# Homological mirror symmetry for chain type polynomials 

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joint work with Alexander Polishchuk

## Chain type polynomials

- For an $n$-tuple of positive integers $a=\left(a_{1}, . ., a_{n}\right) \in \mathbb{Z}_{>1}^{n}$, $n \geq 1$, we define the chain polynomial:

$$
p_{a}\left(z_{1}, \ldots, z_{n}\right):=\sum_{i=1}^{n-1} z_{i}^{a_{i}} z_{i+1}+z_{n}^{a_{n}}
$$

- Isolated singularity at the origin for $p_{a}: \mathbb{C}^{n} \rightarrow \mathbb{C}$ (tame map)
- The group of (diagonal) symmetries up to scaling of $p_{a}$

$$
\Gamma_{a}:=\left\{\lambda_{1}^{a_{1}} \lambda_{2}=\ldots=\lambda_{n-1}^{a_{n-1}} \lambda_{n}=\lambda_{n}^{a_{n}}=\lambda\right\} \subset\left(\mathbb{C}^{*}\right)^{n+1}
$$

Group of symmetries $\Gamma_{a}^{0} \subset \Gamma_{a}$ given by $\lambda=1$

- Using the $\Gamma_{a}$ action on $\mathbb{C}$ by multiplication with $\lambda$, $p_{a}: \mathbb{C}^{n} \rightarrow \mathbb{C}$ becomes $\Gamma_{a}$-equivariant.


## Berglund-Hubsch-Henningson duality for chain polynomials

- Define $a^{\vee}=\left(a_{n}, \ldots, a_{1}\right)$
- B-Hu: $p_{a}$ and $p_{\mathrm{a}}$ are " mirror" LG models?
- B-He: Non-trivial check + clarified role of the symmetry groups
- Takahashi: there should exist a triangulated equivalence

$$
\operatorname{DFuk}\left(p_{a}\right) \simeq H M F_{L_{\mathrm{a}} \vee}\left(p_{a \vee}\right)
$$

- $L_{a^{\vee}}:=\operatorname{Hom}\left(\Gamma_{a^{\vee}}, \mathbb{C}^{*}\right)$ grading group, acts on the RHS
- Canonical short exact sequence

$$
0 \rightarrow \mathbb{Z} \rightarrow L_{a \vee} \rightarrow \operatorname{Hom}\left(\Gamma_{a^{\vee}}^{0}, \mathbb{C}^{*}\right) \underset{\text { can. }}{\simeq} \Gamma_{a}^{0} \rightarrow 0
$$

## Equivariant HMS conjecture

- $\Gamma_{a}^{0}$ acts on $\mathbb{C}^{n}$ by symplectomorphisms and preserves $p_{a}$ by definition. Taking graded lifts we obtain an action of $L_{a} \vee$ on $D^{b} F u k\left(p_{a}\right)$.
- Equivariant conjecture: there is an HMS equivalence intertwining these actions
- Would this imply the statements with $G \subset \Gamma_{a}^{0}$ ?
- There exists $F \in L_{a} \vee$ whose action on either side equals $T^{2}$, where $T$ is the shift functor
- There exists $P \in L_{a} \vee$ whose action on either side satisfies

$$
\begin{equation*}
P^{\mu(a)}=T^{m(a)} S^{-1} \tag{1}
\end{equation*}
$$

where $S$ is the Serre functor, $\mu(a), m(a)$ explicit integers

- $L_{a} \vee=<F, P \mid d(a) P=\mu(-a) F>$ - gen. by $T^{2}$ and $S$ if $(d(a), \mu(a))=1$, so equivariance is automatic in that case


## Exceptional collections: A-side I

- Milnor number of the singularity of $p_{a}$ :

$$
\mu(a):=a_{1} \ldots a_{n}-a_{2} \ldots a_{n}+\ldots+(-1)^{n-1} a_{n}+(-1)^{n} .
$$

- Let $d(a):=a_{1} \ldots a_{n}$ and $-a:=\left(a_{2}, \ldots, a_{n}\right)$.
- Useful: $\mu(a)+\mu(-a)=d(a)$ and $\Gamma_{a} \simeq\left\{\lambda_{1}^{d(a)}=\lambda^{\mu(-a)}\right\}$.
- Morsification: $\epsilon z_{1}+p_{a}\left(z_{1}, \ldots, z_{n}\right)=\epsilon z_{1}+\sum_{i=1}^{n-1} z_{i}^{a_{i}} z_{i+1}+z_{n}^{a_{n}}$
- $\epsilon z_{1}+p_{a}: \mathbb{C}^{n} \rightarrow \mathbb{C}$ tame for all $\epsilon \in \mathbb{C}$
- 0 is a regular value for $\epsilon \neq 0$.
- The order $\mu(a)$ cyclic subgroup of $\Gamma_{a}$ given by $\lambda_{1}=\lambda$ makes $z_{1}+p_{a}$ equivariant (maximal possible for such perturbation)
- $D\left(F\left(p_{a}\right)\right) \simeq D\left(F\left(\epsilon z_{1}+p_{a}\right)\right)$ ("same at infinity")
- We can geometrically reconstruct the action of $P$ on $D\left(F\left(\epsilon z_{1}+p_{a}\right)\right)$. On objects up to grading:



## Exceptional collections: A-side III



- Blue paths give our distinguished basis of thimbles.
- Yellow ones give the geometric left dual distinguished basis. Purple ones are in the geometric helix of the blue distiguished basis (up to grading).
- Blue and purple thimbles are generated by a single thimble using $P^{ \pm 1}$ (up to grading)


## Exceptional collections: A-side IV

- Let us denote the directed Fukaya-Seidel $A_{\infty}$-category with an exceptional collection corresponding to the blue paths by

$$
\left(\mathcal{A}_{a}, e_{a}\right)
$$

- $D\left(\mathcal{A}_{a}\right) \simeq D\left(F\left(\epsilon z_{1}+p_{a}\right)\right)$ - work in progress, but should follow from GPS
- In previous work, I computed the Euler pairing in $K_{0}\left(\mathcal{A}_{a}\right)$, which is the same as the Seifert form of $p_{a}$, with respect to the basis corresponding to $e_{a}$. This confirms conjecture by Orlik-Randell '77.
- Set $\left(\mathcal{A}_{\emptyset}, e_{\emptyset}\right)$ to be the $A_{\infty}$ cat. with one object - with $\operatorname{Hom}^{*}(\cdot, \cdot)=k$
- We prove that $\left(\mathcal{A}_{a}, e_{a}\right)$ can be obtained from $\left(\mathcal{A}_{-a}, e_{-a}\right)$ by an explicit recursive procedure $\mathcal{R}$ for $a \neq \emptyset$


## Exceptional collections: B-side

- Aramaki-Takahashi consider the following graded matrix factorization of $p_{a}$ :

$$
E:= \begin{cases}\operatorname{stab}\left(x_{2}, x_{4}, \ldots, x_{n}\right), & n \text { even } \\ \operatorname{stab}\left(x_{1}, x_{3}, \ldots, x_{n}\right), & n \text { odd }\end{cases}
$$

- The collection $\left(E, P(E), \ldots, P^{\mu\left(a^{\vee}\right)-1}(E)\right)$ is a full exceptional collection in $\operatorname{HMF}_{L_{a}}\left(p_{a}\right)$.
- AT compute the Euler pairing wrt this basis. My computation for $p_{a}$ and their computation for $p_{a} \vee$ give exactly the same result.
- Let us denote their subcategory in an enhancement by

$$
\left(\mathcal{B}_{a}, e_{a}\right)
$$

- We prove that $\left(\mathcal{B}_{a}, e_{a}\right)$ can be obtained from $\left(\mathcal{B}_{a-}, e_{a-}\right)$ by $\mathcal{R}$
- Start with a directed $A_{\infty}$-category $(\mathcal{C}, e)$.
- Extend $e$ to a helix inside $T w(\mathcal{C})$ and take the segment $f$ of length $N$ in this helix such that $e$ is the rightmost subsegment of $f$.
- Note that $f$ is no longer an exceptional collection in general. We define $\mathcal{C}^{\prime}$ as the directed $A_{\infty}$-category defined by the directed $A_{\infty}$-subcategory of $f$ (keeping track of only morphisms from left to right in the order of the helix).
- Inside $T_{w}\left(\mathcal{C}^{\prime}\right)$, we consider the right dual exceptional collection $f^{+}$.
- We say $\left(\mathcal{C}^{+}, e^{+}\right)$is obtained from $(\mathcal{C}, e)$ by $\mathcal{R}$ if one can take shifts of the objects of $f^{+}$and find an $A_{\infty}$-quasi-isomorphism from $(\mathcal{C}, e)$ to their directed $A_{\infty}$-subcategory.
- Mutations of exceptional collections in a triangulated $A_{\infty}$-category:

$$
E_{1}, \ldots E_{n} \mapsto E_{1}, \ldots, \mathbb{L}_{E_{i}} E_{i+1}, E_{i}, \ldots E_{n}
$$

$\mathbb{L}$ is the twist functor.

- Given full exceptional collection $E_{1}, \ldots E_{n}$, we can extend it to the left by adding the object $\mathbb{L}_{E_{1}} \ldots \mathbb{L}_{E_{n-1}} E_{n}$. Then the $n$ leftmost elements form a FEC, and we do the same. Extend in both directions to get an infinite collection of objects, called the helix of $E_{1}, \ldots E_{n}$.
- We have $E_{i-n}=S\left(E_{i}\right)$ (up to shift), where $S$ is the Serre functor.
- Left (resp. right) dual collection is obtained by applying mutations to the right (resp. left) until the order is fully reversed.
- The initial $A_{\infty}$-subcategory determines all the others.
- $\left(\mathcal{A}_{\emptyset}, e_{\emptyset}\right)=\left(\mathcal{B}_{\emptyset}, e_{\emptyset}\right)$
- Theorem (Polishchuk - V): $\left(\mathcal{B}_{a}, e_{a}\right)$ can be obtained from $\left(\mathcal{B}_{a_{-}}, e_{a-}\right)$ by $\mathcal{R}$. Shifts can be computed.
- Theorem* (Polishchuk - V): $\left(\mathcal{A}_{a}, e_{a}\right)$ can be obtained from $\left(\mathcal{A}_{-a}, e_{-a}\right)$ by $\mathcal{R}$. Only up to undetermined shifts.
- This proves Takahashi's HMS conjecture.
- Equivariant version in the interesting case $(d(a), \mu(a)) \neq 1$ needs a bit more work.


## Recursion in the A-side I

- Compute the vanishing cycles of the dual basis in $f_{s, r}:\left\{\left(z_{1}, \ldots, z_{n}\right) \mid z_{1}-s z_{2}-r z_{1}^{a_{1}} z_{2}-p_{-a}\left(z_{2}, \ldots z_{n}\right)=0\right\} \xrightarrow{z_{1}} \mathbb{C}$, as matching cycles. $s$ is a small positive real, $r=1$ for now. Sign changes for convenience.



## Recursion in the A-side II

- First get the vanishing cycle for the positive real critical value using real solutions.
- For each other critical value the vanishing cycle is similarly easy to compute for $s$ with a specific argument. Then we move the $s$ back to positive real axis by rotating it clockwise (and use braid parallel transport).
- The key point is that as $s$ rotates, the inner critical values of $f_{s, 1}$ are very close to rotating with it (with different speed) and the outer ones don't move much - rigorously proved using Rouche's theorem.
- Now fix the $s$, and let $r \rightarrow 0$. Inner critical values don't move much, outer ones go off the infinity roughly radially.
- When $r=0$, the map is the same as $s z_{2}+p_{-a}$ and we have part of the helix of the distinguished collection in maximally moved position (up to grading that we cannot yet compute).


## Recursion in the A-side III



## Recursion in the B-side (only the first step)

- Consider $W=x_{1}^{a_{1}} x_{2}+x_{2}^{a_{2}} x_{3}+\ldots+x_{n}^{a_{n}} x_{n+1}^{a_{n}}$. We also define a $\mathbb{G}_{m}$-action on $\mathbb{C}^{n+1}$ leaving $W$ invariant
- The VGIT machinery of Ballard-Favero-Katzarkov gives the diagram (except $\Phi$ )

where $w_{-}=\left.W\right|_{x_{n}=1}$ and $w_{+}=\left.W\right|_{x_{n+1}=1}$, and the categories at the top row are "window subcategories" of $M F_{\Gamma}(W)$.
- We get a functor $M F_{\Gamma_{a-}}\left(p_{a-}\right) \rightarrow M F_{\Gamma_{a}}\left(p_{a}\right)$.
$a=\phi$

$$
\alpha=\left(a_{1}\right)
$$

$$
\xrightarrow[x]{x} \quad \begin{aligned}
& x^{2}=0 \\
& \operatorname{deg} x=1
\end{aligned}
$$

$$
a=\left(a_{1}, a_{2}\right)
$$



$$
x^{a_{2}}=0
$$

$$
\operatorname{deg} x=0
$$

$$
a=\left(a_{1}, a_{2}, a_{3}\right)
$$


$y=\mu_{a_{1}-1}(x, x, \ldots x)$

Conjecture: The endomorphism algebra of $\mathcal{A}_{a}$ is generated as a vector space by elements obtained by applying the $A_{\infty}$-operations iteratively to $x$ 's. There are relations between such elements starting from $n=4$.

## References - with links

- Original Berglund-Hubsch paper (1993)
- Berglund-Henningson (1994)
- Takahashi, HMS (2007)
- Aramaki-Takahashi, B-side $K_{0}$ level (2019)
- Varolgunes, A-side $K_{0}$ level (2020)
- Ballard-Favero-Katzarkov, VGIT (2012)
- Seidel, Directed Fukaya-Seidel $L a g \rightarrow(\mathbb{V})(2000)$
- Seidel, Fuk ( $p_{a}$ ) by localization (2018)
- Gorodentsev-Kuleshov, Helix theory in triangulated categories (developed in late 80's)
- Seidel's book for mutations in $A_{\infty}$-categories and matching cycles (no link)

