Triangulated persistence categories

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A fews words to start...

Slogan: study topology on the object-set of a category!

$$\begin{array}{ccc} & & \text{triangulated category} \\ & & \downarrow \\ \text{persistence } \mathbf{k}\text{-module} & \longrightarrow & \text{triangulated persistence category (TPC)} \end{array}$$

⋆ TPC summarizes to be a refinement of a triangulated category, taking account of a *filtration structure* of the hom-sets.

⋆ Recent studies in the similar directions:

- Orlov: *Remarks on generators and dimensions of triangulated categories* (2009).
- Dimitrov-Haiden-Katzarkov-Kontsevich: *Dynamical systems and categories* (2014).
- Fan-Filip: *Asymptotic shifting numbers in triangulated categories* (2020). (Suggested by L. Polterovich)

⋆ (Communicated with M. Usher) TPC is different from *persistence triangulated category* (PTC).

Main result

Theorem (Biran-Cornea-Z., 2020)

Given a triangulated persistence category $\mathscr{C}$, for any $\mathscr{F} \subset \mathrm{Ob}(\mathscr{C})$, the following quantities are well-defined,

- *for $A, B \in \mathrm{Ob}(\mathscr{C})$, the fragmentation pseudo-metric $d^{\mathscr{F}}(A, B)$;*
- *for endofunctor F on $\mathscr{C}$, the categorical entropy $h^{\mathscr{F}}(F) \in [0, \infty)$.*

Moreover, $(\mathrm{Ob}(\mathscr{C}), \mathrm{Top}_{d^{\mathscr{F}}})$ is an H-space.

Example (Examples of TPC)

(a) Homology category of a filtered pre-triangulated dg-category.

(b) (*In progress*) Derived Fukaya category in an exact case.

(c) Tamarkin category (constructed from microlocal sheaf theory).

(d) Homotopy category of the cofibration category of topological spaces with potentials.

(e) (*In progress*) Filtered singularity category via filtered homotopy category of matrix factorization.

In this talk, we will only focus on (a). Application (in discussion)

Outline of the talk

$$\begin{array}{c}\text{Definition of a persistence category}\\ \downarrow \\ \text{Definition of a TPC and propositions}\\ \downarrow \\ \text{Quantities - } d^{\mathscr{F}}(A,B) \text{ and } h^{\mathscr{F}}(F)\\ \downarrow \\ \text{The limit category of a TPC}\\ \downarrow \\ \text{Example (a) - filtered pre-triangulated dg-category (if we have time)}\end{array}$$

Persistence category

View $(\mathbb{R}, \leq)$ as a category, where $\mathrm{Mor}_{\mathbb{R}}(r,s) = \{i_{r,s}\}$ only if $r \leq s$. It admits an additive structure, i.e. a bifunctor $\oplus : (\mathbb{R}, \leq) \times (\mathbb{R}, \leq) \to (\mathbb{R}, \leq)$.

Definition

A category $\mathscr{C}$ is called a **persistence category** if for any $A, B \in \mathrm{Ob}(\mathscr{C})$, there exists a functor $E_{A,B} : (\mathbb{R}, \leq) \to \mathrm{Vect}_{\mathbf{k}}$ satisfying

(i) the hom-set in $\mathscr{C}$ is $\mathrm{Hom}_{\mathscr{C}}(A,B) = \{(f,r) \,|\, f \in E_{A,B}(r)\}$. For later use, denote $\mathrm{Mor}^r(A,B) := E_{A,B}(r)$, and $\bar{f} = (f,r)$ where $\lceil \bar{f} \rceil = r$.

(ii) for $r \leq r'$ and $s \leq s'$, we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Mor}^r(A,B) \times \mathrm{Mor}^s(B,C) \arrow[r, "\circ_{(r,s)}"] \arrow[d, "E_{A,B}(i_{r,r'}) \times E_{B,C}(i_{s,s'})"'] & \mathrm{Mor}^{r+s}(A,C) \arrow[d, "E_{A,C}(i_{r+s,r'+s'})"] \\
\mathrm{Mor}^{r'}(A,B) \times \mathrm{Mor}^{s'}(B,C) \arrow[r, "\circ_{(r',s')}"] & \mathrm{Mor}^{r'+s'}(A,C)
\end{tikzcd}$$

Each $E_{A,B}$ is called a persistence $\mathbf{k}$-module (without regularity conditions) and $\{E_{A,B}(i_{r,s})\}_{r \leq s}$ are called structure maps. In general, given any persistence $\mathbf{k}$-module $\mathbb{V} = \{V_t\}_{t \in \mathbb{R}}$, denote by $\mathbb{V}[r] := \{V_{t+r}\}_{t \in \mathbb{R}}$.

Induced categories from a persistence category

Definition

Let $\mathscr{C}$ be a persistence category.

(a) Denote by $\mathscr{C}_0$ the category with the same objects as $\mathscr{C}$ but

$$\mathrm{Mor}_{\mathscr{C}_0}(A, B) := \mathrm{Mor}^0(A, B).$$

(b) Denote by $\mathscr{C}_\infty$ the category with the same objects as $\mathscr{C}$ but

$$\mathrm{Mor}_{\mathscr{C}_\infty}(A, B) := \varinjlim_{r \to \infty} \mathrm{Mor}^r(A, B)$$

where the direct limit is taken with respect to the morphisms $i_{r,s} : \mathrm{Mor}^r(A, B) \to \mathrm{Mor}^s(A, B)$ for any $r \leq s$.

Remark

The induced categories $\mathscr{C}_0$ and $\mathscr{C}_\infty$ are pre-additive, but $\mathscr{C}$ is not.

Remark

When $\mathrm{Ob}(\mathscr{C}) = \{*\}$, then $\mathscr{C}$ is identified with a persistence $\mathbf{k}$-algebra.

Other ingredients in a persistence category

- Persistence functor $\mathscr{F} : \mathscr{C} \to \mathscr{C}'$:

$$\mathscr{F}_{A,B} = \left\{ (\mathscr{F}_{A,B})_r : \mathrm{Mor}^r_{\mathscr{C}}(A,B) \to \mathrm{Mor}^r_{\mathscr{C}'}(\mathscr{F}(A), \mathscr{F}(B)) \right\}_{r \in \mathbb{R}}.$$

In other words, a persistence morphism.

- Persistence natural transformation $\eta : \mathscr{F} \to \mathscr{G}$ of **shift** $\alpha \in \mathbb{R}$ is a natural transformation where

$$\eta_A \in \mathrm{Mor}^{\alpha}(\mathscr{F}(A), \mathscr{G}(A)) \ + \ \text{other compatibility properties.}$$

- Shift functor on $\mathscr{C}$ is a $\mathbb{R}$-parametrized family of persistence functors on $\mathscr{C}$ denoted by $\Sigma = \{\Sigma^r\}_{r \in \mathbb{R}}$ such that $\Sigma^r \circ \Sigma^s = \Sigma^{r+s}$ and for **any** $r, s \in \mathbb{R}$,

$$\eta_{r,s} : \Sigma^r \to \Sigma^s \text{ is a natural transformation of shift } s - r.$$

In particular, for $A \in \mathrm{Ob}(\mathscr{C})$, we have $\eta_{r,0}(A) \in \mathrm{Mor}^{-r}(\Sigma^r A, A)$.

Acyclic objects

Definition

For $r \geq 0$, an object K in $\mathscr{C}$ is r-**acyclic** if $i_{0,r}(e_K) = 0$.

Remark

An equivalent way to define K to be r-acyclic is $i_{-r,0}((\eta_{r,0})_K) = 0$. Recall that $(\eta_{r,0})_K \in \mathrm{Mor}^{-r}(\Sigma^r K, K)$ and $i_{-r,0}$ shifts it to level 0. For brevity, denote $\eta^r_K = i_{-r,0}((\eta_{r,0})_K)$.

Remark (An equivalence relation on morphisms)

For $f, g \in \mathrm{Mor}^{\alpha}(A, B)$, we call f and g are r-**equivalent** for some $r \geq 0$ if $i_{\alpha,\alpha+r}(f - g) = 0$. Denote by $f \simeq_r g$ if f and g are r-equivalent. (Then K is r-acyclic if $e_K \simeq_r 0$.)

Example

Let $\mathscr{P}$ be the category of persistence **k**-modules (which admit barcodes). Then $\mathscr{P}$ can be enriched to be a persistence category $\mathscr{C}$. The r-acyclic objects are precisely those persistence **k**-modules with only finite bars and boundary depths no greater than r.

Towards TPC

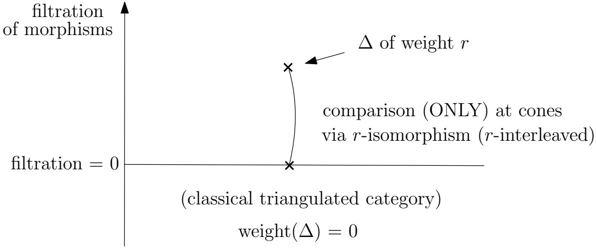
TPC $\approx$ persistence category + triangulated structure.

$\star$ Naive goal: use *filtrations of morphisms* to associate weights to exact triangles Δ (such that certain properties are satisfied).

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "?"', "\simeq"] & B \arrow[r, "g"] \arrow[d, "?"', "\simeq"] & C \arrow[r, "h"] \arrow[d, "?"', "\simeq"] & A \arrow[d, "?"', "\simeq"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A' \end{tikzcd}$$

? : iso $\simeq$ in a per. cat admits a filtration

$\star$ Our approach:



Triangulated persistence category

Definition

A **triangulated persistence category** (TPC) is a persistence category $\mathscr{C}$ endowed with a shift functor Σ satisfying

(i) the category $\mathscr{C}_0$ is triangulated, and for any $X \in \mathrm{Ob}(\mathscr{C})$ and any $r \in \mathbb{R}$, the functors $\mathrm{Mor}^r(X, \cdot)$ and $\mathrm{Mor}^r(\cdot, X)$ are exact on $\mathscr{C}_0$;

(ii) for any $r \geq 0$ and any $A \in \mathrm{Ob}(\mathscr{C})$, the morphism $\eta_r^A : \Sigma^r A \to A$ embeds into an exact triangle of $\mathscr{C}_0$ as follows,

$$\Sigma^r A \xrightarrow{\eta_r^A} A \to K \to \Sigma^r A$$

which has the property that K is r-acyclic.

One should view condition (ii) above as a generalization of the axiom that $A \xrightarrow{e_A} A \to 0 \to A$ is an exact triangle in $\mathscr{C}_0$.

Theorem

TPC admits a weighted version of triangulated axioms.

In this talk, we will only state the Weighted Octahedral Axiom.

Weight r exact triangle in $\mathscr{C}$ denoted by $w(\Delta) = r$

Definition ("oversimplified")

A **strict exact triangle** Δ **of weight** r **in** $\mathscr{C}$ is the following diagram

$$A \xrightarrow{\bar{u}} B \xrightarrow{\bar{v}} C \xrightarrow{\bar{w}} \Sigma^{-r} A$$

for $r \geq 0$ with $\bar{u} \in \mathrm{Mor}^0(A, B)$, $\bar{v} \in \mathrm{Mor}^0(B, C)$, $\bar{w} \in \mathrm{Mor}^0(C, \Sigma^{-r} A)$ such that there exists an exact triangle

$$A \xrightarrow{u} B \xrightarrow{v} C' \xrightarrow{w} A \quad \text{in } \mathscr{C}_0,$$

and $\phi \in \mathrm{Mor}^0(C', C)$ and $\psi \in \mathrm{Mor}^0(\Sigma^r C, C')$ such that in the following diagram

$$\begin{tikzcd}
 & & \Sigma^r C \arrow[d, "\psi"'] \arrow[dr, "\Sigma^r \bar{w}"] & \\
A \arrow[r, "u"] & B \arrow[r, "v"] \arrow[dr, "\bar{v}"'] & C' \arrow[r, "w"] \arrow[d, "\phi"] & A \\
 & & C &
\end{tikzcd}$$

we have $\bar{u} = u$, $\bar{v} = \phi \circ v$, $\Sigma^r \bar{w} = w \circ \psi$, and $\phi \circ \psi = \eta_r^C$.

Weighted Octahedral Axiom

Proposition

Consider the following diagram formed by two strict exact triangles $\Delta_1 : E \to F \to X \xrightarrow{k} \Sigma^{-r}E$ *and* $\Delta_2 : X \to A \to B \xrightarrow{b} \Sigma^{-s}X$ *which satisfy* $w(\Delta_1) = r$ *and* $w(\Delta_2) = s$.

$$\begin{tikzcd}
E \arrow[d] & & & \\
F \arrow[d] & & & \\
X \arrow[r] \arrow[d, "k"'] & A \arrow[r] & B \arrow[r, "b"] & \Sigma^{-s}X \\
\Sigma^{-r}E & & &
\end{tikzcd}$$

It can be completed into the following commutative diagram

Weighted Octahedral Axiom (continue)

Proposition

except that the right bottom square is r-commutative

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & E \arrow[d] & \\
F \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & F \\
X \arrow[r] \arrow[d] & A \arrow[r] & B \arrow[r] \arrow[d] & \Sigma^{-s}X \arrow[d] \\
\Sigma^{-r}E & & \Sigma^{-r-s}E \arrow[r] & \Sigma^{-r-s}E
\end{tikzcd}$$

where $\Delta_3 : F \to A \to C \to F$ *and* $\Delta_4 : E \to C \to B \to \Sigma^{-r-s}E$ *are strict exact triangles satisfying* $w(\Delta_3) = 0$ *and* $w(\Delta_4) = r + s$.

The way we chose to define the weight of a triangle is to guarantee that this weighted octahedral axiom behaves in the expected "additive" way.

Iterated cone decomposition in $\mathscr{C}$

The following definition is an analogue of the cone-decomposition from Biran-Cornea's Lagrangian cobordism theory.

Definition

Let $\mathscr{C}$ be a TPC and $X \in \mathrm{Ob}(\mathscr{C})$. An **iterated cone decomposition** D **of** X **with the linearization** $(X_0, X_1, ..., X_n)$ where $X_i \in \mathrm{Ob}(\mathscr{C})$ consists of a family of strict exact triangles in $\mathscr{C}$

$$\begin{cases} \Delta_1: & X_1 \to X_0 \to Y_1 \to \Sigma^{-r_1} X_1 \\ \Delta_2: & X_2 \to Y_1 \to Y_2 \to \Sigma^{-r_2} X_2 \\ & \vdots \\ \Delta_{n-1}: & X_{n-1} \to Y_{n-2} \to Y_{n-1} \to \Sigma^{-r_{n-1}} X_{n-1} \\ \Delta_n: & X_n \to Y_{n-1} \to X \to \Sigma^{-r_n} X_n. \end{cases}$$

The weight of a cone decomposition is denoted by

$$w(D) = \sum_{i=1}^{n} w(\Delta_i).$$

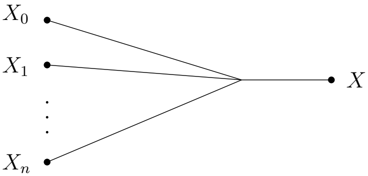
For brevity, a linearization in a cone decomposition D is a denoted by $\ell(D) = (X_0, X_1, ..., X_n)$.

Iterated cone decomposition in $\mathscr{C}$ (cont.)

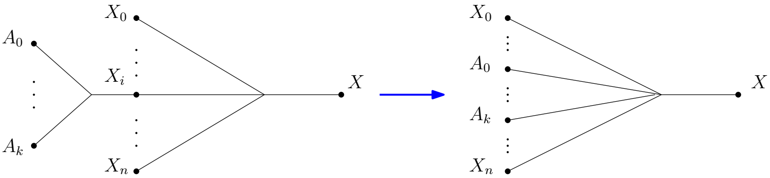
- ⋆ Passing to the K_0-group (over $\mathbb{Z}_2$),

$$[X] = [X_n] + [Y_{n-1}] = [X_n] + ([X_{n-1}] + [Y_{n-2}]) = \cdots = [X_n] + \cdots + [X_0].$$

- ⋆ A standard picture:



- ⋆ Refinement:



Refinement of cone decomposition in $\mathscr{C}$

Proposition

Suppose X admits an iterated cone decomposition D with linearization $(X_0, X_1, ..., X_n)$ and for some $i \in \{0, ..., n\}$, and X_i admits an iterated cone decomposition D' with linearization $(A_0, A_1, ..., A_k)$. Then X admits an iterated cone decomposition D'' of linearization

$$(X_0, ..., X_{i-1}, A_0, ..., A_k, X_{i+1}, ..., X_n).$$

Moreover, the weights of these cone decompositions satisfy the relation that $w(D'') = w(D) + w(D')$. D'' is called a **refinement** *of D with respect to D'.*

Example

The Weighted Octahedral Axiom is an example of a refinement. In fact, B admits a cone decomposition D with linearization $\ell(D) = (A, X)$ and X admits a cone decomposition D' with linearization $\ell(D') = (F, E)$. Then consider refinement

$$D'' := \begin{cases} F \to A \to C \to F \\ E \to C \to B \to \Sigma^{-r-s} E \end{cases}$$

with $\ell(D'') = (A, F, E)$. Moreover, $w(D'') = r + s = w(D) + w(D')$.

Fragmentation pseudo-metric

Definition (Corollary)

Let $\mathscr{F} \subset \mathrm{Ob}(\mathscr{C})$ be a family of objects of $\mathscr{C}$. For objects $X, X' \in \mathrm{Ob}(\mathscr{C})$,

$$\delta^{\mathscr{F}}(X, X') := \inf \left\{ w(D) \,\middle|\, \begin{array}{l} D \text{ is an iterated cone decomposition} \\ \text{of } X \text{ (in } \mathscr{C}\text{) with the linearization} \\ \ell(D) = (F_0, F_1, ..., X', ..., F_k) \text{ and} \\ \text{with the objects } F_i \in \mathscr{F},\ k \in \mathbb{N}. \end{array} \right\}.$$

Define the **fragmentation pseudo-metric on $\mathscr{C}$ associated to $\mathscr{F}$** by

$$d^{\mathscr{F}}(X, X') = \max\{\delta^{\mathscr{F}}(X, X'), \delta^{\mathscr{F}}(X', X)\}.$$

Example

When $\mathscr{F} = \{0\}$, one can check that $d^{\mathscr{F}}(X, X') = \inf\{r \geq 0 \,|\, X \simeq_r X'\}$.
When $\mathscr{F} = \mathrm{Ob}(\mathscr{C})$, one can check that $d^{\mathscr{F}}(X, X') \equiv 0$.

Remark

(1) A geometric motivation of $d^{\mathscr{F}}$ above is the *shadow metric* from Biran-Cornea-Shelukhin's work on Lagrangian cobordism. (2) It's an interesting question that for which $\mathscr{F}$, the metric $d^{\mathscr{F}}$ is non-degenerate.

Definition (Corollary)

Let $\mathscr{C}$ be a TPC and $F \in \mathscr{P}\mathrm{End}(\mathscr{C})$. Suppose $0 \in \mathscr{F} \subset \mathrm{Ob}(\mathscr{C})$. Then define **categorical entropy of F (with respect to $\mathscr{F}$)** by

$$h^{\mathscr{F}}(F) = \lim_{n\to\infty} \frac{d^{\mathscr{F}}(X, F^n(X))}{n}$$

where $0 \neq X \in \mathscr{F}$.

One needs to show that $\lim$ exists and is independent of $X \neq 0$ in $\mathscr{F}$.

• Recall that a topological space is called an *H-space* if there exists a continuous map $\mu : X \times X \to X$ with an identity element e such that $\mu(e,x) = \mu(x,e) = x$ for any $x \in X$.

Corollary

Set $\mathrm{Ob}(\mathscr{C})$ together with the topology induced from the fragmentation pseudo-metric $d^{\mathscr{F}}$ is an H-space.

This is due to the following property. For $A, B, A', B' \in \mathrm{Ob}(\mathscr{C})$, we have

$$d^{\mathscr{F}}(A \oplus B, A' \oplus B') \leq d^{\mathscr{F}}(A, A') + d^{\mathscr{F}}(B, B').$$

Passing to the limit category $\mathscr{C}_\infty$

- When passing to the limit category $\mathscr{C}_\infty$, any r-isomorphism will be an isomorphism since r-acyclic K in $\mathscr{C}$ is isomorphic to 0 in $\mathscr{C}_\infty$.
- The upshot is the following theorem.

Theorem

The limit category $\mathscr{C}_\infty$ is a triangulated category (in the classical sense).

What are the exact triangles in $\mathscr{C}_\infty$?

Definition

A triangle $\Delta : A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A$ is called an **exact triangle in $\mathscr{C}_\infty$** if there is a triangle $\bar{\Delta} : A \xrightarrow{\bar{u}} B \xrightarrow{\bar{v}} C \xrightarrow{\bar{w}} A$ in $\mathscr{C}$ with $\max\{\lceil \bar{u} \rceil, \lceil \bar{v} \rceil, \lceil \bar{w} \rceil\} \geq 0$ and $[\bar{u}] = u$, $[\bar{v}] = v$, $[\bar{w}] = w$ such that the shifted triangle

$$\widetilde{\Delta} = \Sigma^{0, -\lceil \bar{u} \rceil, -\lceil \bar{u} \rceil - \lceil \bar{v} \rceil, -\lceil \bar{u} \rceil - \lceil \bar{v} \rceil - \lceil \bar{w} \rceil} \bar{\Delta}$$

is a strict exact triangle of weight $\lceil \bar{u} \rceil + \lceil \bar{v} \rceil + \lceil \bar{w} \rceil$ in $\mathscr{C}$. For brevity, the existence of such $\widetilde{\Delta}$ is denoted by $\Delta = [\widetilde{\Delta}]$.

Here, $\Sigma^{a,b,c,d}(A \to B \to C \to A) := \Sigma^a A \to \Sigma^b B \to \Sigma^c C \to \Sigma^d A$.

Quantitative aspect of $\mathscr{C}_\infty$

• One advantage of defining exact triangles in $\mathscr{C}_\infty$ from those weighted strict exact triangles in $\mathscr{C}$ is that we can associate a "reduced" weight for each exact triangle in $\mathscr{C}_\infty$. Explicitly,

$$w_\infty(\Delta) := \inf\left\{ w(\tilde{\Delta}) \,\middle|\, \Delta = [\tilde{\Delta}] \right\}.$$

Here, we take the convention that if there does not exist any $\tilde{\Delta}$ such that $[\tilde{\Delta}] = \Delta$, then $w_\infty(\Delta) = \infty$.

• By the same scheme as in $\mathscr{C}$, one can consider

(i) iterated cone decomposition of object in $\mathscr{C}_\infty$ (by using exact triangles in $\mathscr{C}_\infty$);

(ii) the weight of the iterated cone decomposition, which admits sub-additive property under refinement;

(iii) fragmentation pseudo-metric over $\mathscr{C}_\infty$.

• The upshot is that if a triangulated category $\mathscr{T} = \mathscr{C}_\infty$ for a TPC $\mathscr{C}$. Then we can study topology and dynamics on $\mathrm{Ob}(\mathscr{T})$.

Filtered pre-triangulated dg-category

Filtered pre-triangulated dg-category $\mathscr{C}$ is defined in three steps.

$$\text{filtered dg-category } \mathscr{A} \longrightarrow \text{filtered pre-triangulated hull } \mathscr{A}^{\text{hull}} \xrightarrow{\simeq} \mathscr{C},$$

where the second arrow is an equivalence, so let's elaborate $\mathscr{A}$ and $\mathscr{A}^{\text{hull}}$.

Definition

A **filtered dg-category** is a preadditive category $\mathscr{A}$ where

(i) for $A, B \in \mathrm{Ob}(\mathscr{A})$, hom-set $\mathrm{Hom}_{\mathscr{A}}(A, B)$ is a filtered chain complex;

(ii) the composition morphism "$\circ$" is a 0-morphism and a chain map;

(iii) for any inclusions $\iota_{r,r'}^{AB}$ and $\iota_{s,s'}^{BC}$, the composition morphism satisfies the compatibility condition

$$\iota_{s,s'}^{BC}(g) \circ \iota_{r,r'}^{AB}(f) = \iota_{r+s,r'+s'}^{AC}(g \circ f)$$

for any $f \in \mathrm{Hom}_{\mathscr{A}}^{\leq r}(A, B)$ and $g \in \mathrm{Hom}_{\mathscr{A}}^{\leq s}(B, C)$.

One can check that the homology category of $\mathscr{A}$, denoted by $\mathrm{H}_{\bullet}(\mathscr{A})$, i.e., replacing $\mathrm{Hom}_{\mathscr{A}}(A, B)$ by its associated persistence $\mathbf{k}$-module, is a persistence category.

Filtered pre-triangulated dg-category (continue)

Enlarge $\mathscr{A}$ to be $\Sigma\mathscr{A}$ (**suspended category of** $\mathscr{A}$) where we add formal objects $\Sigma^r A[d]$ for any $r \in \mathbb{R}$, $d \in \mathbb{Z}$ and $A \in \mathrm{Ob}(\mathscr{A})$, and the filtration function is defined by $\ell_{\Sigma^r A[d_A], \Sigma^s B[d_B]} = \ell_{A,B} + s - r$ for the filtered chain complex $\mathrm{Hom}_{\Sigma\mathscr{A}}(\Sigma^r A[d_A], \Sigma^s B[d_B])$.

Definition (Variation of Bondal-Kapranov's definition)

Let $\mathscr{A}$ be a filtered dg-category. A **filtered one-sided twisted complex** of $\Sigma\mathscr{A}$ is a pair $A = \left(\bigoplus_{i=1}^n \Sigma^{r_i} A_i[d_i], q = (q_{ij})_{1 \leq i,j \leq n}\right)$ such that the following conditions hold.

(i) $\Sigma^{r_i} A_i[d_i] \in \mathrm{Ob}(\Sigma\mathscr{A})$, where $r_i \in \mathbb{R}$ and $d_i \in \mathbb{Z}$.

(ii) $q_{ij} \in \mathrm{Hom}_{\Sigma\mathscr{A}}(\Sigma^{r_j} A_j[d_j], \Sigma^{r_i} A_i[d_i])$ with degree 1, and $q_{ij} = 0$ for $i \geq j$.

(iii) $d_{\mathrm{Hom}} q_{ij} + \sum_{k=1}^n q_{ik} \circ q_{kj} = 0$.

(iv) For any q_{ij}, $\ell_{\Sigma^{r_j} A_j[d_j] \Sigma^{r_i} A_i[d_i]}(q_{ij}) \leq 0$.

Remark

The only essentially new ingredient is the item (iv), and one can show that there is an inductive way to choose $\{r_i\}_{1 \leq i \leq n}$ such that (i) - (iv) are satisfied.

Filtered pre-triangulated dg-category (continue)

Definition

Given a filtered dg-category $\mathscr{A}$, define its **filtered pre-triangulated hull**, denoted by $\mathscr{A}^{\text{hull}}$, is a category where the following conditions hold.

(i) $\text{Ob}(\mathscr{A}^{\text{hull}}) :=$ {filtered one-sided twisted complex of $\Sigma\mathscr{A}$}.

(ii) For $A = \left(\bigoplus \Sigma^{r_j} A_j[d_j], q\right)$ and $A' = \left(\bigoplus \Sigma^{r'_i} A'_i[d'_i], q'\right)$ in $\text{Ob}(\mathscr{A}^{\text{hull}})$, a morphism $f \in \text{Hom}_{\mathscr{A}^{\text{hull}}}(A, A')$ is a matrix of morphisms denoted by $f = (f_{ij}) : A \to A'$, where $f_{ij} \in \text{Hom}_{\Sigma\mathscr{A}}\left(\Sigma^{r_j} A_j[d_j], \Sigma^{r'_i} A'_i[d'_i]\right)$.

(iii) For any $f \in \text{Hom}_{\mathscr{A}^{\text{hull}}}(A, A')$ as in (ii) above, define

$$d_{\mathscr{A}^{\text{hull}}}(f) := (d_{\text{Hom}} f_{ij}) + q' f + f q$$

where the right-hand side is written in terms of matrices. Moreover, the composition $f' \circ f$ is given by the matrix multiplication.

Example

Let $\mathscr{A}$ be the category of filtered chain complexes, then $\mathscr{A}$ is a filtered dg-category. Filtered one-sided twisted complexes include the classical filtered mapping cones and filtered mapping cylinders. For this case, $\mathscr{A}^{\text{hull}} = \mathscr{A}$.

One should view a (filtered) pre-triangulated hull as an algebraic structure built up from a given (filtered) dg-category such that one can work with it just like working with the category of (filtered) chain complexes.

Theorem

Let $\mathscr{A}$ be a filtered dg-category, then

(i) $\mathscr{A}^{\mathrm{hull}}$ *is a filtered dg-category. In particular,* $\mathrm{H}_\bullet(\mathscr{A}^{\mathrm{hull}})$ *is a persistence category.*

(ii) $\mathrm{H}_0(\mathscr{A}^{\mathrm{hull}})$ *is a triangulated persistence category.*

Remark

(1) The proof of (i) essentially uses the filtration condition (iv) in the definition of filtered one-side twisted complexes. (2) The proof of (ii) is very similar to the classical argument that a pre-triangulated hull of a dg-category is triangulated.

Remark

The example of a triangulated persistence category $\mathscr{C}$ from a Tamarkin category is obtained in an opposite direction. We first have a well-defined derived category (denoted by $\mathscr{T}(M)$) over a closed manifold M, defined as a triangulated subcategory of $\mathscr{D}(\mathbf{k}_{M\times\mathbb{R}})$. This $\mathscr{T}(M)$ serves as $\mathscr{C}_0$. On this category, there is a well-defined shift functor with the help of the $\mathbb{R}$-component, which results in a persistence category $\mathscr{C}$.

Thank you!