Infinite staircases of symplectic embeddings of ellipsoids into Hirzebruch surfaces

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Montreal, PU/IAS, Paris & Tel-Aviv Symplectic Zoominar
June 5, 2020
Gromov nonsqueezing

Let $\omega = \sum_{i=1}^{2} dx_i \wedge dy_i$ be the std. symplectic form on $\mathbb{R}^4 = \mathbb{C}^2$.

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Define the ball

$$B(c) := \{(z_1, z_2) \in \mathbb{C}^2 | \pi|z_1|^2 + \pi|z_2|^2 \leq c\}$$

and the cylinder

$$Z(C) := \{(z_1, z_2) \in \mathbb{C}^2 | \pi|z_1|^2 \leq C\}$$

**Theorem (Gromov ’84)**

$B(c) \hookrightarrow Z(C) \Rightarrow c \leq C$ (notice: no volume obstruction!).
The McDuff-Schlenk Fibonacci stairs

Generalize the ball to the **ellipsoid**

\[ E(a, b) := \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid \frac{\pi |z_1|^2}{a} + \frac{\pi |z_2|^2}{b} \leq 1 \right\} \]

Define the **ellipsoid embedding function** of the ball

\[ c_0(a) := \inf \left\{ \mu > 0 \mid E(1, a) \xhookrightarrow{s} B(\mu) \right\} \]

We know \( c_0(a) \geq \sqrt{a} \) from the volume obstruction.

**Theorem (McDuff-Schlenk '12)**

\( c_0 \) is piecewise linear or smooth and nonsmooth at infinitely many points. A subsequence of nonsmooth points accumulates from below at \((\tau^4, \tau^2)\), where \( \tau = \frac{1+\sqrt{5}}{2} \). For large \( a \), \( c_0(a) = \sqrt{a} \).
The steps ascend from below.

The $x$-coordinates of the outer corners are $2, 5, \frac{13}{2}, \frac{34}{5}, \frac{89}{13}, \ldots$
A toric domain $X_\Omega$ in $\mathbb{C}^2$ is the preimage of a region $\Omega \subset \mathbb{R}_{\geq 0}^2$ under the map $(z_1, z_2) \mapsto (\pi |z_1|^2, \pi |z_2|^2)$.

Let $M_\Omega$ be the symplectic toric manifold with one of the above polytopes. Using Cristofaro-Gardiner–Holm–Mandini–Pires ’20:

$$E(a, b) \hookrightarrow X_\Omega \iff E(a, b) \hookrightarrow M_\Omega$$
Other ellipsoid embedding functions

Let $\mu X_\Omega = X_{\mu \Omega}$ (i.e. $|z_i|^2$ scales by $\mu$.)

Define the **ellipsoid embedding function** of $X_\Omega$ by

$$c_{X_\Omega}(a) := \inf \left\{ \mu > 0 \left| E(1, a) \stackrel{s}{\rightarrow} \mu X_\Omega \right\} \geq \sqrt{\frac{a}{\text{vol}(X_\Omega)}}$$

For $a$ large enough $c_{X_\Omega}(a) = \sqrt{\frac{a}{\text{vol}(X_\Omega)}}$.

We say $c_{X_\Omega}(a)$ has an **infinite staircase** if it is nonsmooth at infinitely many points.
What is known

Based on the vertices and edges of $\Omega$, we know:

$\Omega$ has integer vertices: The most is known.

Cristofaro-Gardiner–Holm–Mandini–Pires ’20 find 12 $\Omega$s with infinite staircases, all ascending, including $\mathbb{C}P^2 \# \mathbb{C}P_{1/3}^2$. They conjecture there are no others.

$\Omega$ has rational edge slopes, irrational vertices: One result to date.

Usher ’18 found infinitely many ascending infinite staircases for polydisks $P(1, b)$, $b \in \mathbb{R} - \mathbb{Q}$.

$\Omega$ has irrational edge slopes: Nothing is known.
New infinite staircases

Let $\Omega_b$ be the Delzant polytope of the Hirzebruch sfc. $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2_b$, i.e., the trapezoid with corner $(b, 1 - b)$. Let $c_b := c_{\chi_{\Omega_b}}$.

**Theorem (Bertozzi-Holm-Maw-McDuff-Mwakyoma-Pires-W i.p.)**

Let

$$b_0 = \frac{5(165 - 7\sqrt{5})}{2698} \approx 0.2767745073$$

$c_{b_0}$ has an infinite staircase whose steps descend to accumulate at

$$\left(\frac{2443 + 3\sqrt{5}}{418}, \frac{\sqrt{281981 - 2124\sqrt{5}}}{209}\right) \approx (5.86054594, 2.51927208)$$

Figure: $\Omega_{b_0}$
$c_{b_0}$ near the accumulation point

Figure: Max of blue and the many reds is $c_{b_0}$. Orange: volume constraint. Green: crosses $c_{b_0}$ at the accumulation point. The stairs descend instead of ascending like $c_0$’s.
Accumulation points

Recall $\Omega_b$ is the convex hull of $(0, 1 - b), (b, 1 - b), (1, 0), (0, 0)$.

Theorem (C-G–H–M–P ’20)

If $c_b$ has an infinite staircase, the $x$-coordinate of its accumulation point, denoted by $\text{acc}(b)$, is the larger of the solutions to

$$x^2 - \left( \frac{(3 - b)^2}{1 - b^2} - 2 \right) x + 1 = 0$$

The accumulation point will always be on the volume obstruction.
The accumulation point curve

**Figure**: Green: the parameterized curve \( \left( \text{acc}(b), \sqrt{\frac{\text{acc}(b)}{\text{vol}(X_{\Omega_b})}} \right) \). Black: the accumulation point \((\tau^4, \tau^2)\) of the Fibonacci stairs. Red: accumulation point of the \(b = \frac{1}{3}\) stairs.
There are five infinite sequences of bs where $c_b$ has an $\infty$ staircase:

Figure: Orange, pink, and yellow are ascending staircases ($x$-values of nonsmooth points increase). Cyan and brown are descending.

There are likely many more such sequences of infinite staircases.
Zooming in near $b = \frac{1}{5}$

The accumulation point of $c_{b_0}$ is the leftmost cyan point.

The minimum of $\sqrt{\frac{\text{acc}(b)}{\text{vol}(X_{\Omega_b})}}$ occurs at $b = \frac{1}{5}$.

$c_{\frac{1}{5}}$ likely does not have an infinite staircase:

Figure: $c_{\frac{1}{5}}$
ECH capacities and ellipsoid embedding functions

$X_\Omega$ has ECH capacities $0 = c_0(X) < c_1(X) \leq c_2(X) \cdots \leq \infty$.

If $\Omega$ is convex,
$c_{X_\Omega}(a) = \sup_k \left\{ \frac{c_k(E(1,a))}{c_k(X_\Omega)} \right\}$
(Cristofaro-Gardiner '19).

This is handy, because $c_k(X_\Omega)$ is combinatorial.

**Figure:** Orange: volume obstruction.
Blue: $c_0$. The obstructions $\frac{c_2(E(1,a))}{c_2(B(1))}$, $\frac{c_5(E(1,a))}{c_5(B(1))}$, $\frac{c_20(E(1,a))}{c_20(B(1))}$. 
**ECH capacities and ellipsoid embedding functions**

$X_\Omega$ has **ECH capacities** $0 = c_0(X) < c_1(X) \leq c_2(X) \cdots \leq \infty$.

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But $c_b$ is still a supremum over an infinite set!

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**Blue:** $c_0$. The obstructions \( \frac{c_2(E(1,a))}{c_2(B(1))}, \frac{c_5(E(1,a))}{c_5(B(1))}, \frac{c_{20}(E(1,a))}{c_{20}(B(1))} \).
Identifying obstructive capacities

We ruled out many $b$ for which

$$c_b(\text{acc}(b)) \geq \max_{k=1,\ldots,25,000} \left\{ \frac{c_k(E(1, \text{acc}(b)))}{c_k(X_{\Omega b})} \right\} > \sqrt{\frac{\text{acc}(b)}{\text{vol}(X_{\Omega b})}}$$

Figure: Orange: volume obstruction. Blue: $c_b$. Green: accumulation point curve. Cyan: $\frac{c_k(E(1,a))}{c_k(X_{\Omega b})}$.
Unviable regions of $b$

Each capacity rules out an infinite staircase for an interval of $bs$.

For example, \[
\frac{c_{125}(E(1,\text{acc}(b)))}{c_{125}(X_{\Omega_b})} > \sqrt{\frac{\text{acc}(b)}{\text{vol}(X_{\Omega_b})}} \quad \text{for at least}\quad 0.277 < b < 0.32475
\]

And \[
\frac{c_{2564}(E(1,\text{acc}(b)))}{c_{2564}(X_{\Omega_b})} > \sqrt{\frac{\text{acc}(b)}{\text{vol}(X_{\Omega_b})}} \quad \text{for at least}\quad 0.274398 < b < 0.27643
\]
Periodic continued fractions: hidden structure of the steps

Now we’ve ruled out many bs, we look for staircases in what’s left.

The **continued fraction expansion** of a number $a$ is the sequence $[a_0, a_1, a_2, \ldots]$ where $a = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}$.

In known $\infty$ staircases, $x$-coords of the outer corners of stairs have periodic CFs. E.g. the Fibonacci stairs have outer corners

$$\frac{13}{2} = [6, 2], \quad \frac{34}{5} = [6, 1, 4], \quad \frac{89}{13} = [6, 1, 5, 2], \quad \frac{233}{34} = [6, 1, 5, 1, 4], \ldots$$

i.e. $[6, \{1, 5\}^k, 2]$ or $[6, \{1, 5\}^k, 1, 4]$

Accumulation points have infinite periodic CFs: $\tau^4 = [6, \{1, 5\}^\infty]$. 
Climb (or descend) the periodic CFs to infinity!

We looked for sequences $a_k$ such that:

- $a_k$ has a periodic continued fraction
- $a_k \to \text{acc}(b)$ and $\max_{k=1,\ldots,25,000} \left\{ \frac{c_k(E(1,\text{acc}(b))))}{c_k(X_{\Omega_b})} \right\} < \sqrt{\frac{\text{acc}(b)}{\text{vol}(X_{\Omega_b})}}$

Such $a_k$ could be outer corners of stairs in $\infty$ staircases. It worked!
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$$0 \leq b < \frac{1}{5}: \quad [6, \{1 + 2n, 5 + 2n\}^k, \text{End}_i(n)], \text{ where } \text{End}_1(n) = 2 + 2n, \text{End}_2(n) = \{1 + 2n, 4 + 2n\}$$
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$\frac{1}{5} < b < \frac{1}{3}$: $[5, 1, 6 + 2n, \{5 + 2n, 1 + 2n\}^k, \text{end}_i(n)]$, where $\text{end}_1(n) = 4 + 2n, \text{end}_2(n) = \{5 + 2n, 2 + 2n\}$

$c_{b_0}$ is the $n = 0$ case, $b_0 = \text{acc}^{-1}([5, 1, 6, \{5, 1\}^\infty])$
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$\frac{1}{3} < b < 1$: yellow $[\{7 + 2n, 5 + 2n, 3 + 2n, 1 + 2n\}^k, 6 + 2n]$; $[7 + 2n, \{5 + 2n, 1 + 2n\}^k, \text{end}_i(n)]$; $[\{5 + 2n, 1 + 2n\}^k, \text{end}_i(n)]$
Proving we have a staircase
Proving we have a staircase

Would take us too long for today!
Thanks!

Thank you!