

A divergent horocycle in
the horoboundary of the
Teichmüller metric

Maxime Fortier Bourque
Université de Montréal

Compactifications of Teichmüller space

$S = \text{closed oriented surface w/ possible punctures, } \chi(S) < 0$

$T(S) = \text{Teichmüller space of } S$

$= \{ \text{complex structures on } S \} / \{ \text{biholomorphisms homotopic to } id_S \}$

$\cong \{ \text{hyperbolic metrics on } S \} / \{ \text{isometries homotopic to } id_S \}$

Compactifications of Teichmüller space

$S = \text{closed oriented surface w/ possible punctures}, \chi(S) < 0$

$\mathcal{T}(S) = \text{Teichmüller space of } S$

$= \{ \text{complex structures on } S \} / \{ \text{biholomorphisms homotopic to } \text{id}_S \}$

$\cong \{ \text{hyperbolic metrics on } S \} / \{ \text{isometries homotopic to } \text{id}_S \}$

Thurston compactification

$\mathcal{C}(S) = \text{homotopy classes of essential simple closed curves in } S$

The map $\mathcal{T}(S) \xrightarrow{\Phi} \mathbb{P}(\mathbb{R}_+^{\mathcal{C}(S)})$ is an embedding

$$X \mapsto [\ell_{hyp}(\alpha, X)]_{\alpha \in \mathcal{C}(S)}$$

The Thurston compactification is defined as $\overline{\mathcal{T}(S)}^{\text{Th}} := \overline{\Phi(\mathcal{T}(S))}$

Thurston compactification

$\mathcal{C}(S)$ = homotopy classes of essential simple closed curves in S

The map $T(S) \xrightarrow{\Phi} P(\mathbb{R}_+^{\mathcal{C}(S)})$ is an embedding

$$X \mapsto [\ell_{hyp}(\alpha, X)]_{\alpha \in \mathcal{C}(S)}$$

The Thurston compactification is defined as $\overline{T(S)}^{\text{th}} := \overline{\Phi(T(S))}$

(this records how relatively fast hyperbolic lengths blow up as $X \rightarrow \infty$)

Thurston compactification

$\mathcal{C}(S)$ = homotopy classes of essential simple closed curves in S

The map $T(S) \xrightarrow{\Phi} P(\mathbb{R}_+^{\mathcal{C}(S)})$ is an embedding

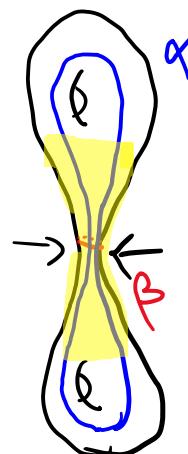
$$X \mapsto [\ell_{hyp}(\alpha, X)]_{\alpha \in \mathcal{C}(S)}$$

The **Thurston compactification** is defined as $\overline{T(S)}^{\text{th}} := \overline{\Phi(T(S))}$

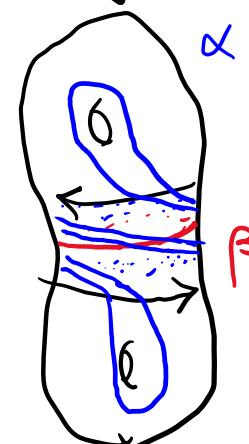
(this records how relatively fast hyperbolic lengths blow up as $X \rightarrow \infty$)

Ex

Pinching



twisting



In either case, the deformation converges to $[i(\alpha, \beta)]_{\alpha \in \mathcal{C}(S)}$

In either case, the deformation converges to $[i(\alpha, \beta)]_{\beta \in \mathcal{C}(S)}$

We can then replace β by a sequence of curves β_n that converge projectively to an arbitrary measured foliation F on S .

In either case, the deformation converges to $[i(\alpha, \beta)]_{\alpha \in C(S)}$

We can then replace β by a sequence of curves β_n that converge projectively to an arbitrary measured foliation F on S .

$$\Rightarrow \partial^{\text{Th}} T(S) \supseteq \text{all points of the form } [i(\alpha, F)]_{\alpha \in C(S)} = \text{PMF}(S).$$

In either case, the deformation converges to $[i(\alpha, \beta)]_{\alpha \in C(S)}$

We can then replace β by a sequence of curves β_n that converge projectively to an arbitrary measured foliation F on S .

$$\Rightarrow \partial^{\text{Th}} T(S) \supseteq \text{all points of the form } [i(\alpha, F)]_{\alpha \in C(S)} = \text{PMF}(S).$$

Thm (Thurston) $\partial^{\text{Th}} T(S) = \text{PMF}(S) \cong \text{sphere}$ and
 $\overline{T(S)}^{\text{Th}} \cong \text{closed ball}.$

In either case, the deformation converges to $[i(\alpha, \beta)]_{\alpha \in \mathcal{C}(S)}$

We can then replace β by a sequence of curves β_n that converge projectively to an arbitrary measured foliation F on S .

$$\Rightarrow \partial^{\text{Th}} T(S) \supseteq \text{all points of the form } [i(\alpha, F)]_{\alpha \in \mathcal{C}(S)} = \text{PMF}(S).$$

Thm (Thurston) $\partial^{\text{Th}} T(S) = \text{PMF}(S) \cong \text{sphere}$ and
 $\overline{T(S)}^{\text{Th}} \cong \text{closed ball}.$

Why is it useful?

- $\text{MCG}(S)$ acts continuously on $\overline{T(S)}^{\text{Th}}$
- Can use Brouwer's fixed point thm to prove the classification of mapping classes

Gardiner - Masur compactification

Same thing but with extremal length.

The map $T(S) \xrightarrow{\Psi} P(\mathbb{R}_+^{c(S)})$ is an embedding
 $x \mapsto [\sqrt{\text{EL}(\alpha, x)}]_{\alpha \in \mathcal{C}(S)}$

and the closure of its image $\overline{T(S)}^{\text{GM}} := \overline{\Psi(T(S))}$ is compact.

Gardiner - Masur compactification

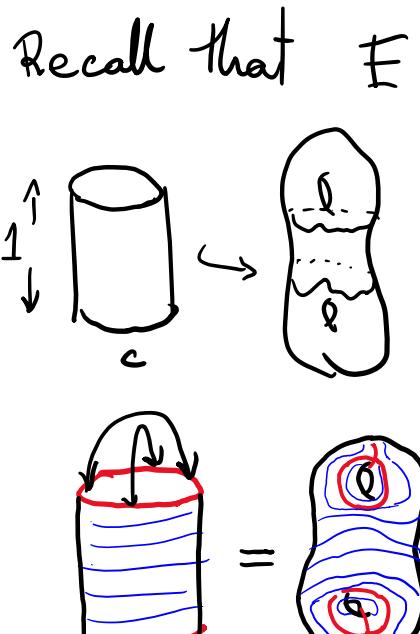
Same thing but with extremal length.

The map $T(S) \xrightarrow{\Psi} P(\mathbb{R}_+^{c(S)})$ is an embedding
 $x \mapsto [\sqrt{EL(\alpha, x)}]_{\alpha \in \mathcal{C}(S)}$

and the closure of its image $\overline{T(S)}^{\text{GM}} := \overline{\Psi(T(S))}$ is compact.

Recall that $EL(\alpha, X) =$ smallest circumference of an open cylinder of height 1 that embeds conformally in the homotopy class of X

$=$ area of unique holomorphic quadratic differential on X whose horizontal foliation is measure-equivalent to α .



Advantage: More compatible with constructions coming from conformal geometry (as opposed to hyperbolic geometry).

Ex Every Teichmüller ray converges in $\overline{\mathcal{T}(S)}^{\text{GM}}$ (Liu-Su, Walsh)
but in $\overline{\mathcal{T}(S)}^{\text{Th}}$ they can accumulate onto

- intervals (Lenzhen)
- circles (Brock - Leininger - Modami - Rafi)
- higher-dim simplices (Lenzhen - Modami - Rafi)

Advantage: More compatible with constructions coming from conformal geometry (as opposed to hyperbolic geometry).

Ex Every Teichmüller ray converges in $\overline{T(S)}^{GM}$ (Liu-Su, Walsh)
but in $\overline{T(S)}^{\text{Th}}$ they can accumulate onto

- intervals (Lenzhen)
- circles (Brock - Leininger - Modami - Rafi)
- higher-dim simplices (Lenzhen - Modami - Rafi)

Drawback: We do not know what $\overline{T(S)}^{GM}$ looks like, nor do we have an explicit description of its boundary points.

Advantage: More compatible with constructions coming from conformal geometry (as opposed to hyperbolic geometry).

Ex Every Teichmüller ray converges in $\overline{T(S)}^{\text{GM}}$ (Liu-Su, Walsh)
but in $\overline{T(S)}^{\text{Th}}$ they can accumulate onto

- intervals (Lenzhen)
- circles (Brock-Leininger-Modami-Rafi)
- higher-dim simplices (Lenzhen-Modami-Rafi)

Drawback: We do not know what $\overline{T(S)}^{\text{GM}}$ looks like, nor do we have an explicit description of its boundary points.

Thm (Gardiner-Masur) $\mathcal{J}^{\text{GM}} T(S) \supsetneq \mathcal{J}^{\text{Th}} T(S)$ (as subsets of $\text{PC}(\mathbb{R}_+^{e(S)})$)

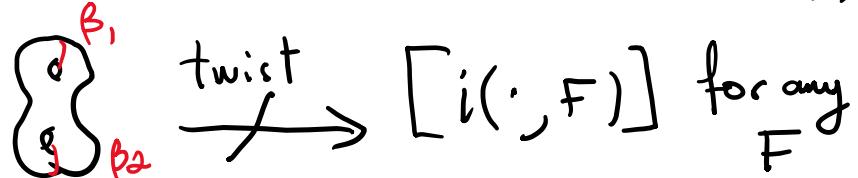
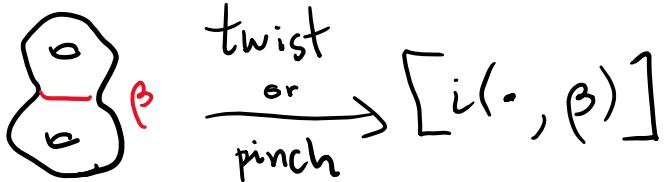
Advantage: More compatible with constructions coming from conformal geometry (as opposed to hyperbolic geometry).

Ex Every Teichmüller ray converges in $\overline{T(S)}^{GM}$ (Liu-Su, Walsh)
 but in $\overline{T(S)}^{\text{Th}}$ they can accumulate onto

- intervals (Lenzhen)
- circles (Brock-Leininger-Modami-Rafi)
- higher-dim simplices (Lenzhen-Modami-Rafi)

Drawback: We do not know what $\overline{T(S)}^{GM}$ looks like, nor do we have an explicit description of its boundary points.

Thm (Gardiner-Masur) $\partial^{GM} T(S) \supsetneq \partial^{\text{Th}} T(S)$ (as subsets of $\text{PC}(R_+^{T(S)})$)



Thm (Liu-Su) The Gardiner-Masur compactification is isomorphic to the horofunction compactification of the Teichmüller metric.

Thm (Liu-Su) The Gardiner-Masur compactification is isomorphic to the horofunction compactification of the Teichmüller metric.

The horofunction compactification

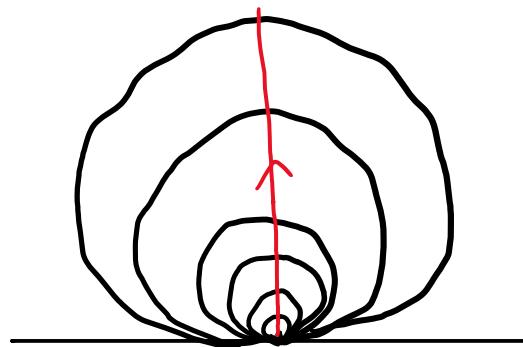
Def" A **horoball / horosphere** ^{in a metric space} is a nonempty Hausdorff limit of a sequence of balls / spheres whose centers go off to infinity.

Thm (Liu-Su) The Gardiner-Masur compactification is isomorphic to the horofunction compactification of the Teichmüller metric.

The horofunction compactification

Def" A **horoball / horosphere** ^{in a metric space} is a nonempty Hausdorff limit of a sequence of balls / spheres whose centers go off to infinity.

Ex ① In \mathbb{R}^n , a horoball is a half-space

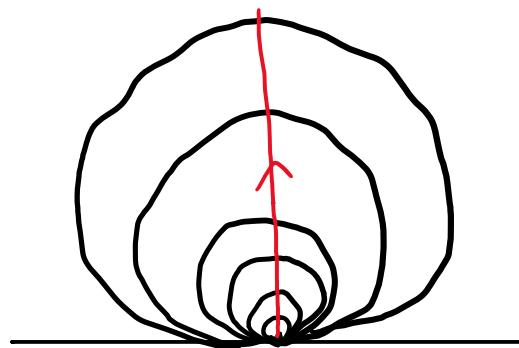


Thm (Liu-Su) The Gardiner-Masur compactification is isomorphic to the horofunction compactification of the Teichmüller metric.

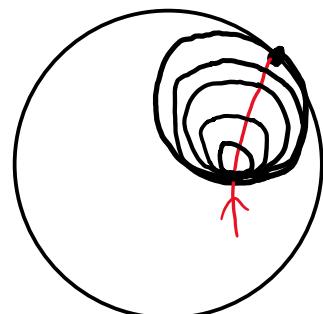
The horofunction compactification

Def" A **horoball / horosphere** ^{in a metric space} is a nonempty Hausdorff limit of a sequence of balls / spheres whose centers go off to infinity.

Ex ① In \mathbb{R}^n , a horoball is a half-space



② In H^2 :

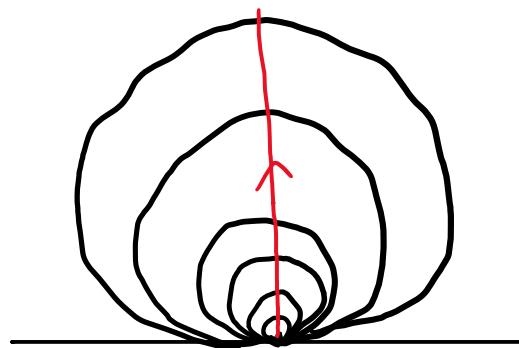


Thm (Liu-Su) The Gardiner-Masur compactification is isomorphic to the horofunction compactification of the Teichmüller metric.

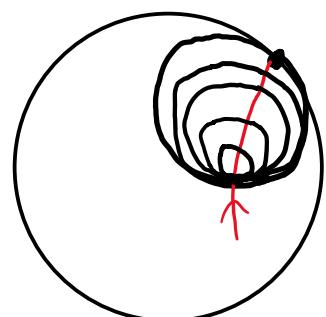
The horofunction compactification

Def" A **horoball / horosphere** ^{in a metric space} is a nonempty Hausdorff limit of a sequence of balls / spheres whose centers go off to infinity.

Ex ① In \mathbb{R}^n , a horoball is a half-space



② In H^2 :



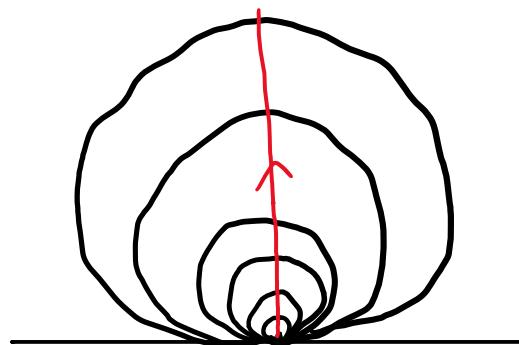
A **horofunction** is a renormalized limit of distance functions $y \mapsto d(x, y)$ as $x \rightarrow \infty$.

Thm (Liu-Su) The Gardiner-Masur compactification is isomorphic to the horofunction compactification of the Teichmüller metric.

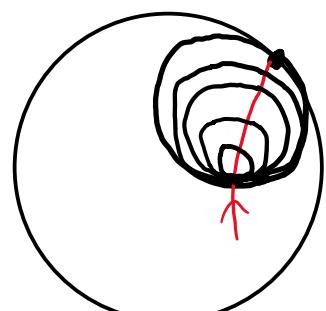
The horofunction compactification

Def" A **horoball / horosphere** ^{in a metric space} is a nonempty Hausdorff limit of a sequence of balls / spheres whose centers go off to infinity.

Ex ① In \mathbb{R}^n , a horoball is a half-space



② In H^2 :



A **horofunction** is a renormalized limit of distance functions $y \mapsto d(x, y)$ as $x \rightarrow \infty$.

If x goes off along a geodesic, then the limit is called a **Busemann point**.

Thm (Liu-Su) The Gardiner-Masur compactification is isomorphic to the horofunction compactification of the Teichmüller metric.

Claim / Hope : The intricate nature of the Teichmüller metric is reflected in the horofunction compactification.

Thm (Liu-Su) The Gardiner-Masur compactification is isomorphic to the horofunction compactification of the Teichmüller metric.

Claim / Hope : The intricate nature of the Teichmüller metric is reflected in the horofunction compactification.

(Miyachi) The horoboundary of $T(S)$ contains non-Busemann points.



(Masur) The Teichmüller metric is not non-positively curved.



(FB-Rafi) \exists non-convex balls in $T(S)$.



(FB-Rafi) \exists non-convex horoballs in $T(S)$.

Paths in Teichmüller space

Quadratic differential \leftrightarrow Half-translation structure
(atlas of charts with transition maps of
the form $z \mapsto \pm z + c$)

Paths in Teichmüller space

Quadratic differential \leftrightarrow Half-translation structure
(atlas of charts with transition maps of
the form $z \mapsto z + c$)

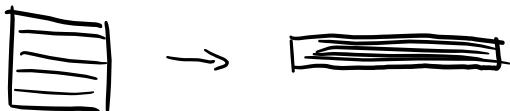
$SL(2, \mathbb{R})$ acts on these by post-composing the charts.

Paths in Teichmüller space

Quadratic differential \leftrightarrow Half-translation structure
(atlas of charts with transition maps of
the form $z \mapsto z + c$)

$SL(2, \mathbb{R})$ acts on these by post-composing the charts.

Teichmüller geodesic = image of a given QD / HTS under the
diagonal sbgp $\left\{ \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R} \right\}$

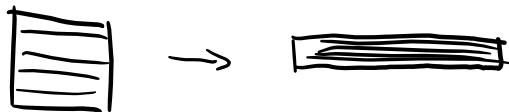


Paths in Teichmüller space

Quadratic differential \leftrightarrow Half-translation structure
 (atlas of charts with transition maps of the form $z \mapsto z + c$)

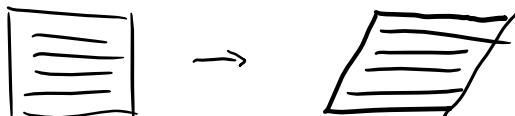
$SL(2, \mathbb{R})$ acts on these by post-composing the charts.

Teichmüller geodesic = image of a given QD / HTS under the diagonal sbgp $\left\{ \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R} \right\}$



horocycle = image by the upper triangular sbgp

$$\left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} : s \in \mathbb{R} \right\}$$

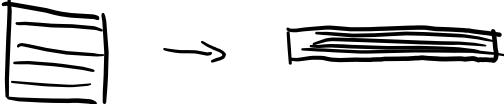


Paths in Teichmüller space

Quadratic differential \leftrightarrow Half-translation structure
 (atlas of charts with transition maps of the form $z \mapsto z + c$)

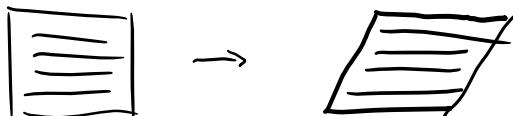
$SL(2, \mathbb{R})$ acts on these by post-composing the charts.

Teichmüller geodesic = image of a given QD / HTS under the diagonal sbgp $\left\{ \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R} \right\}$

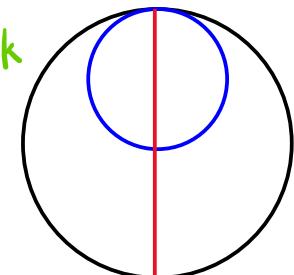


horocycle = image by the upper triangular sbgp

$$\left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} : s \in \mathbb{R} \right\}$$



The whole $SL(2, \mathbb{R})$ -orbit of a QD / HTS is a Teichmüller disk isomorphic to $SL(2, \mathbb{R}) / SO(2) \cong \mathbb{H}^2$, within which these paths are a geodesic and a horocycle respectively



Convergence of paths

Thurston compactification	Gardiner-Masur compactification
stretch paths	Teichmüller rays
earthquakes	horocycles

Convergence of paths

Thurston compactification	Gardiner-Masur compactification
stretch paths ✓	Teichmüller rays ✓
earthquakes ✓	horocycles

Convergence of paths

Thurston compactification	Gardiner-Masur compactification
stretch paths ✓	Teichmüller rays ✓
earthquakes ✓	horocycles ✗

Convergence of paths

Thurston compactification	Gardiner-Masur compactification
stretch paths ✓	Teichmüller rays ✓
earthquakes ✓	horocycles ✗

Thm (FB) If $\dim_{\mathbb{C}} \Gamma(s) > 1$, then $\Gamma(s)$ contains a horocycle that does not have a limit in the Gardiner-Masur compactification.

Convergence of paths

Thurston compactification	Gardiner-Masur compactification
stretch paths ✓	Teichmüller rays ✓
earthquakes ✓	horocycles ✗

Thm (FB) If $\dim_{\mathbb{C}} \Gamma(s) > 1$, then $\Gamma(s)$ contains a horocycle that does not have a limit in the Gardiner-Masur compactification.

Rmk If the horizontal foliation is a single cylinder or uniquely ergodic, then the horocycle converges
(Albrege)

Convergence of paths

Thurston compactification	Gardiner-Masur compactification
stretch paths ✓	Teichmüller rays ✓
earthquakes ✓	horocycles ✗

Thm (FB) If $\dim_{\mathbb{C}} \Gamma(s) > 1$, then $\gamma(s)$ contains a horocycle that does not have a limit in the Gardiner-Masur compactification.

Rmk If the horizontal foliation is a single cylinder or uniquely ergodic, then the horocycle converges (Alberge)

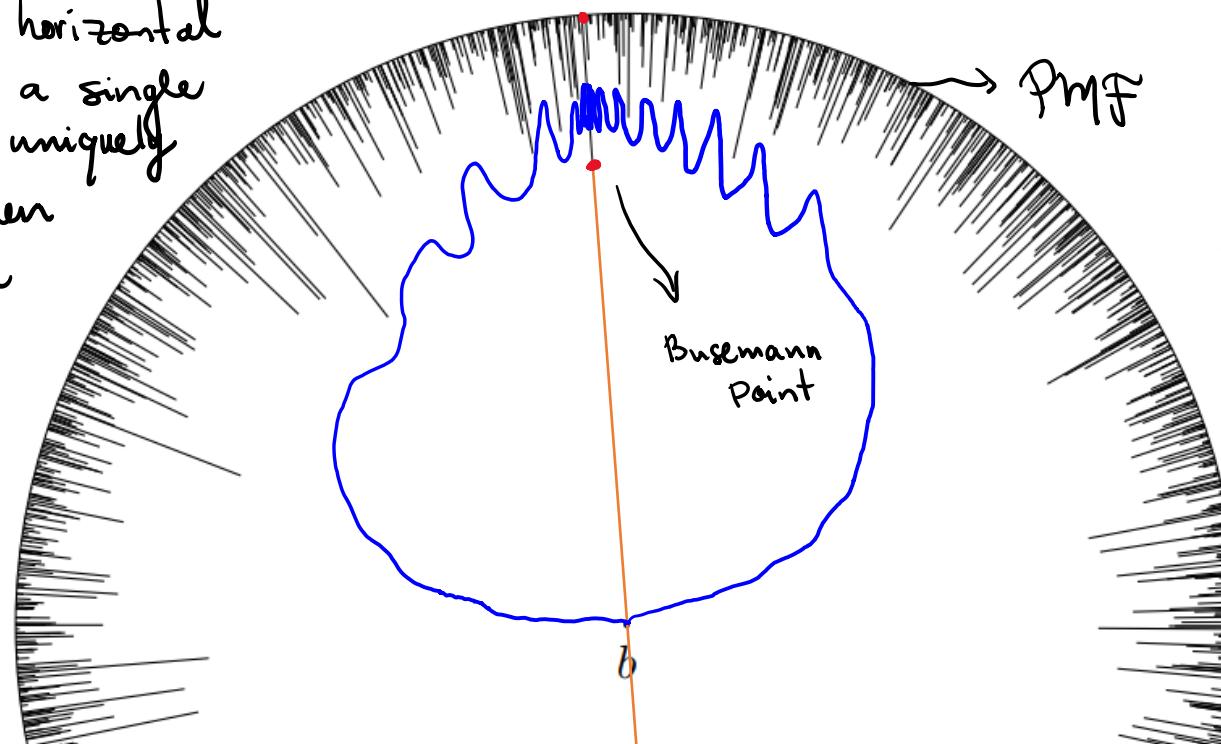


figure
courtesy
of

Aitor Azemar

Convergence of paths

Thurston compactification	Gardiner-Masur compactification
stretch paths ✓	Teichmüller rays ✓
earthquakes ✓	horocycles ✗

Thm (FB) If $\dim_{\mathbb{C}} \Gamma(s) > 1$, then $\gamma(s)$ contains a horocycle that does not have a limit in the Gardiner-Masur compactification.

Rmk If the horizontal foliation is a single cylinder or uniquely ergodic, then the horocycle converges (Alberge)

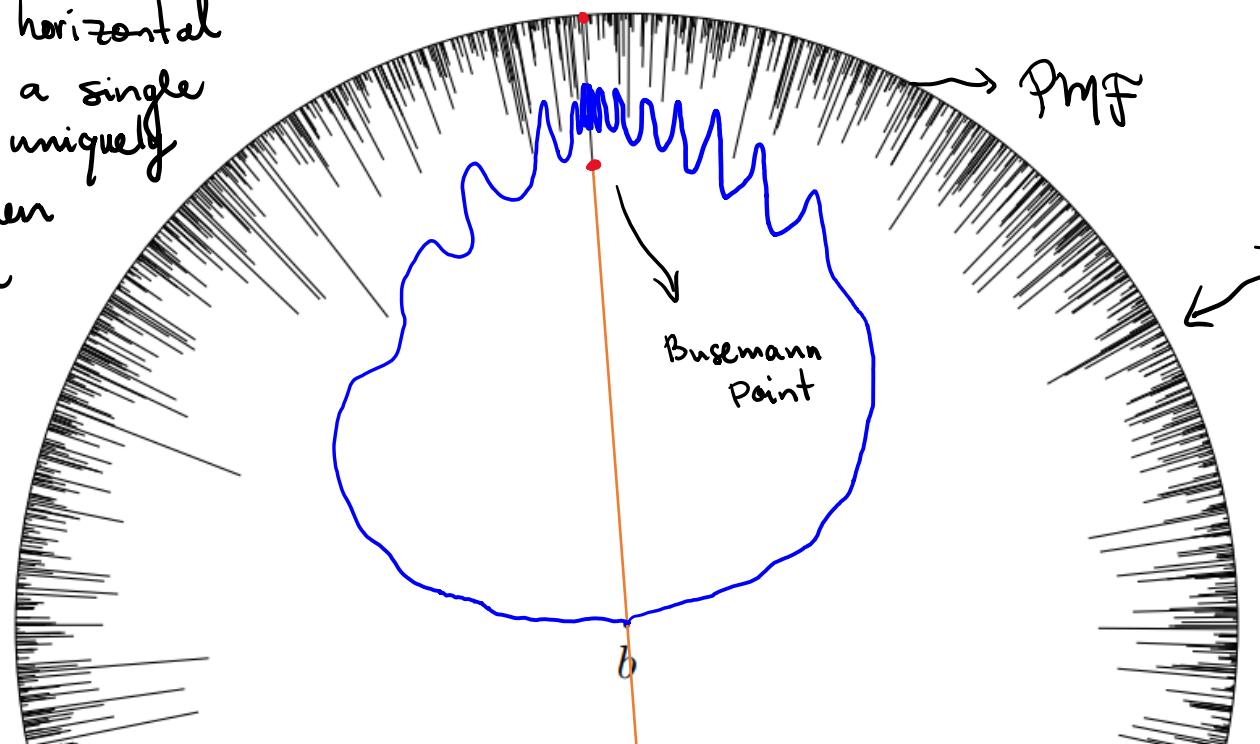


figure
courtesy
of

Aitor Azemar

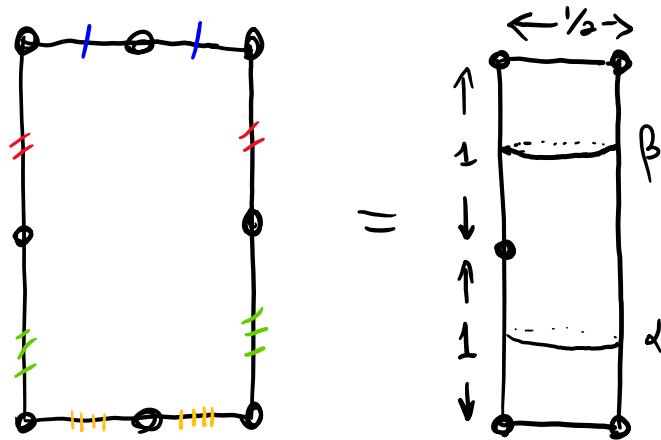
The example

By covering constructions, it suffices to find an example for the 5-times-punctured sphere. ($T(S_{0,5}) \hookrightarrow T(S_{g,n})$ whenever $3g - 3 + n > 1$)

The example

By covering constructions, it suffices to find an example for the 5-times-punctured sphere. ($T(S_{0,5}) \hookrightarrow T(S_{g,n})$ whenever $3g - 3 + n > 1$)

Consider the following QD/TTS:

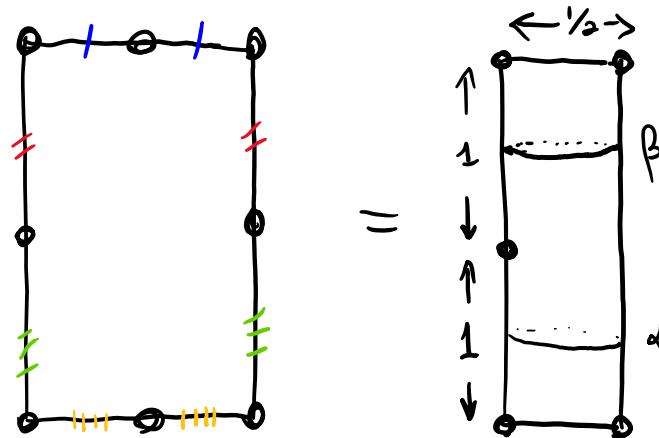


Call the quadratic diff q
and the complex structure X .

The example

By covering constructions, it suffices to find an example for the 5-times-punctured sphere. ($T(S_{0,5}) \hookrightarrow T(S_{g,n})$ whenever $3g - 3 + n > 1$)

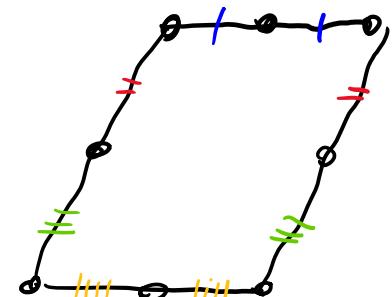
Consider the following QD/HTS:



Call the quadratic diff q
and the complex structure X .

Applying the matrix $h_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ simply shifts the punctures with respect to each other.

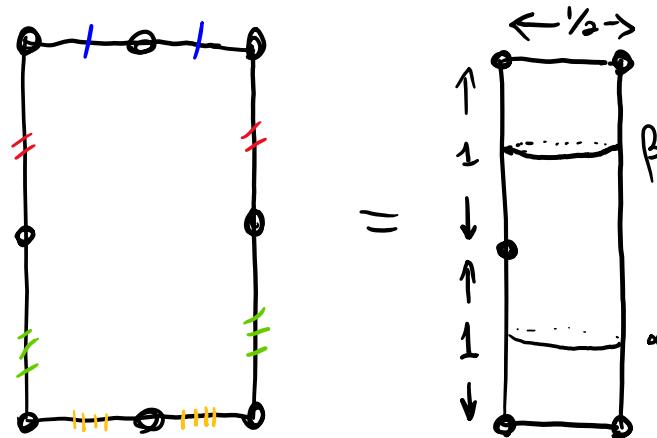
Moreover, $h_1 = T_\alpha \circ T_\beta$ ↪ Dehn twists



The example

By covering constructions, it suffices to find an example for the 5-times-punctured sphere. ($T(S_{0,5}) \hookrightarrow T(S_{g,n})$ whenever $3g - 3 + n > 1$)

Consider the following QD/TTS:

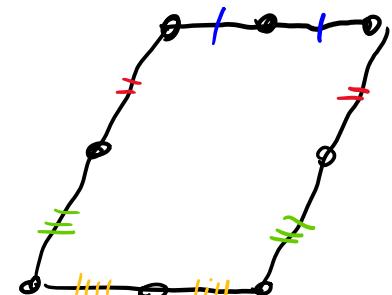


Call the quadratic diff q
and the complex structure X .

Applying the matrix $h_S = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ simply shifts the punctures with respect to each other.

Moreover, $h_1 = T_\alpha \circ T_\beta$ ↪ Dehn twists

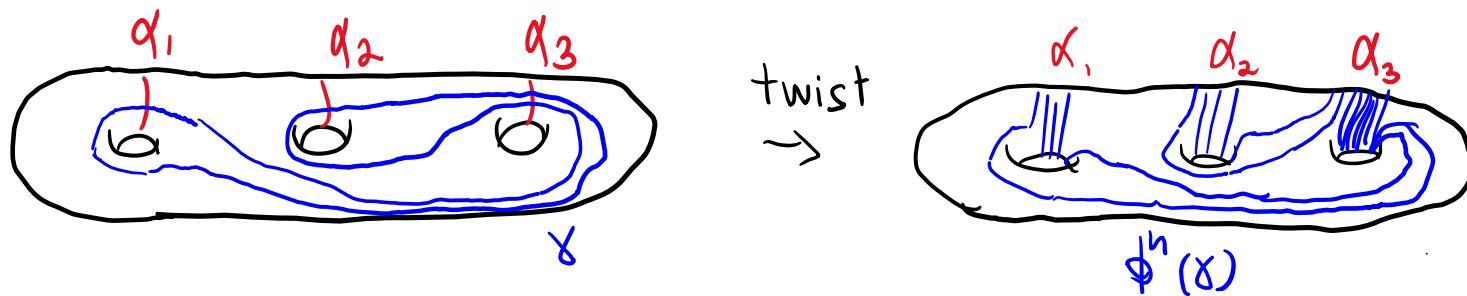
By the argument of Gardiner-Masur, $h_n q = (T_\alpha^n \circ T_\beta^n) \cdot X$ does not converge to a point in PMF. In fact, we can compute its limit.



Limits of Dehn multi-twists

Lemma (Ivanov) Let $\phi = T_1 \circ \dots \circ T_k$ be a Dehn multi-twist about a multicurve $\{\alpha_1, \dots, \alpha_k\}$. Then for any $\gamma \in \mathcal{C}(S)$,

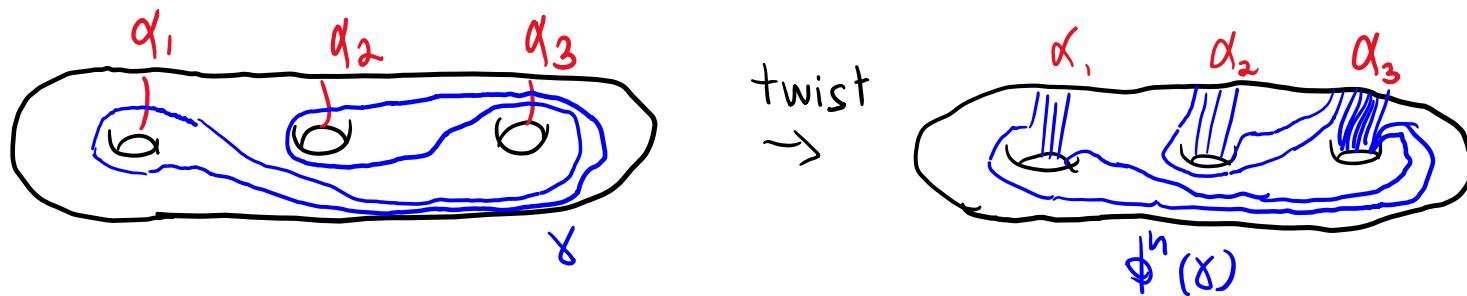
$$\frac{1}{n} \phi^n(\gamma) \rightarrow \sum_{j=1}^k i(\gamma, \alpha_j) \cdot \alpha_j \quad \text{as } n \rightarrow \infty.$$



Limits of Dehn multi-twists

Lemma (Ivanov) Let $\phi = \tau_1 \circ \dots \circ \tau_k$ be a Dehn multi-twist about a multicurve $\{\alpha_1, \dots, \alpha_k\}$. Then for any $\gamma \in \mathcal{C}(S)$,

$$\frac{1}{n} \phi^n(\gamma) \rightarrow \sum_{j=1}^k i(\gamma, \alpha_j) \cdot \alpha_j \quad \text{as } n \rightarrow \infty.$$

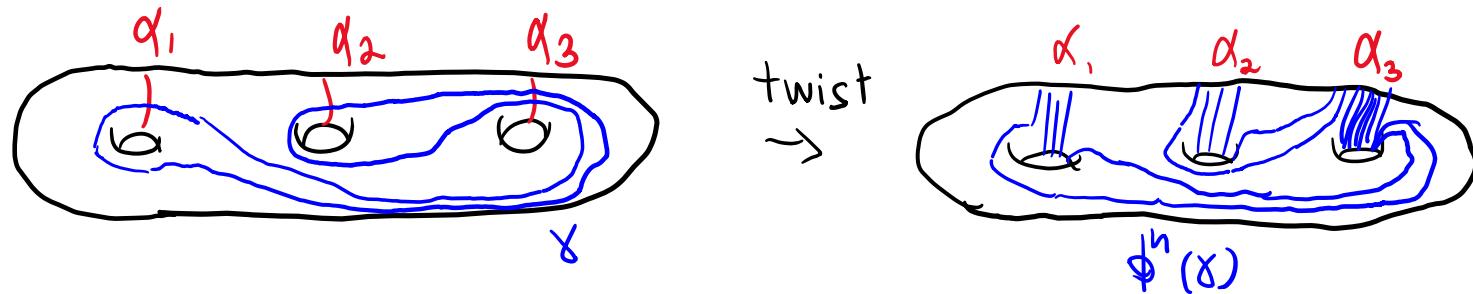


Cor If $\phi = \tau_1 \circ \dots \circ \tau_k$ and $\gamma \in \mathcal{T}(S)$, then $\phi^{-n}\gamma$ converges to $\left[\sqrt{\text{EL}\left(\sum_{j=1}^k i(\gamma, \alpha_j) \alpha_j, \gamma \right)} \right]_{\gamma \in \mathcal{C}(S)}$ in $\overline{\mathcal{T}(S)}^{\text{GM}}$ as $n \rightarrow \infty$.

Limits of Dehn multi-twists

Lemma (Ivanov) Let $\phi = T_1 \circ \dots \circ T_k$ be a Dehn multi-twist about a multicurve $\{\alpha_1, \dots, \alpha_k\}$. Then for any $\gamma \in \mathcal{C}(S)$,

$$\frac{1}{n} \phi^n(\gamma) \rightarrow \sum_{j=1}^k i(\gamma, \alpha_j) \cdot \alpha_j \quad \text{as } n \rightarrow \infty.$$



Cor If $\phi = T_1 \circ \dots \circ T_k$ and $y \in \mathcal{T}(S)$, then $\phi^{-n}y$ converges to

$$\left[\sqrt{\text{EL}} \left(\sum_{j=1}^k i(\gamma, \alpha_j) \alpha_j, y \right) \right]_{\gamma \in \mathcal{C}(S)} \text{ in } \overline{\mathcal{T}(S)}^{\text{GM}} \text{ as } n \rightarrow \infty.$$

Pf Under the GM embedding, $\phi^{-n}y$ maps to $\left[\sqrt{\text{EL}} (\gamma, \phi^{-n}y) \right]_{\gamma \in \mathcal{C}(S)}$

$$= \left[\sqrt{\text{EL}} (\phi^n(\gamma), y) \right]_{\gamma \in \mathcal{C}(S)} = \left[\sqrt{\text{EL}} \left(\frac{\phi^n(\gamma)}{n}, y \right) \right]_{\gamma \in \mathcal{C}(S)} \xrightarrow{n \rightarrow \infty} \left[\sqrt{\text{EL}} \left(\sum i(\gamma, \alpha_j) \alpha_j, y \right) \right]_y$$

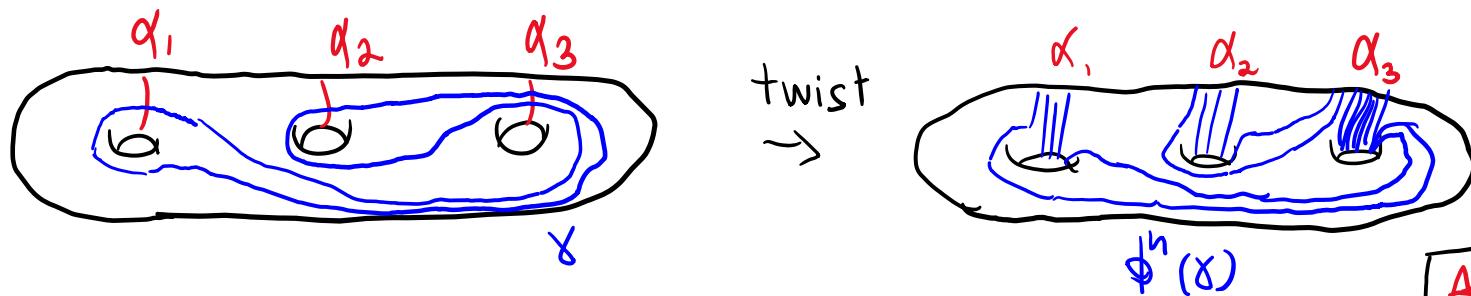
by continuity of EL on MF

□

Limits of Dehn multi-twists

Lemma (Ivanov) Let $\phi = T_1 \circ \dots \circ T_k$ be a Dehn multi-twist about a multicurve $\{\alpha_1, \dots, \alpha_k\}$. Then for any $\gamma \in \mathcal{C}(S)$,

$$\frac{1}{n} \phi^n(\gamma) \rightarrow \sum_{j=1}^k i(\gamma, \alpha_j) \cdot \alpha_j \quad \text{as } n \rightarrow \infty.$$



Cor If $\phi = T_1 \circ \dots \circ T_k$ and $y \in T(S)$, then $\phi^{-n}y$ converges to

$$\left[\sqrt{\text{EL}\left(\sum_{j=1}^k i(y, \alpha_j) \alpha_j, y \right)} \right]_{y \in \mathcal{C}(S)} \text{ in } \overline{T(S)}^{\text{GM}} \text{ as } n \rightarrow \infty.$$

Pf Under the GM embedding, $\phi^{-n}y$ maps to $\left[\sqrt{\text{EL}(y, \phi^{-n}y)} \right]_{y \in \mathcal{C}(S)}$

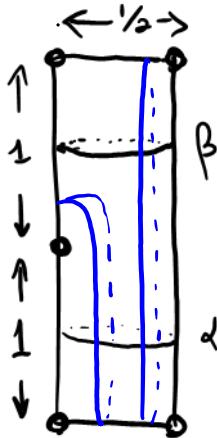
$$= \left[\sqrt{\text{EL}(\phi^n(y), y)} \right]_{y \in \mathcal{C}(S)} = \left[\sqrt{\text{EL}\left(\frac{\phi^n(y)}{n}, y \right)} \right]_{y \in \mathcal{C}(S)} \xrightarrow{n \rightarrow \infty} \left[\sqrt{\text{EL}\left(\sum i(y, \alpha_j) \alpha_j, y \right)} \right]_y$$

by continuity of EL on MF

Azemar used this to show that Bugemann points are not dense in the horoboundary

□

Back to the example



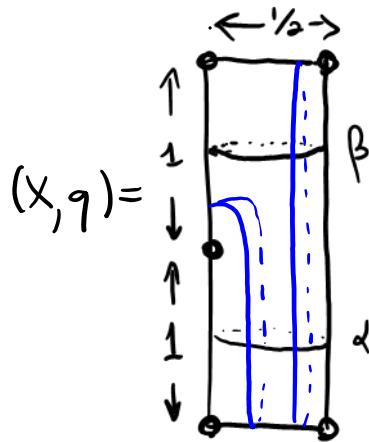
$(X, q) =$

Write X_s for the surface underlying $h_S q$ and $\phi = \tau_\alpha \circ \tilde{\tau}_\beta$.

Then $X_{S+n} = \phi^n \cdot X_s$ converges to

$$V_s = [JEL(i(\gamma, \alpha)\alpha + i(\gamma, \beta)\beta, X_s)]_{\gamma \in C(S)} \quad \text{as } n \rightarrow \infty$$

Back to the example



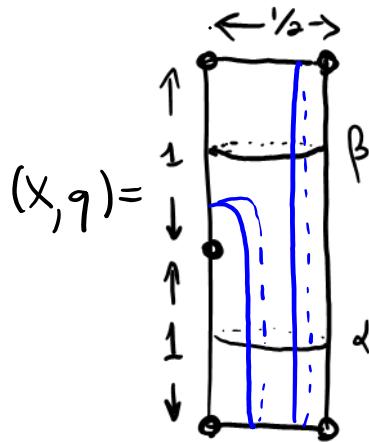
Write X_s for the surface underlying $h_S q$ and $\phi = \tau_\alpha \circ \tilde{\tau}_\beta$.

Then $X_{s+n} = \phi^n \cdot X_s$ converges to

$$V_s = [JEL(i(\delta, \alpha)\alpha + i(\delta, \beta)\beta, X_s)]_{\delta \in C(S)} \quad \text{as } n \rightarrow \infty$$

We want to show that this limit depends on s .

Back to the example



Write X_s for the surface underlying (X, q) and $\phi = \tau_\alpha \circ \tau_\beta$.

Then $X_{s+n} = \phi^n \cdot X_s$ converges to

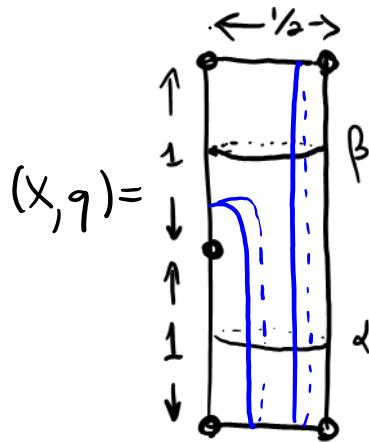
$$V_s = [\int_{\gamma \in C(S)} EL(i(\gamma, \alpha)\alpha + i(\gamma, \beta)\beta, X_s)] \quad \text{as } n \rightarrow \infty$$

We want to show that this limit depends on s .

By taking a curve γ intersecting α and β the same # of times we get

$$\sqrt{EL(\alpha + \beta, X_s)} = \sqrt{\text{area}(X_s)} = \sqrt{2}.$$

Back to the example



Write X_s for the surface underlying (X, q) and $\phi = \tau_\alpha \circ \tilde{\tau}_\beta$.

Then $X_{s+n} = \phi^n \cdot X_s$ converges to

$$V_s = [\int_{\gamma \in C(S)} \text{EL}(\gamma, \alpha) \alpha + i(\gamma, \beta) \beta, X_s] \quad \text{as } n \rightarrow \infty$$

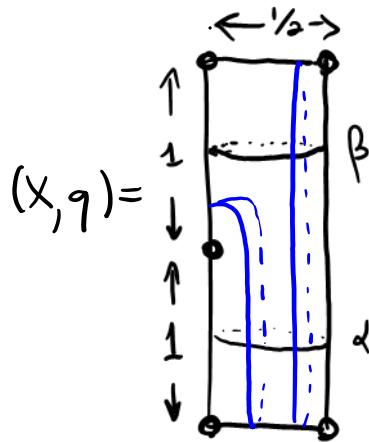
We want to show that this limit depends on s .

By taking a curve γ intersecting α and β the same # of times we get

$$\sqrt{\text{EL}(\alpha + \beta, X_s)} = \sqrt{\text{area}(X_s)} = \sqrt{2}.$$

If γ intersects only α , we get $\sqrt{\text{EL}(\alpha, X_s)}$.

Back to the example



Write X_s for the surface underlying $h_S q$ and $\phi = \tau_\alpha \circ \tilde{\tau}_\beta$.

Then $X_{s+n} = \phi^n \cdot X_s$ converges to

$$V_s = [\text{JEL}(i(\gamma, \alpha)\alpha + i(\gamma, \beta)\beta, X_s)]_{\gamma \in C(S)} \quad \text{as } n \rightarrow \infty$$

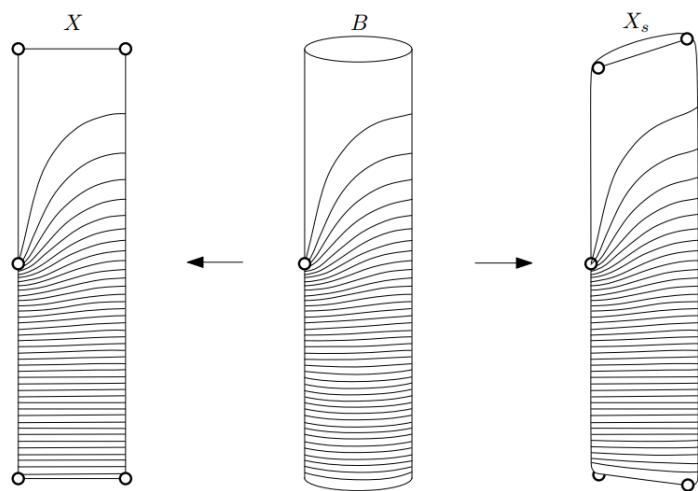
We want to show that this limit depends on s .

By taking a curve γ intersecting α and β the same # of times we get $\sqrt{\text{EL}(\alpha + \beta, X_s)} = \sqrt{\text{area}(X_s)} = \sqrt{2}$.

If γ intersects only α , we get $\sqrt{\text{EL}(\alpha, X_s)}$.

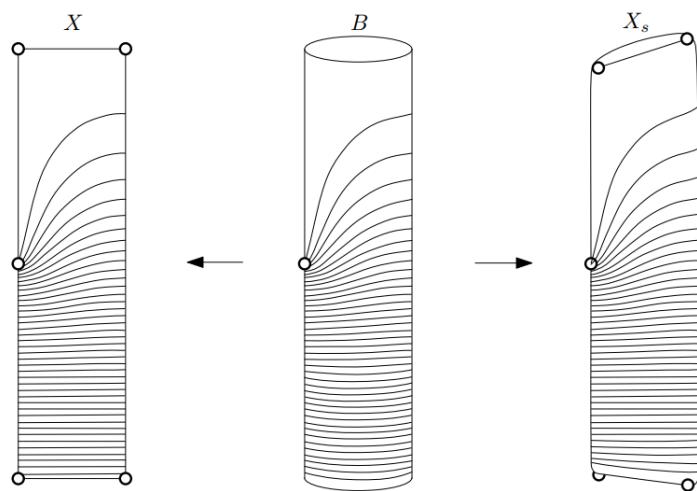
To prove that V_s is not constant, it suffices to show that $\text{EL}(\alpha, X_s)$ is not constant.

Finishing the proof



By symmetry, the extremal annulus A for α on X is disjoint from the top and bottom segments.

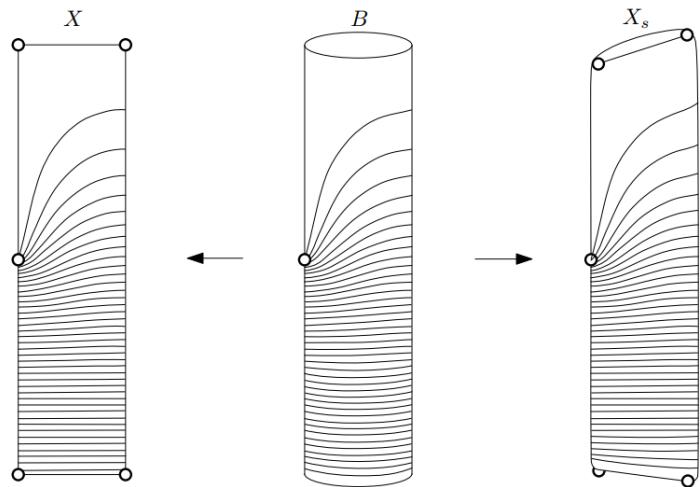
Finishing the proof



By symmetry, the extremal annulus A for α on X is disjoint from the top and bottom segments.

$\Rightarrow A$ embeds conformally into X_s for any $s \in \mathbb{R}$.

Finishing the proof

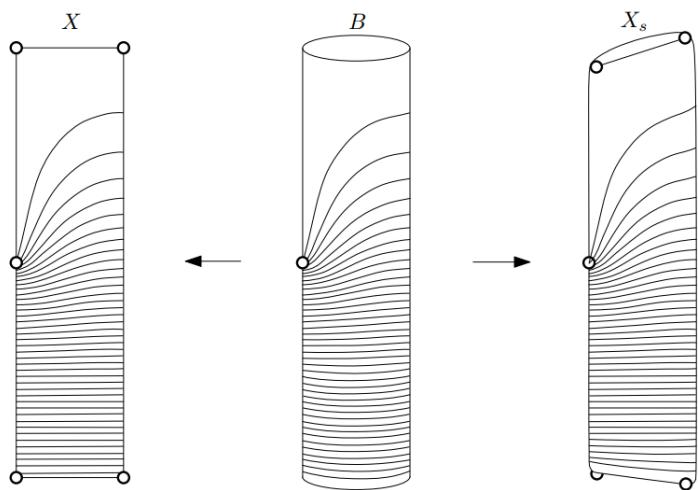


By symmetry, the extremal annulus A for α on X is disjoint from the top and bottom segments.

$\Rightarrow A$ embeds conformally into X_s for any $s \in \mathbb{R}$.

However, one can show that for small $s \neq 0$, $g|_A$ does not extend to a quadratic differential on X_s (the gluing is not isometric), hence A is not extremal on X_s .

Finishing the proof



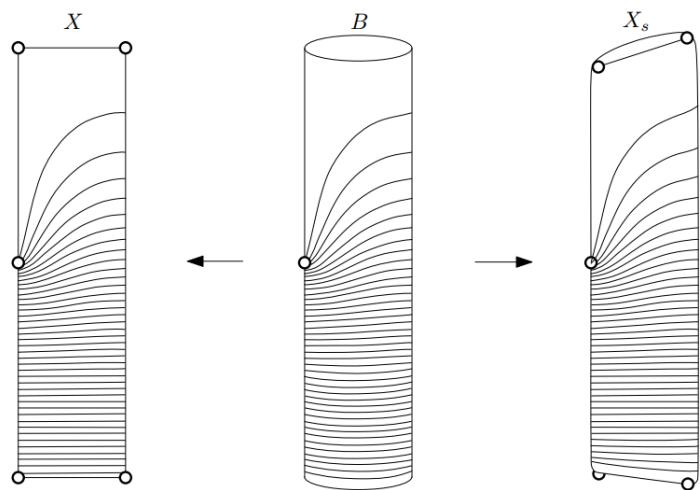
By symmetry, the extremal annulus A for α on X is disjoint from the top and bottom segments.

$\Rightarrow A$ embeds conformally into X_s for any $s \in \mathbb{R}$.

However, one can show that for small $s \neq 0$, $g|_A$ does not extend to a quadratic differential on X_s (the gluing is not isometric), hence A is not extremal on X_s .

$\Rightarrow EL(\alpha, X_s) < EL(\alpha, X_0) \Rightarrow EL(\alpha, X_s)$ is not constant

Finishing the proof



By symmetry, the extremal annulus A for α on X is disjoint from the top and bottom segments.

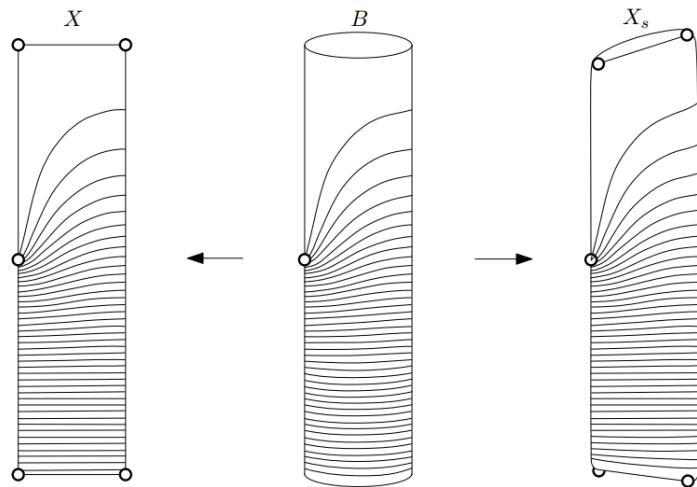
$\Rightarrow A$ embeds conformally into X_s for any $s \in \mathbb{R}$.

However, one can show that for small $s \neq 0$, $g|_A$ does not extend to a quadratic differential on X_s (the gluing is not isometric), hence A is not extremal on X_s .

$\Rightarrow EL(\alpha, X_s) < EL(\alpha, X_0) \Rightarrow EL(\alpha, X_s)$ is not constant

$\Rightarrow (X_{s+n})_{n \geq 1}$ and $(X_n)_{n \geq 1}$ have different limits

Finishing the proof



By symmetry, the extremal annulus A for α on X is disjoint from the top and bottom segments.

$\Rightarrow A$ embeds conformally into X_s for any $s \in \mathbb{R}$.

However, one can show that for small $s \neq 0$, $g|_A$ does not extend to a quadratic differential on X_s (the gluing is not isometric), hence A is not extremal on X_s .

$\Rightarrow EL(\alpha, X_s) < EL(\alpha, X_0) \Rightarrow EL(\alpha, X_s)$ is not constant

$\Rightarrow (X_{s+n})_{n \geq 1}$ and $(X_n)_{n \geq 1}$ have different limits

\Rightarrow the horocycle diverges.

□

Thank
you !