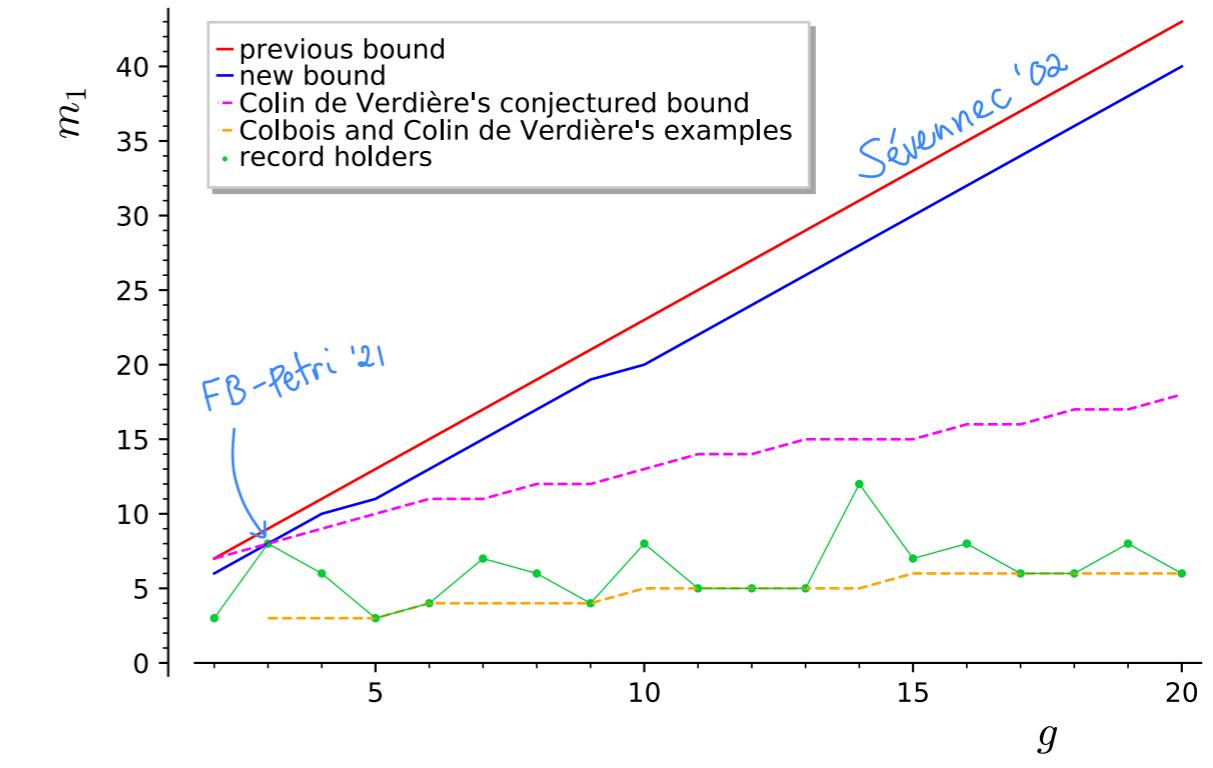
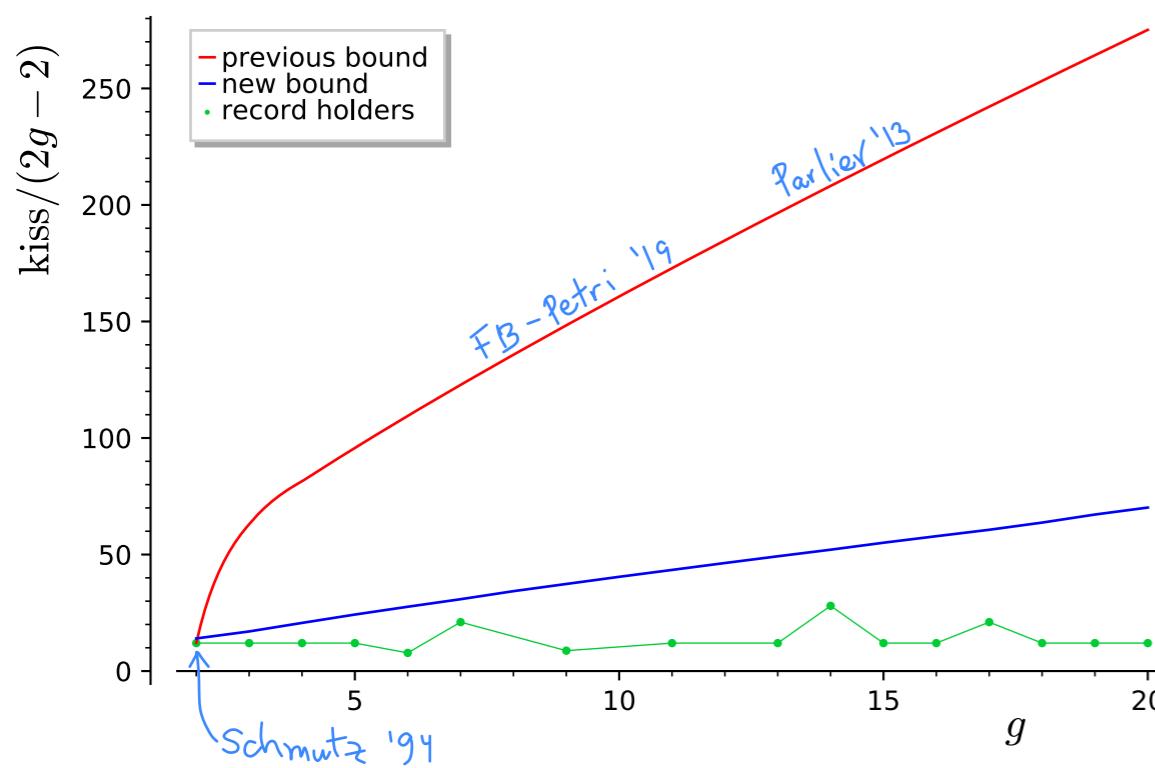
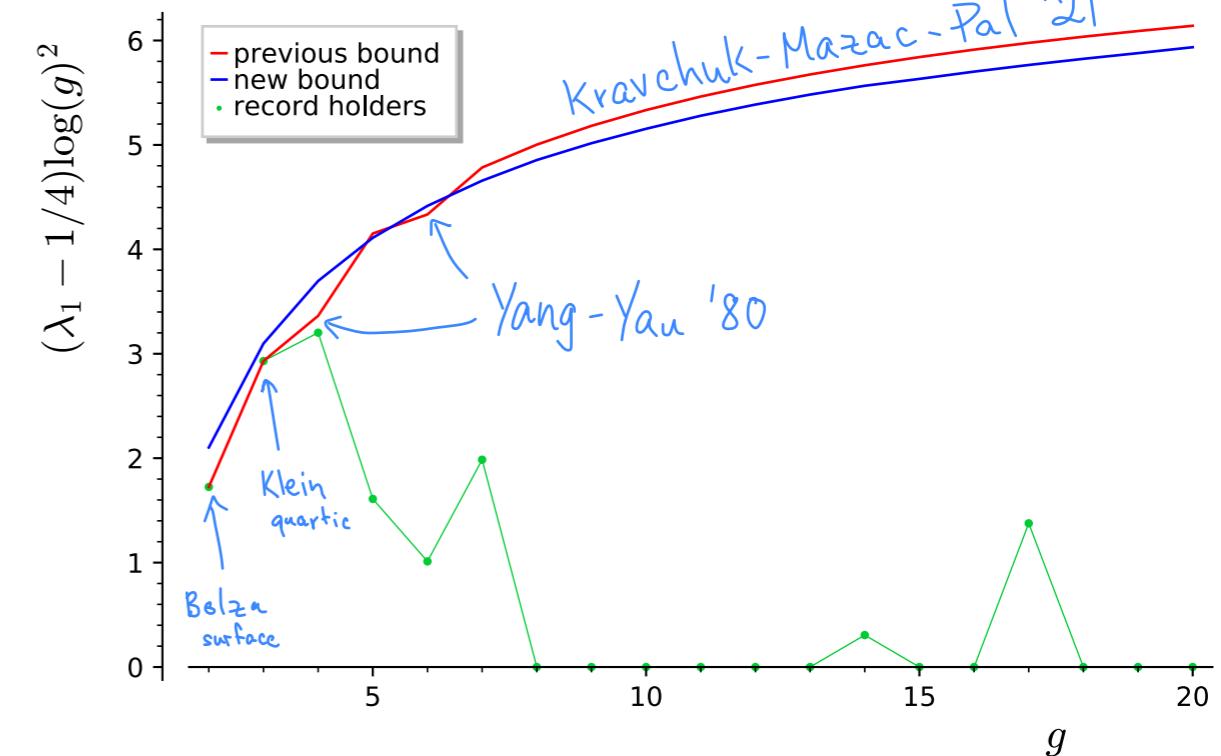
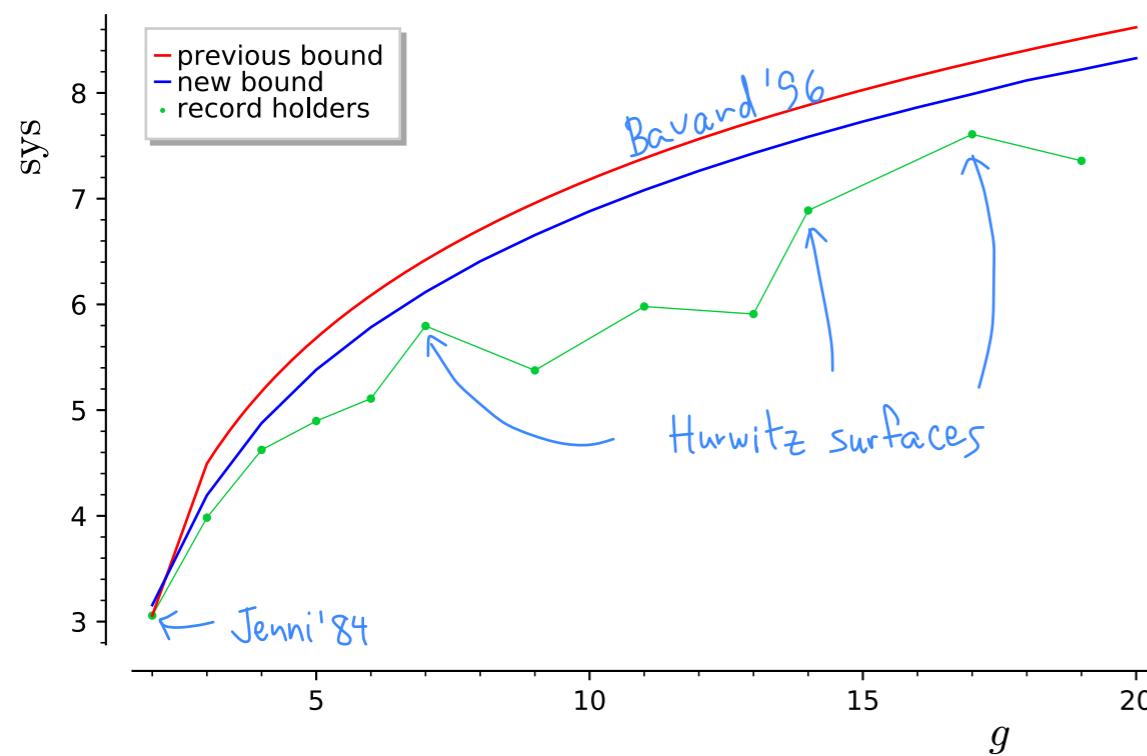
primitive closed geodesics of length $\leq 2\operatorname{arcsinh}(1)$ is $\leq 3g - 3$

Buser '78

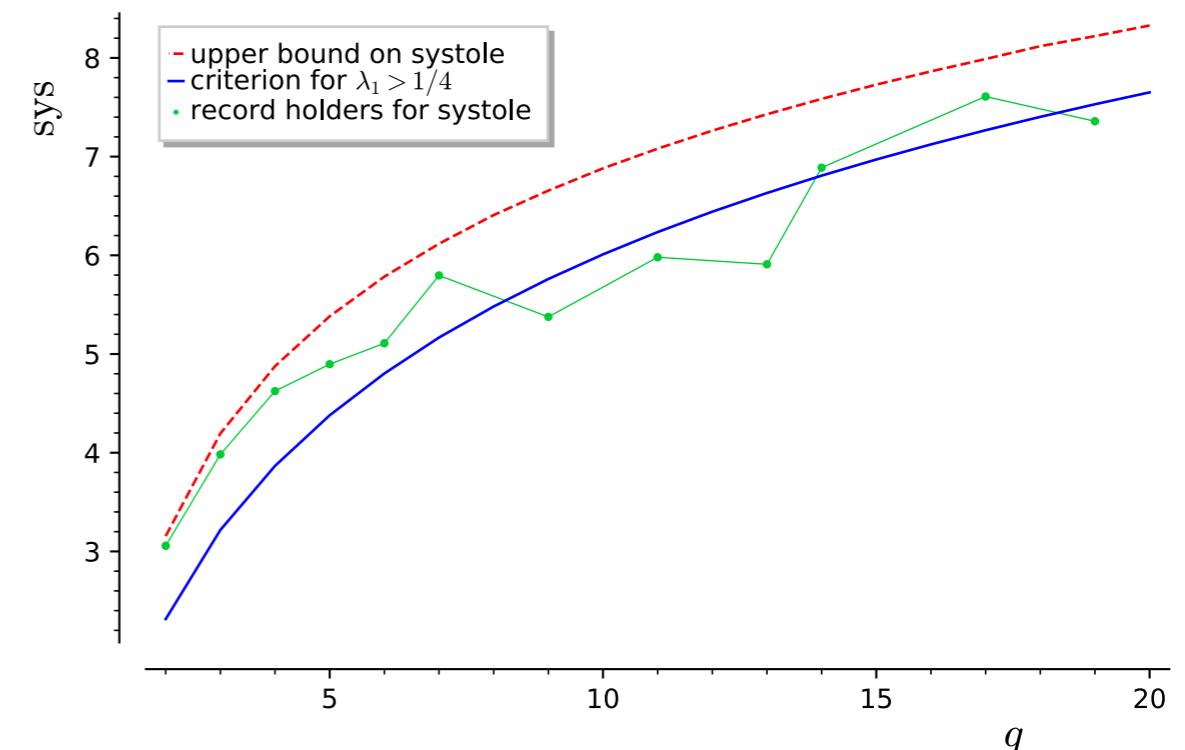
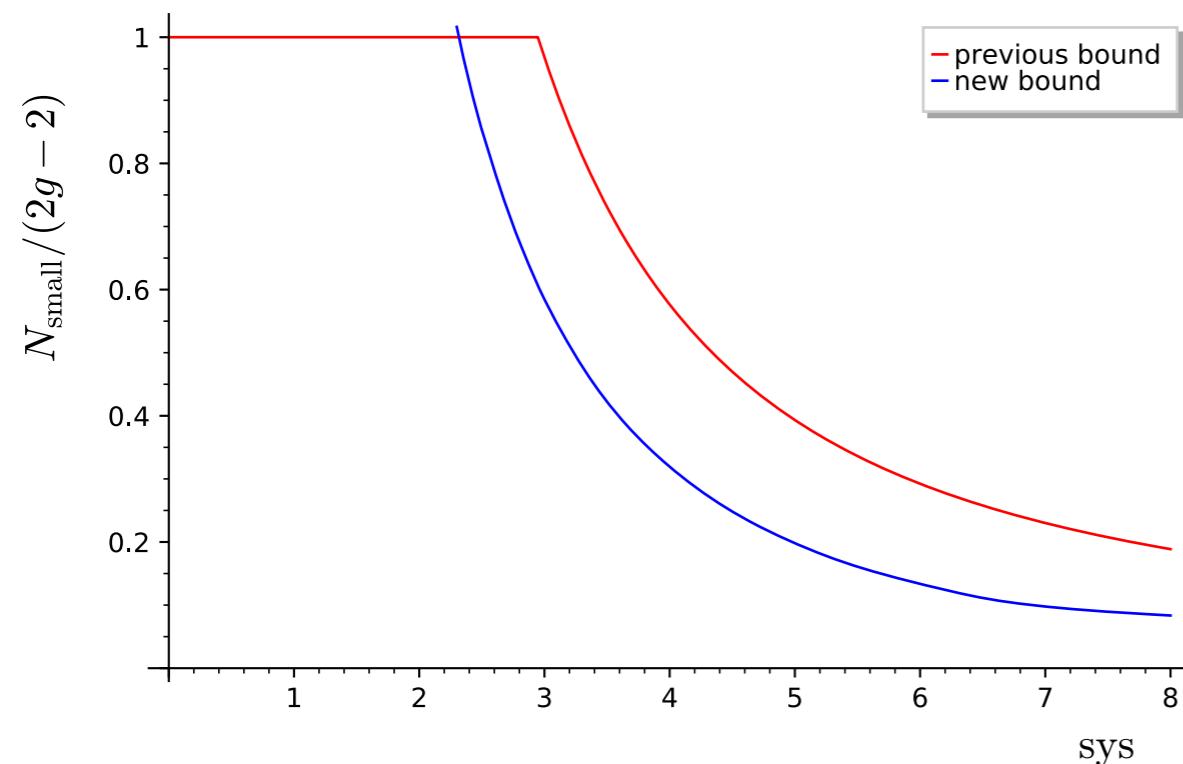
New bounds



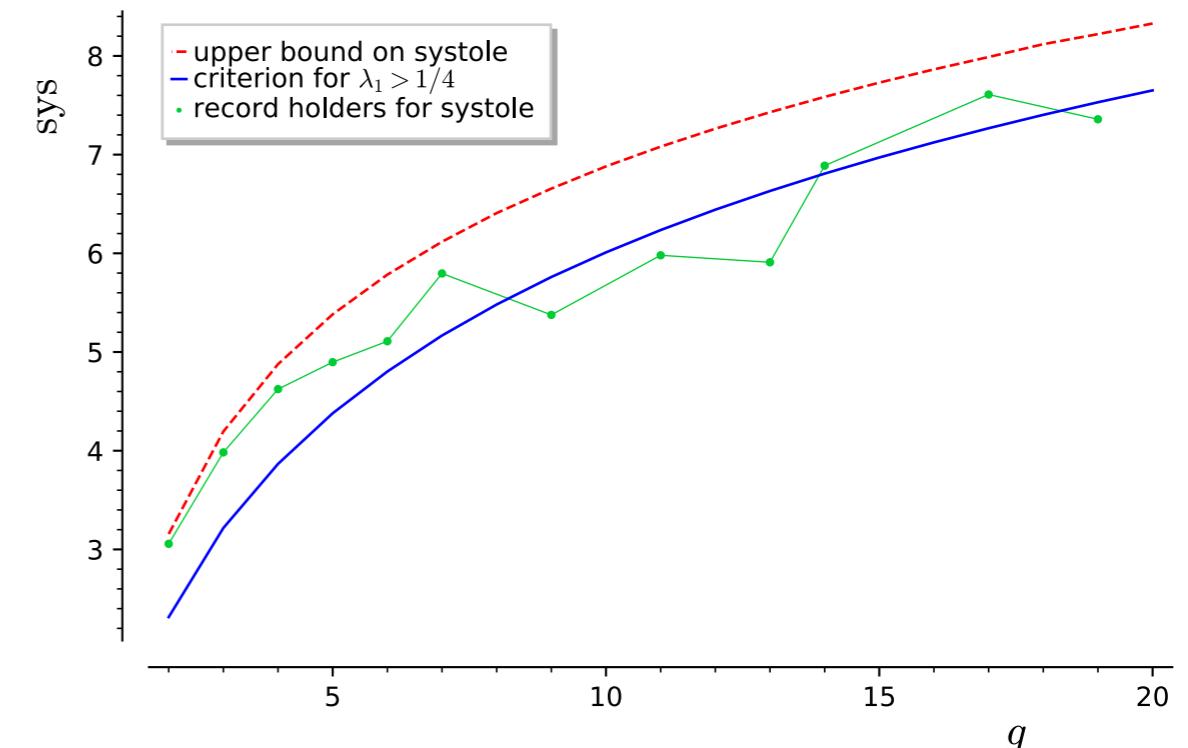
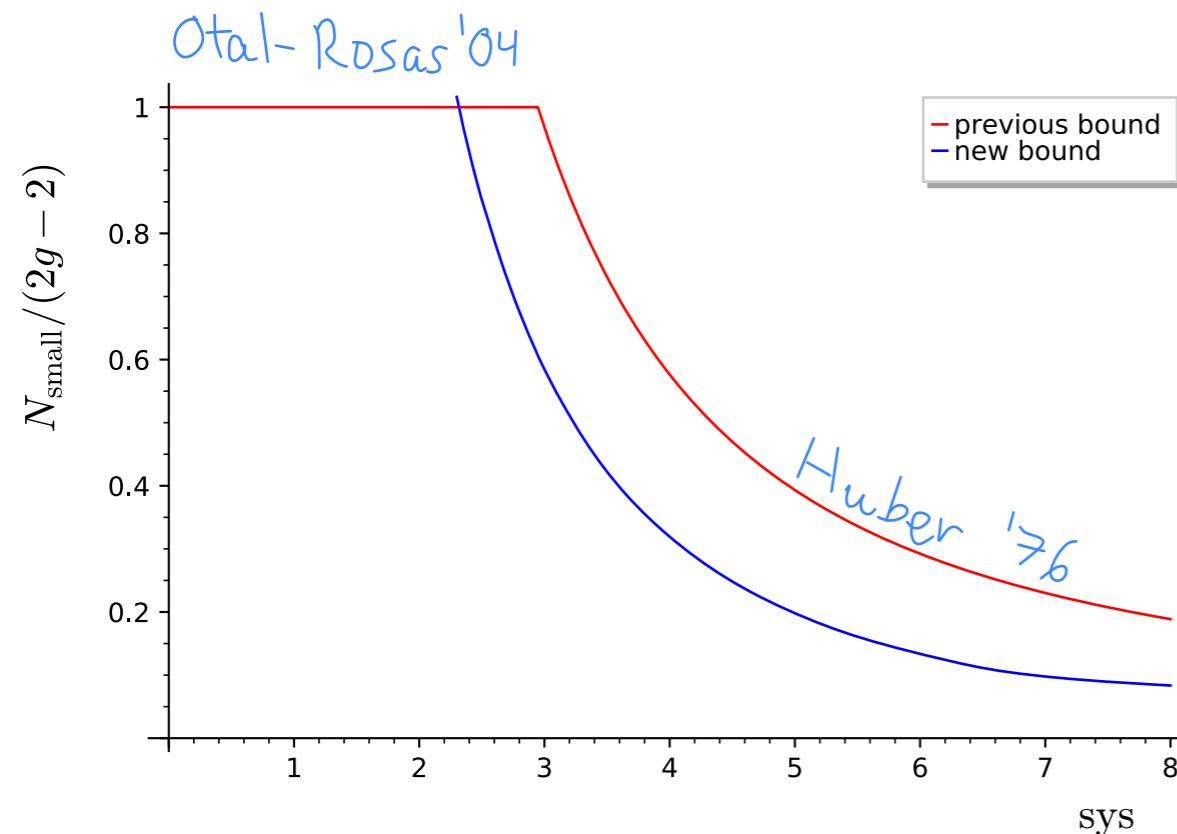
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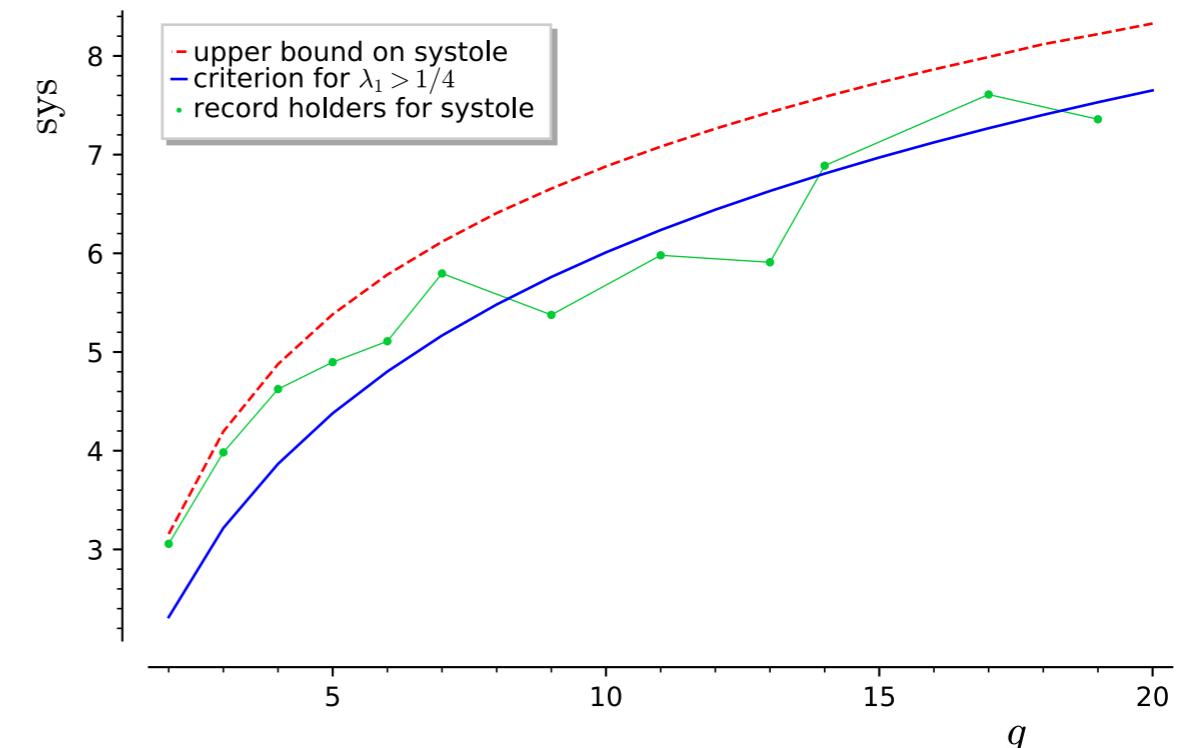
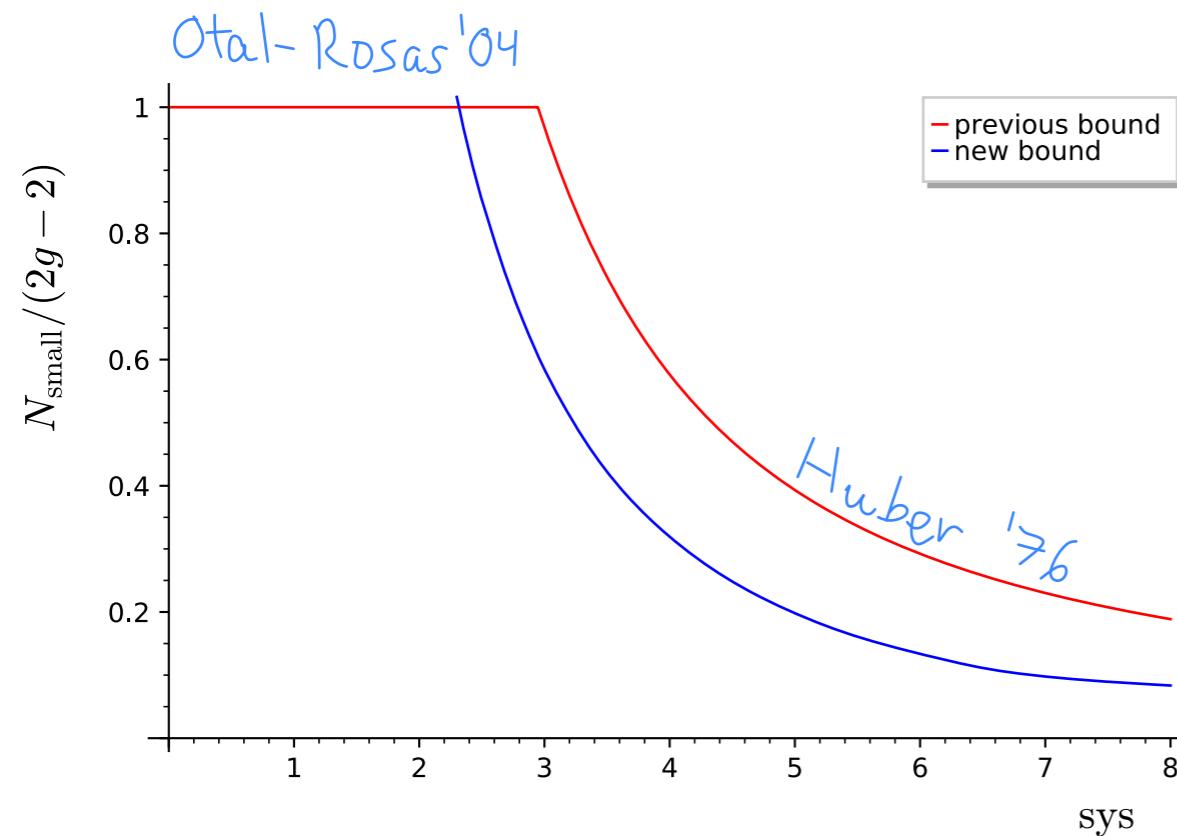
New bounds



New bounds



New bounds



Q (Selberg, Mondal, Wright)

Does there exist $X \in \mathcal{M}_g$ with $\lambda_1(X) \geq \frac{1}{4}$ for every $g \geq 2$?

A: Yes for $g \in \{2, \dots, 7, 14, 17\}$.

Hide-Magee '21 $\forall \varepsilon > 0, \forall g \geq 2, \exists X \in \mathcal{M}_g$ s.t. $\lambda_1(X) \geq \frac{1}{4} - \varepsilon$.

Computers are our friends



Computers are our friends



How do we extrapolate as $g \rightarrow \infty$?

Asymptotics

New

Old

Theorem 6.4. There exists some $g_0 \geq 2$ such that every closed hyperbolic surface M of genus $g \geq g_0$ satisfies

$$\text{sys}(M) < 2 \log(g) + 2.409. \quad < \quad 2 \log(g) + 2.681 + o(1)$$

Barard '96

Theorem 7.8. There exists some $g_0 \geq 2$ such that every closed hyperbolic surface M of genus $g \geq g_0$ satisfies

$$\text{kiss}(M) < \frac{4.873 \cdot g^2}{\log(g) + 1.2045}. \quad < \quad \frac{121.66 g^2}{\log(g) + 1.34}$$

FB-Petri '19
Parlier '13

Theorem 8.3. There exists some $g_0 \geq 2$ such that every closed hyperbolic surface M of genus $g \geq g_0$ satisfies

$$\lambda_1(M) < \frac{1}{4} + \left(\frac{\pi}{\log(g) + 0.7436} \right)^2. \quad < \quad \frac{1}{4} + \left(\frac{2\pi}{\log(g-1)} \right)^2$$

Cheng '75
+
Gage '80

Theorem 9.5. There exists some $g_0 \geq 2$ such that every closed hyperbolic surface M of genus $g \geq g_0$ satisfies

$$m_1(M) \leq 2g - 1. \quad < \quad 2g + 3$$

Sévennec '02

Theorem 10.2. If M is a closed hyperbolic surface of genus $g \geq 2$, then

$$N_{\text{small}}(M) < \min \left(\boxed{\frac{24\pi^2(g-1)}{\text{sys}(M)^3}}, \frac{16(g-1)}{\text{sys}(M)^2} \right) \cdot \frac{3\pi^2(g-1)}{8 \left(\log(\cosh(\frac{\text{sys}(M)}{4})) \right)^3}$$

Huber '76

Sphere packings

Annals of Mathematics, **157** (2003), 689–714

New upper bounds on sphere packings I

By HENRY COHN and NOAM ELKIES*

Abstract

We develop an analogue for sphere packing of the linear programming bounds for error-correcting codes, and use it to prove upper bounds for the density of sphere packings, which are the best bounds known at least for dimensions 4 through 36. We conjecture that our approach can be used to solve the sphere packing problem in dimensions 8 and 24.

→ proved by Viazovska et al in 2016

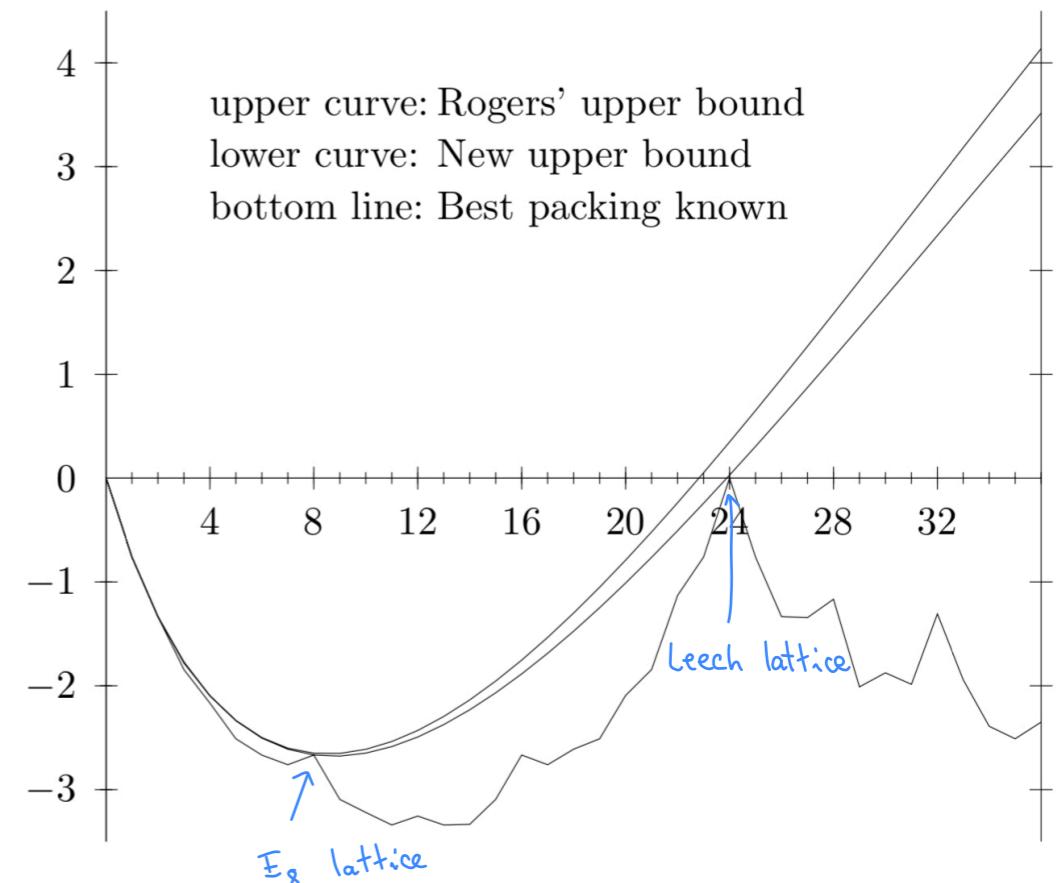


Figure 1. Plot of $\log_2 \delta + n(24 - n)/96$ vs. dimension n .

Poisson summation formula

$$\sum_{x \in \Lambda} f(x + v) = \frac{1}{|\Lambda|} \sum_{t \in \Lambda^*} e^{-2\pi i \langle v, t \rangle} \widehat{f}(t)$$

The Selberg trace formula

Fourier transform

$$\widehat{f}(y) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-iyx} dx$$

Defⁿ $f : \mathbb{R} \rightarrow \mathbb{C}$ is admissible if even, integrable,
and \widehat{f} is holomorphic in $\left\{ z \in \mathbb{C} : |\operatorname{Im}(z)| < \frac{1}{2} + \varepsilon \right\}$
for some $\varepsilon > 0$ and satisfies $|\widehat{f}(z)| = O\left(\frac{1}{1 + |z|^p}\right)$
for some $p > 2$ in that strip.

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$$0 = \lambda_0(M) < \lambda_1(M) \leq \lambda_2(M) \leq \dots$$

$$\mathcal{C}(M)$$

$$\ell(\gamma)$$

$$\Lambda(\gamma)$$

- eigenvalues of Laplacian on M
- oriented closed geodesics in M
- length of γ
- primitive length $\ell(\alpha)$ s.t. $\gamma = \alpha^k$, k maximal

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STF $\forall M \in \mathcal{M}_g, \forall f \text{ admissible}$

$$\sum_{j=0}^{\infty} \widehat{f}\left(\sqrt{\lambda_j(M) - \frac{1}{4}}\right) = 2(g-1) \int_0^{\infty} \widehat{f}(x) x \tanh(\pi x) dx + \frac{1}{\sqrt{2\pi}} \sum_{\gamma \in \mathcal{C}(M)} \frac{\Lambda(\gamma) f(\ell(\gamma))}{2 \sinh(\ell(\gamma)/2)}$$

Spectral = Integral + Geometric

Criterion for λ_1

Theorem 8.1. Let $g \geq 2$. Suppose that f is an admissible function for which there exists an $L > 0$ such that

- $f(x) \geq 0$ for all $x \in \mathbb{R}$;
- $\widehat{f}\left(\sqrt{\lambda - \frac{1}{4}}\right) \leq 0$ whenever $\lambda \geq L$;
- $\widehat{f}(i/2) < 2(g-1) \int_0^\infty \widehat{f}(x)x \tanh(\pi x) dx$;

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$$\sum_{j=0}^{\infty} \hat{f}\left(\sqrt{\lambda_j(M) - \frac{1}{4}}\right) \leq \hat{f}(i/2)$$

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$$= \sum_{j=0}^{\infty} \widehat{f}\left(\sqrt{\lambda_j(M) - \frac{1}{4}}\right), \quad \text{contradiction}$$

□

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□

Goal

Minimize the last sign change of \widehat{f} under the hypothesis that the first & third bullet points hold.

Double zeros

Goal

Minimize the last sign change of \hat{f} under the hypothesis
that $\overset{(1)}{f(x) \geq 0} \quad \forall x \in \mathbb{R}$ and $\overset{(2)}{\hat{f}'(i/2) < 2(g-1) \int_0^\infty \hat{f}(x) \times \tanh(\pi x) dx}$.

Double zeros

Goal

Minimize the last sign change of \hat{f} under the hypothesis that $\overset{\textcircled{1}}{f(x) \geq 0 \quad \forall x \in \mathbb{R}}$ and $\overset{\textcircled{2}}{\hat{f}(i/2) < 2(g-1) \int_0^\infty \hat{f}(x) x \tanh(\pi x) dx}$.

Strategy (for small g) Take $f(x) = p(x^2) e^{-x^2/2}$ where p is a polynomial
 $\Rightarrow \hat{f}(x) = q(x^2) e^{-x^2/2}$ and $p \leftrightarrow q$ easy to compute

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- ↳ force ② by setting LHS = 0.999 RHS
- ↳ impose double zeros to f & \hat{f}

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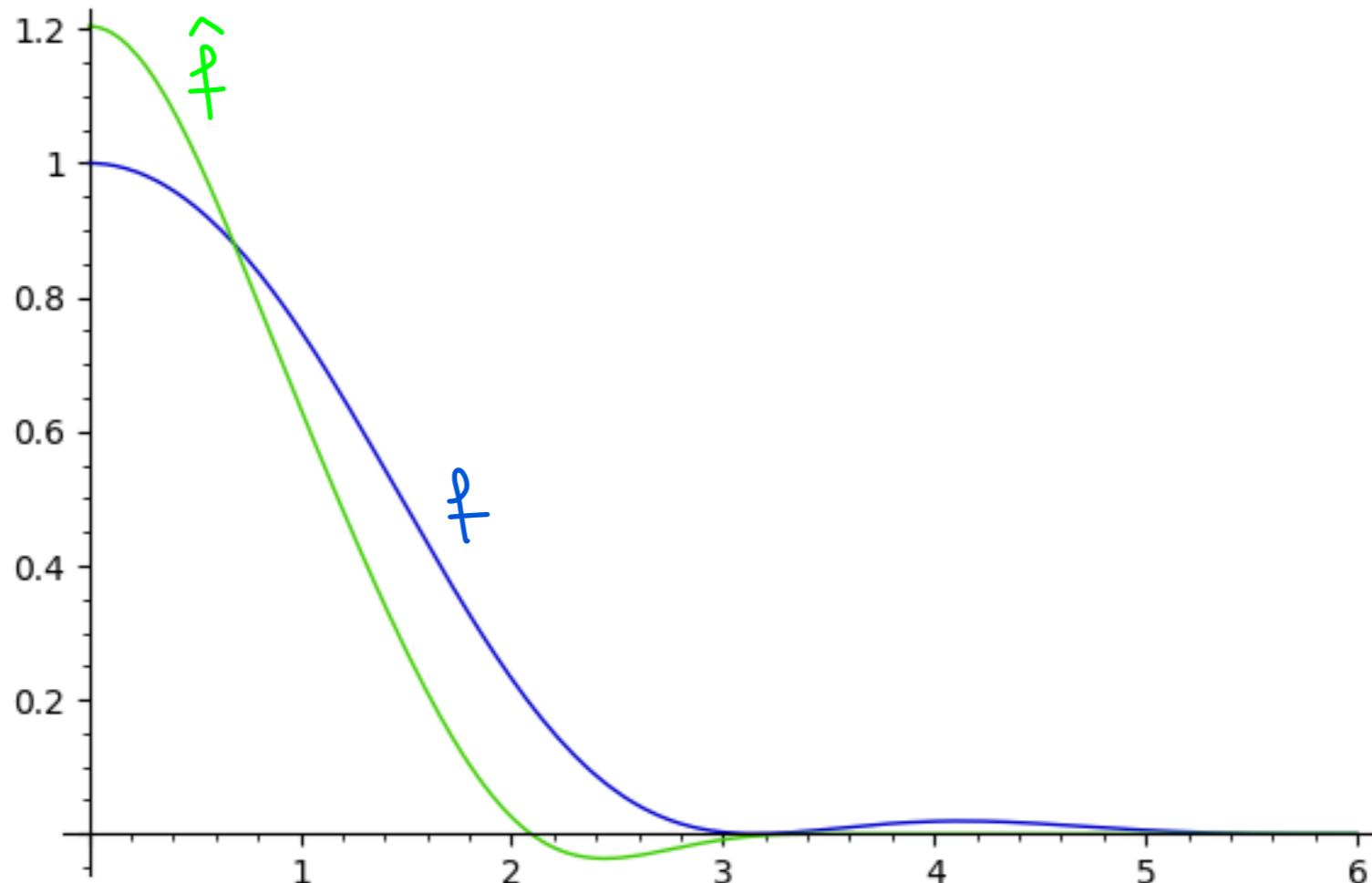
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} linear system of equations

If all conditions satisfied, wiggle the zeros and optimize.

What the functions look like



The scaling trick

To prove an asymptotic bound as $g \rightarrow \infty$, we look for a single function f_0 such that

- ① $f_0(x) \geq 0 \quad \forall x \in \mathbb{R}$
- ② $\hat{f}_0(x) \leq 0 \quad \forall x \geq 1$

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and then use $\widehat{f}(x) = \widehat{f}_0(Rx)$ for some $R > 0$ to be chosen in terms of g . Then $f(x) = f_0(x/R)/R \geq 0 \quad \forall x \in \mathbb{R}$

The scaling trick

To prove an asymptotic bound as $g \rightarrow \infty$, we look for a single function \hat{f}_0 such that

- ① $\hat{f}_0(x) \geq 0 \quad \forall x \in \mathbb{R}$
- ② $\hat{f}_0'(x) \leq 0 \quad \forall x \geq 1$

and then use $\hat{f}(x) = \hat{f}_0(Rx)$ for some $R > 0$ to be chosen in terms of g . Then $f(x) = f_0(x/R)/R \geq 0 \quad \forall x \in \mathbb{R}$ and we want to choose R such that

$$\begin{aligned}\hat{f}_0(Ri/2) &= \hat{f}(i/2) < 2(g-1) \int_0^\infty \hat{f}(x) x \tanh(\pi x) dx \\ &= 2(g-1) \int_0^\infty \hat{f}_0(Rx) x \tanh(\pi x) dx \\ &= \frac{2(g-1)}{R^2} \int_0^\infty \hat{f}_0(y) y \tanh\left(\frac{\pi y}{R}\right) dy\end{aligned}$$

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$O\left(\frac{1}{R^2}\right)$

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if f_0 has support $[-a, a]$

$$= 2(g-1) \int_0^\infty \hat{f}_0(Rx)x \tanh(\pi x) dx$$

then $\hat{f}_0(it) \leq Ce^{at}$

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Want R large, hence a small

The optimal function

By results of Gorbachev-Ivanov-Tikhonov '20, the best f_o to use is

$$f_o(x) = \sqrt{\frac{\pi}{8}}(2\pi - |x| + \sin|x|)\chi_{[-2\pi, 2\pi]}(x)$$

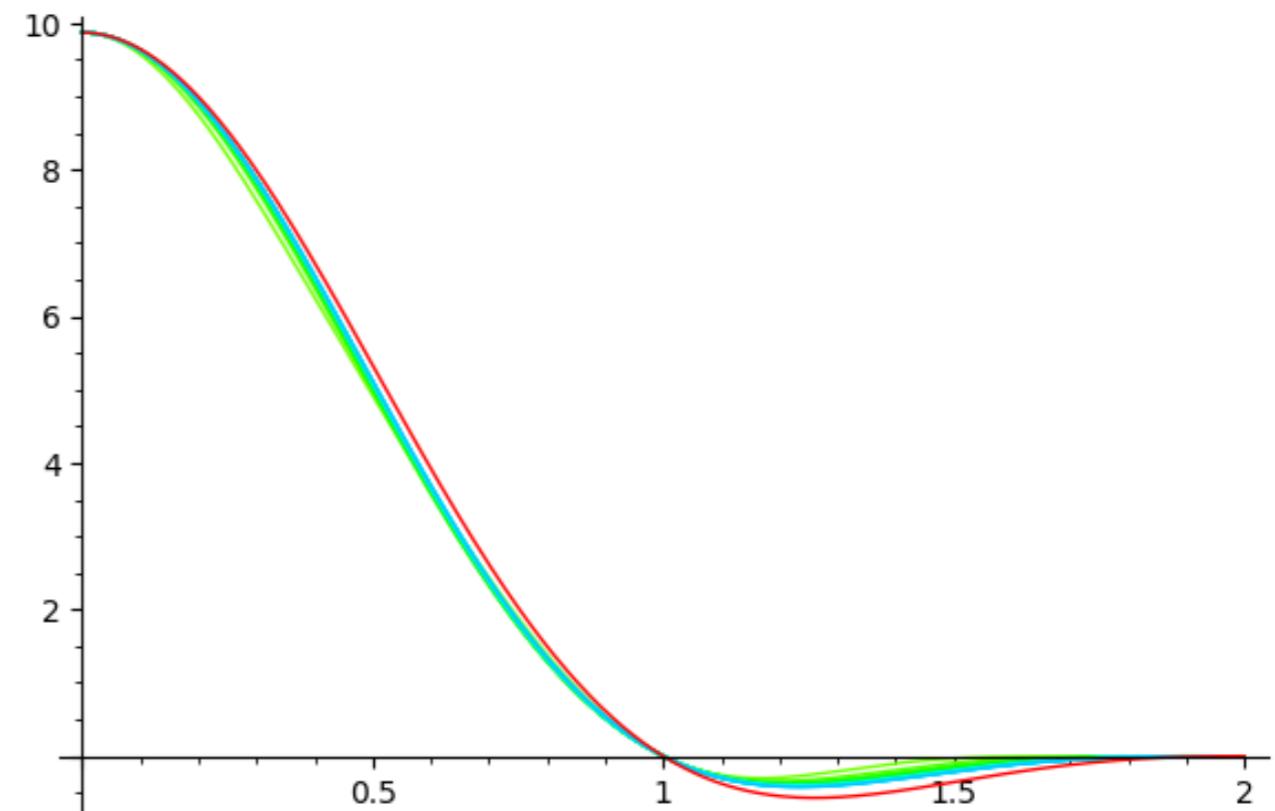
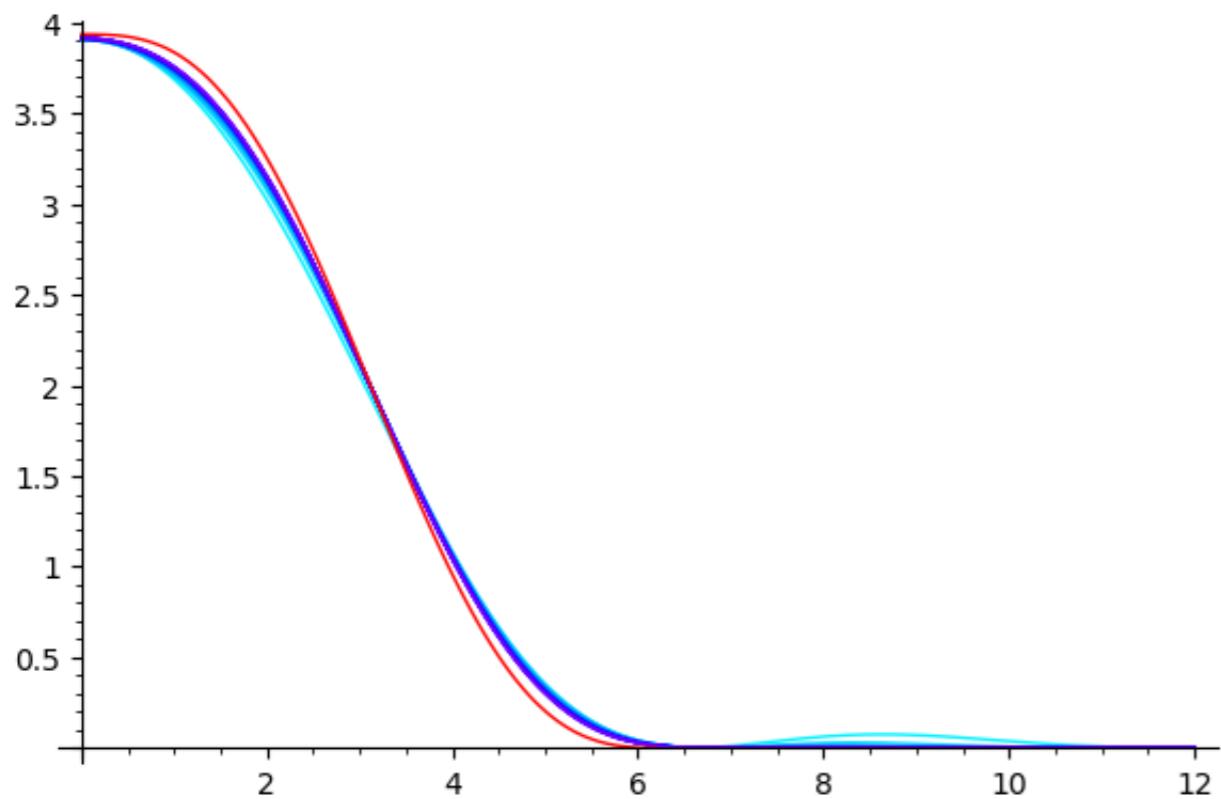
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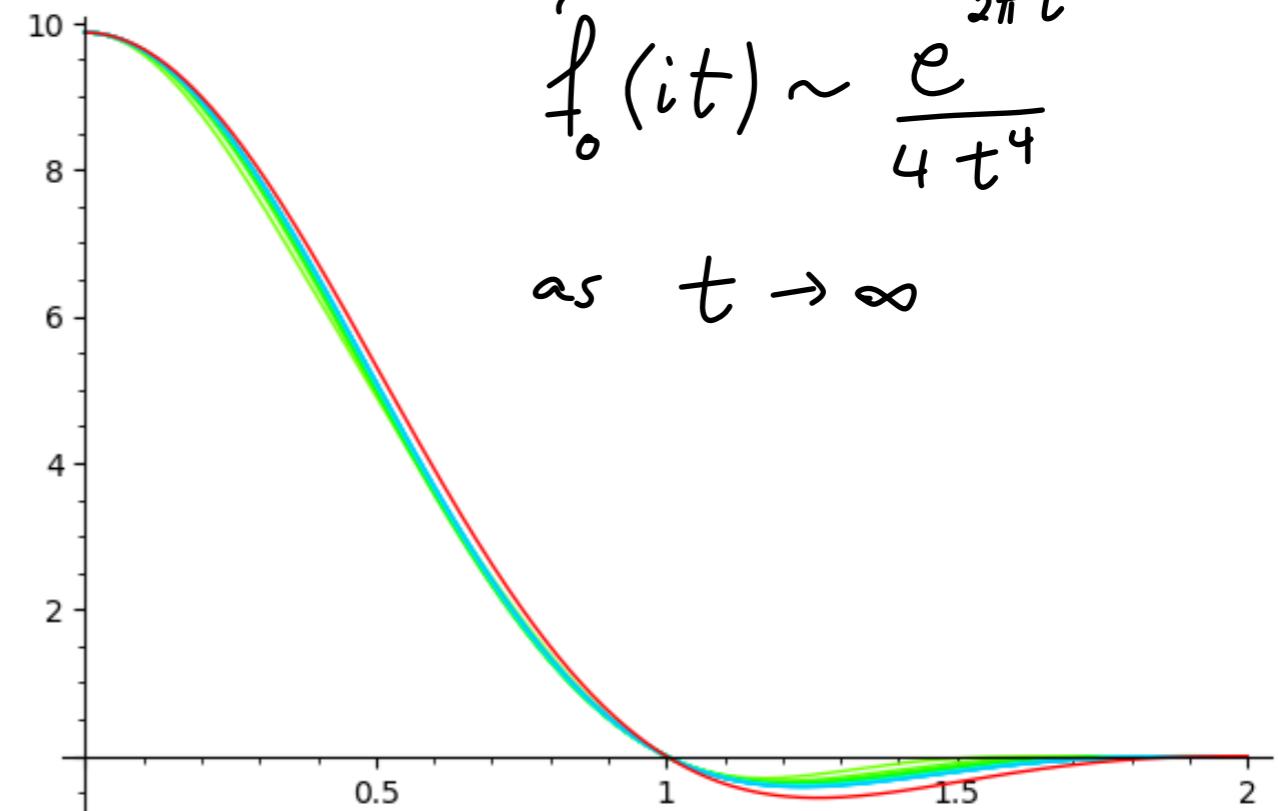
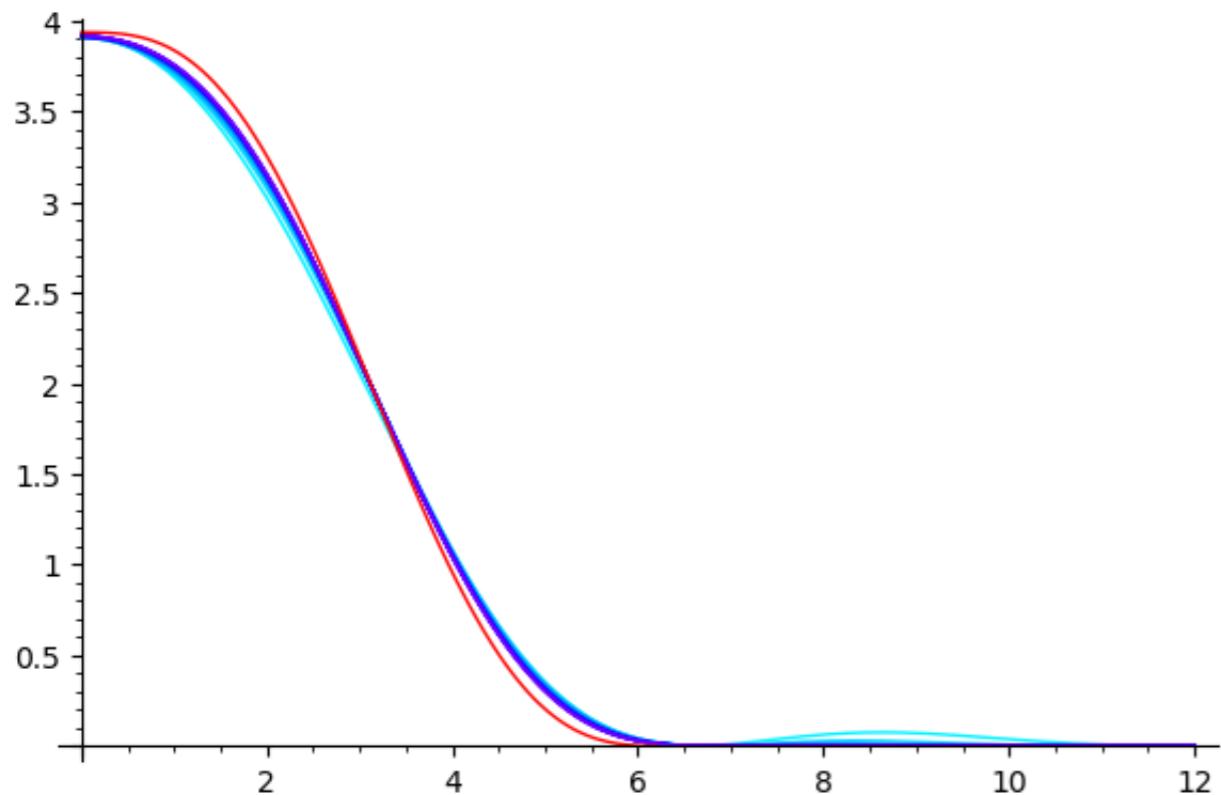


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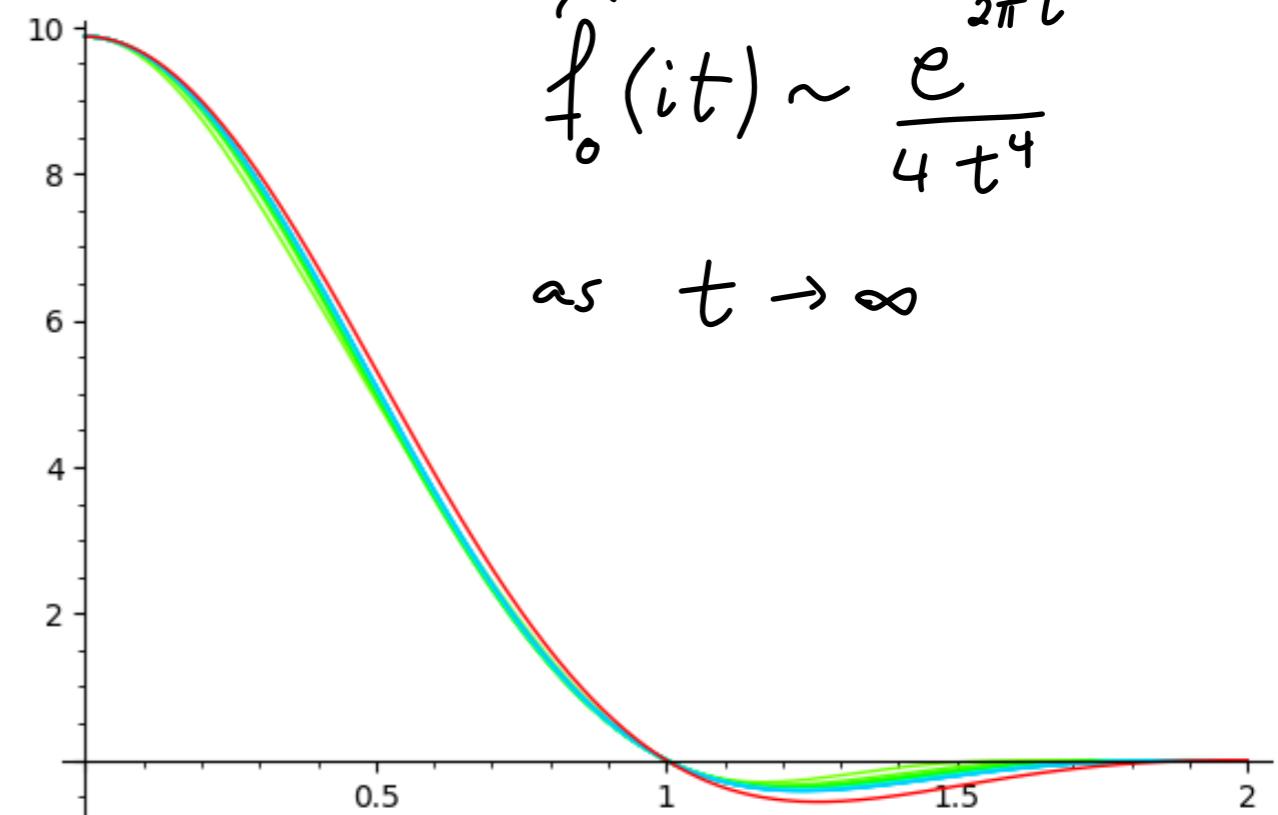
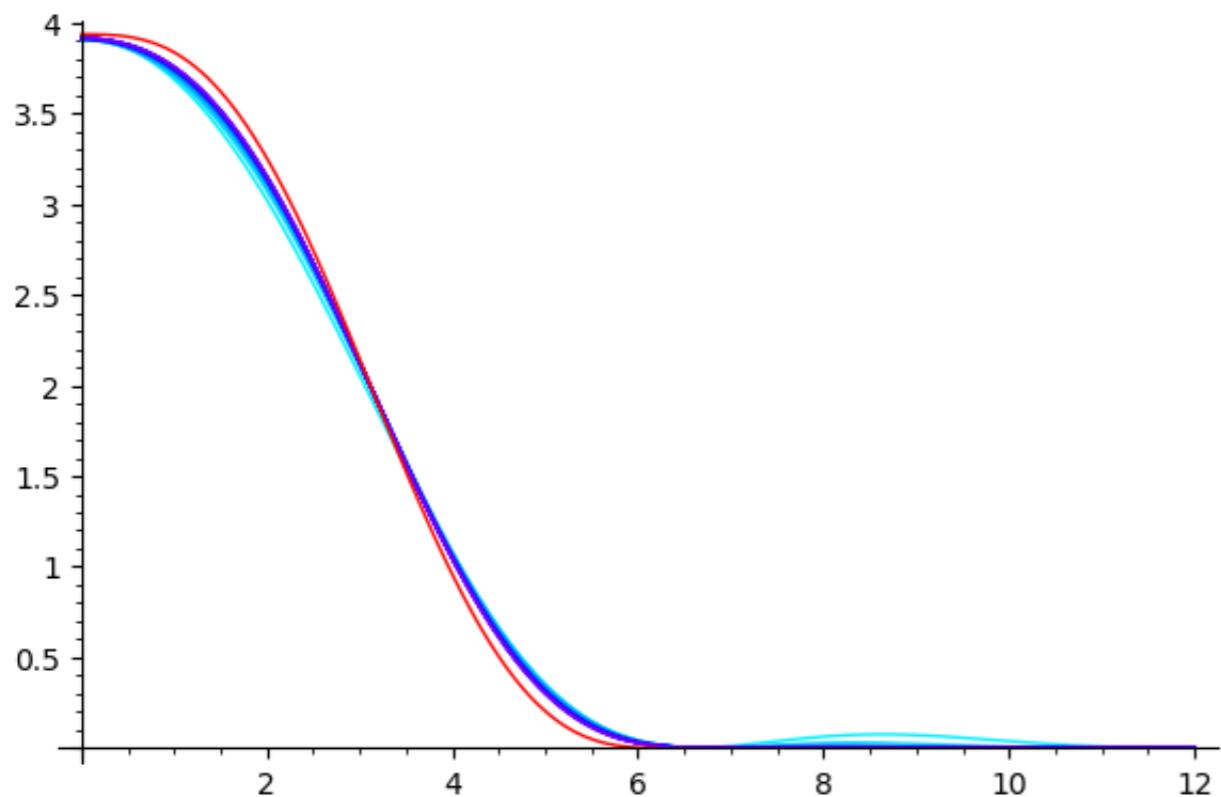


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It remains to estimate the integral

The hardest limit I've ever computed

Lemma 8.6. *We have*

$$\lim_{R \rightarrow \infty} R^4 \int_0^\infty \frac{\sin^2(\pi Rx)}{1 - (Rx)^2} \frac{x \tanh(\pi x)}{(Rx)^2} dx = \frac{\pi^2}{2} \int_0^\infty \frac{\sinh(x) \cosh(x) - x}{x^3 \cosh^2(x)} dx > 4.20718.$$

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Conclusion: Can take $R = \frac{\log(g-1) + \log(4.20718/2)}{\pi}$

leading to the bound

$$\lambda_1 < L = \frac{1}{4} + \frac{1}{R^2} = \frac{1}{4} + \left(\frac{\pi}{\log(g-1) + 0.7436} \right)^2$$

if g is large enough.