The Geography of Immersed Lagrangian Fillings of Legendrian Submanifolds

Lisa Traynor

Bryn Mawr College

April 2020
Joint work with Samantha Pezzimenti (PhD Bryn Mawr ’18), Assistant Teaching Professor at Penn State Brandywine.
Outline

1. Geography of Fillings

2. Legendrians, Lagrangians, and Lagrangian Cobordisms
   - Generating Families

3. Obstructions to Lagrangian Fillings

4. Constructions
Topological Problem: Given a smooth knot $K \subset S^3$, find smooth "filling" surfaces.

$F \subset B^4$ with $\partial F = K$. 
**Topological Problem:** Given a smooth knot $K \subset S^3$, find smooth “filling” surfaces.

\[ F \subset B^4 \text{ with } \partial F = K. \]

There are many options!
**Topological Problem:** Given a smooth knot $K \subset S^3$, find smooth “filling” surfaces.

$$F \subset B^4 \text{ with } \partial F = K.$$ 

There are many options! Find **minimal genus**:

$$g_4(K) := \min \{\text{genus}(F) : \partial F = K\}.$$
One can also try to find immersed fillings with transverse double points.
One can also try to find immersed fillings with transverse double points.

4-ball (genus 0) crossing number:

\[ c_4^0(K) := \min \{ \text{double points in disk filling of } K \} . \]
Immersed Fillings of Smooth Knots

One can also try to find immersed fillings with transverse double points.

4-ball (genus 0) crossing number:

\[ c_4^0(K) := \min \{ \text{double points in disk filling of } K \} . \]

Example: \( c_4^0(m(5_2)) = 1; c_4^0(7_4) = 2. \) [Strle-Owens (’15)]
Smooth Geography Question: Given a smooth knot $K \subset S^3$, what combinations of genus and double points can be realized by smooth fillings?

It is always possible to:
- fix genus and increase the number of double points by 1;
- fix number of double points and increase the genus by 1;
- eliminate a double point at the cost of increasing the genus.
**Smooth Geography Question:** Given a smooth knot $K \subset S^3$, what combinations of genus and double points can be realized by smooth fillings?

It is always possible to:
- fix genus and increase the number of double points by 1;
- fix number of double points and increase the genus by 1;
- eliminate a double point at the cost of increasing the genus.
**Smooth Geography Question:** Given a smooth knot $K \subset S^3$, what combinations of **genus** and **double points** can be realized by smooth fillings?

It is always possible to:

- fix genus and increase the number of double points by 1;
- fix number of double points and increase the genus by 1;
**Smooth Geography Question:** Given a smooth knot $K \subset S^3$, what combinations of **genus** and **double points** can be realized by smooth fillings?

It is always possible to:
- fix genus and increase the number of double points by 1;
- fix number of double points and increase the genus by 1;
- eliminate a double point at the cost of increasing the genus.
Examples of Smooth Geography for Knots

Smooth Geography of the knot $m(5_2)$. 
Examples of Smooth Geography for Knots

Smooth Geography of the knot $m(5_2)$.

Smooth Geography of the knot $7_4$. 
**Smooth Geography Question:** Given a smooth $m$-dimensional submanifold $K^m \subset S^{2m+1}$, what combinations of Betti numbers and double points can be realized by a smooth filling $F^{m+1} \subset B^{2m+2}$?
**Smooth Geography Question:** Given a smooth \( m \)-dimensional submanifold \( K^m \subset S^{2m+1} \), what combinations of Betti numbers and double points can be realized by a smooth filling \( F^{m+1} \subset B^{2m+2} \)?

\( \exists \) restrictions from algebraic topology for embedded fillings.
**Smooth Geography Question:** Given a smooth $m$-dimensional submanifold $K^m \subset S^{2m+1}$, what combinations of Betti numbers and double points can be realized by a smooth filling $F^{m+1} \subset B^{2m+2}$?

There are restrictions from algebraic topology for embedded fillings.

I would like to know if others know anything about this problem!
**Symplectic Problem:** Given a Legendrian knot in a 3-dimensional space, try to find immersed Lagrangian surface fillings in a 4-dimensional space.
**Symplectic Problem:** Given a **Legendrian knot** in a 3-dimensional space, try to find **immersed Lagrangian surface fillings** in a 4-dimensional space.

**Question:** How flexible/rigid are Lagrangian fillings?
**Symplectic Problem:** Given a *Legendrian knot* in a 3-dimensional space, try to find *immersed Lagrangian surface fillings* in a 4-dimensional space.

**Question:** How flexible/rigid are Lagrangian fillings? How does Lagrangian Geography compare to Smooth Geography?
**Symplectic Problem:** Given an $m$-dimensional **Legendrian** try to find $(m+1)$-dimensional **immersed Lagrangian fillings**.

**Question:** How flexible/rigid are Lagrangian fillings?

How does Lagrangian Geography compare to Smooth Geography?

Higher-Dimensional version is also interesting.
Will consider “GF-compatible” fillings. Find much more rigidity!

Smooth Geography

Lagrangian Geography
Will consider “GF-compatible” fillings. Find much more rigidity!
1. Geography of Fillings

2. Legendrians, Lagrangians, and Lagrangian Cobordisms
   - Generating Families

3. Obstructions to Lagrangian Fillings

4. Constructions
Standard Contact Manifold: \((\mathbb{R}^{2n+1}, \xi = \ker \alpha)\)

\[ J^1(\mathbb{R}^n) = T^*\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{2n+1}, \quad \alpha = dz - \sum_i y_i dx_i \]
Contact and Symplectic Setting

Standard Contact Manifold: \((\mathbb{R}^{2n+1}, \xi = \ker \alpha)\)

\[
J^1(\mathbb{R}^n) = T^*\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{2n+1}, \quad \alpha = dz - \sum_i y_i dx_i
\]

Interested in **Legendrians** \(\Lambda^n \subset J^1(\mathbb{R}^n) = \mathbb{R}^{2n+1} \quad (T_p\Lambda \subset \xi, \forall p)\)
Contact and Symplectic Setting

Standard Contact Manifold: \( (\mathbb{R}^{2n+1}, \xi = \ker \alpha) \)

\[ J^1(\mathbb{R}^n) = T^*\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{2n+1}, \quad \alpha = dz - \sum_i y_i dx_i \]

Interested in **Legendrians** \( \Lambda^n \subset J^1(\mathbb{R}^n) = \mathbb{R}^{2n+1} \quad (T_p\Lambda \subset \xi, \forall p) \)

\[ y = \frac{dz}{dx} \]
Contact and Symplectic Setting

Standard Contact Manifold: \((\mathbb{R}^{2n+1}, \xi = \ker \alpha)\)

\[ J^1(\mathbb{R}^n) = T^*\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{2n+1}, \quad \alpha = dz - \sum_i y_i dx_i \]

Interested in Legendrians \(\Lambda^n \subset J^1(\mathbb{R}^n) = \mathbb{R}^{2n+1}\) \((T_p\Lambda \subset \xi, \forall p)\)

Reeb Vector Field of \(\alpha\) is \(\frac{\partial}{\partial z}\):

\[
\begin{align*}
&\text{Reeb Vector Field of } \alpha \\
&\quad \frac{\partial}{\partial z}:
\end{align*}
\]
Standard Contact Manifold: \((\mathbb{R}^{2n+1}, \xi = \ker \alpha)\)

\[ J^1(\mathbb{R}^n) = T^*\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{2n+1}, \quad \alpha = dz - \sum_i y_i dx_i \]

Interested in Legendrians \(\Lambda^n \subset J^1(\mathbb{R}^n) = \mathbb{R}^{2n+1}\) \((T_p\Lambda \subset \xi, \forall p)\)

Reeb Vector Field of \(\alpha\) is \(\frac{\partial}{\partial z}\):
Contact and Symplectic Setting

**Standard Contact Manifold:** \((\mathbb{R}^{2n+1}, \xi = \ker \alpha)\)
Contact and Symplectic Setting

Standard Contact Manifold: \((\mathbb{R}^{2n+1}, \xi = \ker \alpha)\)

Symplectization: \((\mathbb{R} \times \mathbb{R}^{2n+1}, \omega = d(e^s \alpha))\)
Contact and Symplectic Setting

Standard Contact Manifold: \((\mathbb{R}^{2n+1}, \xi = \ker \alpha)\)

Symplectization: \((\mathbb{R} \times \mathbb{R}^{2n+1}, \omega = d(e^s \alpha))\)

Interested in **Lagrangians**: \(L^{n+1}\) s.t. \(d(e^s \alpha)|_L = 0\).
**Contact and Symplectic Setting**

**Standard Contact Manifold:** $\left( \mathbb{R}^{2n+1}, \xi = \ker \alpha \right)$

**Symplectization:** $\left( \mathbb{R} \times \mathbb{R}^{2n+1}, \omega = d(e^s \alpha) \right)$

- Interested in **exact Lagrangians**: $L^{n+1}$ s.t. $(e^s \alpha)|_L$ is exact 1-form.
Contact and Symplectic Setting

Standard Contact Manifold: \((\mathbb{R}^{2n+1}, \xi = \ker \alpha)\)

Symplectization: \((\mathbb{R} \times \mathbb{R}^{2n+1}, \omega = d(e^s \alpha))\)

- Interested in **exact Lagrangians**: \(L^{n+1}\) s.t. \((e^s \alpha)|_L\) is exact 1-form.

There are no closed, exact Lagrangians (Gromov).
Contact and Symplectic Setting

**Standard Contact Manifold:** \((\mathbb{R}^{2n+1}, \xi = \ker \alpha)\)

**Symplectization:** \((\mathbb{R} \times \mathbb{R}^{2n+1}, \omega = d(e^s \alpha))\)

- Interested in **exact Lagrangians**: \(L^{n+1}\) s.t. \((e^s \alpha)|_L\) is exact 1-form.
- For a Legendrian \(\Lambda\), the **cylinder** \(\mathbb{R} \times \Lambda\) is an exact Lagrangian.
A Lagrangian cobordism from $\Lambda_-$ to $\Lambda_+$:

A Lagrangian cobordism from $\Lambda_-$ to $\Lambda_+$:

A Lagrangian cobordism from $\Lambda_-$ to $\Lambda_+$:

A Lagrangian cobordism from $\Lambda_-$ to $\Lambda_+$:

A Lagrangian cobordism from $\Lambda_-$ to $\Lambda_+$:

A Lagrangian cobordism from $\Lambda_-$ to $\Lambda_+$:
A Lagrangian cobordism from $\Lambda_-$ to $\Lambda_+$:

$\Lambda_\pm$ are Legendrian submanifolds in $\{s = s_\pm\}$.
A Lagrangian cobordism from $\Lambda_-$ to $\Lambda_+$:

- $\Lambda_{\pm}$ are *Legendrian submanifolds* in $\{s = s_{\pm}\}$;
- $L$ is *exact Lagrangian* and *cylindrical* over $\Lambda_{\pm}$ at $\pm\infty$:

  $$L \cap [s_-, s_+] \text{ is compact, } \quad L = \mathbb{R} \times \Lambda_{\pm} \text{ outside } [s_-, s_+]$$
A Lagrangian cobordism from $\Lambda_-$ to $\Lambda_+$:

- $\Lambda_\pm$ are Legendrian submanifolds in $\{s = s_{\pm}\}$;
- $L$ is exact Lagrangian and cylindrical over $\Lambda_\pm$ at $\pm \infty$:
  
  $L \cap [s_-, s_+]$ is compact, $L = \mathbb{R} \times \Lambda_\pm$ outside $[s_-, s_+]$;

- $L$ is orientable, Maslov 0 and embedded or immersed.
A Lagrangian cobordism from $\Lambda_-$ to $\Lambda_+$:

- $\Lambda_\pm$ are Legendrian submanifolds in $\{s = s_\pm\}$;
- $L$ is exact Lagrangian and cylindrical over $\Lambda_\pm$ at $\pm\infty$:
  \[ L \cap [s_-, s_+] \text{ is compact, } \quad L = \mathbb{R} \times \Lambda_\pm \text{ outside } [s_-, s_+] \; ; \]
- $L$ is orientable, Maslov 0 and embedded or immersed.
- **Today:** $L$ has a generating family $\implies L$ is exact, Maslov 0.
A Lagrangian cobordism from $\Lambda_-$ to $\Lambda_+$:

- $\Lambda_{\pm}$ are Legendrian submanifolds in $\{s = s_{\pm}\}$;
- $L$ is exact Lagrangian and cylindrical over $\Lambda_{\pm}$ at $\pm\infty$:
  
  \[ L \cap [s_{-}, s_{+}] \text{ is compact, } \quad L = \mathbb{R} \times \Lambda_{\pm} \quad \text{outside } [s_{-}, s_{+}] ; \]

- $L$ is orientable, Maslov 0 and embedded or immersed.
- **Today:** $L$ has a generating family $\implies L$ is exact, Maslov 0.

Arise in relative SFT (Eliashberg-Givental-Hofer)
A Lagrangian filling of $\Lambda$:

$L$ is exact, Maslov 0, embedded or immersed.
Lagrangian filling of a Legendrian

A Lagrangian filling of $\Lambda$:

$L$ is exact, Maslov 0, embedded or immersed

Today: $L$ has a generating family
Geography via technique of “generating families of functions”.

Restricting to:

- Legendrians that can be “generated” by a function $f$;
-Probably corresponds to Legendrians whose DGA admits an augmentation $\varepsilon$.

Lagrangian fillings that can be “generated” by an extension, $F$,

- Maybe corresponds to fillings that induce specified augmentation $\varepsilon$.  

Lisa Traynor (Bryn Mawr)
GF-Compatible Lagrangian Fillings

Geography via technique of “generating families of functions”.

Restricting to:
- Legendrians that can be “generated” by a function $f$;

  Probably corresponds to Legendrians whose DGA admits an augmentation $\varepsilon$. 

Lisa Traynor (Bryn Mawr) Geography of Immersed Lagrangian Fillings Symplectic Zoominar 16/51
GF-Compatible Lagrangian Fillings

Geography via technique of “generating families of functions”.

Restricting to:

- Legendrians that can be “generated” by a function $f$;
  
  Probably corresponds to Legendrians whose DGA admits an augmentation $\varepsilon$.

- Lagrangian fillings that can be “generated” by an extension, $F$, of this function.
  
  Maybe corresponds to fillings that induce specified augmentation $\varepsilon$. 
Generating Families

**Classic** technique;

**Modernized** by Laudenbach, Sikorav, Chaperon, Viterbo
Generating Families

**Classic** technique;

**Modernized** by Laudenbach, Sikorav, Chaperon, Viterbo

For Legendrians $\Lambda \subset J^1 M$, want to study Lagrangian fillings $L \subset \mathbb{R} \times J^1 M$ via generating families.
Generating Families

**Classic** technique;
**Modernized** by Laudenbach, Sikorav, Chaperon, Viterbo

For Legendrians $\Lambda \subset J^1 M$, want to study Lagrangian fillings $L \subset \mathbb{R} \times J^1 M$ via generating families.

- In $J^1 (M)$, describe Legendrian $\Lambda$ as the “1-jet” of function $f : M \times \mathbb{R}^N \to \mathbb{R}$. 
Generating Families

**Classic** technique;

**Modernized** by Laudenbach, Sikorav, Chaperon, Viterbo

For Legendrians $\Lambda \subset J^1 M$, want to study Lagrangian fillings $L \subset \mathbb{R} \times J^1 M$ via generating families.

- In $J^1(M)$, describe Legendrian $\Lambda$ as the “1-jet” of function $f : M \times \mathbb{R}^N \to \mathbb{R}$.

- In $\mathbb{R} \times J^1 M \equiv T^*(\mathbb{R}^+ \times M)$, describe Lagrangian $L$ as the “derivatives” of function $F : \mathbb{R}^+ \times M \times \mathbb{R}^N \to \mathbb{R}$. 

Lisa Traynor (Bryn Mawr)

Geography of Immersed Lagrangian Fillings

Symplectic Zoominar
Generating Families

**Classic** technique;
**Modernized** by Laudenbach, Sikorav, Chaperon, Viterbo

For Legendrians $\Lambda \subset J^1 M$, want to study Lagrangian fillings $L \subset \mathbb{R} \times J^1 M$ via generating families.

- In $J^1 (M)$, describe Legendrian $\Lambda$ as the “1-jet” of function $f : M \times \mathbb{R}^N \to \mathbb{R}$.

- In $\mathbb{R} \times J^1 M \equiv T^*(\mathbb{R}^+ \times M)$, describe Lagrangian $L$ as the “derivatives” of function $F : \mathbb{R}^+ \times M \times \mathbb{R}^N \to \mathbb{R}$.

**Strategy:** Apply analysis/Morse theoretic arguments to these functions to obtain **invariants of** and **relationships between** the Lagrangian and Legendrian submanifolds.
**Idea:** Construct a 1-parameter family of functions $F_x : \mathbb{R} = \{e\} \rightarrow \mathbb{R}$.
**Idea:** Construct a 1-parameter family of functions $F_x : \mathbb{R} = \{e\} \to \mathbb{R}$

$\exists F : \mathbb{R} \times \mathbb{R}^1 \to \mathbb{R}$ so that $\Lambda$ is the “1-jet of $F$ along the fiber critical submanifold”:

$$\Lambda = \left\{ \left( x, \frac{\partial F}{\partial x} (x, e), F(x, e) \right) : \frac{\partial F}{\partial e} (x, e) = 0 \right\}.$$
Existence from Rulings

For Legendrian knots: [Chekanov-Pushkar; Fuchs - Rutherford]

∃ (linear at infinity) generating family ⇐⇒ ∃ graded normal ruling

Graded normal rulings of two different Legendrian knots.

Graded normal ruling of a Legendrian knot.
For Legendrian knots: [Chekanov-Pushkar; Fuchs - Rutherford]

\[ \exists \text{ (linear at infinity) generating family} \iff \exists \text{ graded normal ruling} \]

Graded normal rulings of two different Legendrian $m(5_2)$ knots.
For Legendrian knots: [Chekanov-Pushkar; Fuchs - Rutherford]

∃ (linear at infinity) generating family ⇐⇒ ∃ graded normal ruling

Graded normal rulings of two different Legendrian $m(5_2)$ knots.

Graded normal ruling of a Legendrian $7_4$ knot.
Generating Family Cohomology for Legendrians

\Lambda \subset J^1 M \text{ with generating family } f : M \times \mathbb{R}^N \to \mathbb{R}.

Difference function: 

\delta f : M \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}

\delta f(\mathbf{x}, \eta, \tilde{\eta}) = f(\mathbf{x}, \tilde{\eta}) - f(\mathbf{x}, \eta).

Generating Family Cohomology Groups:

\text{GH}_k(\Lambda, f) = H^k(\delta \infty f, \delta \epsilon f).

GF-Polynomials:

\Gamma_{\Lambda, f}(t) = \sum \text{dim} \text{GH}_k(\Lambda, f) t^k.
Generating Family Cohomology for Legendrians

$\Lambda \subset J^1 M$ with generating family $f : M \times \mathbb{R}^N \to \mathbb{R}$.

**Difference function:**

$$\delta_f : M \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$$

$$\delta_f(x, \eta, \tilde{\eta}) = f(x, \tilde{\eta}) - f(x, \eta).$$
Generating Family Cohomology for Legendrians

\[ \Lambda \subset J^1 M \] with generating family \( f : M \times \mathbb{R}^N \to \mathbb{R} \).

**Difference function:**
\[
\delta_f : M \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}
\]
\[
\delta_f(x, \eta, \tilde{\eta}) = f(x, \tilde{\eta}) - f(x, \eta).
\]

critical points with + value \(\leftrightarrow\) Reeb chords

critical points with 0 value \(\leftrightarrow\) submanifold diffeo to \(\Lambda\)
Generating Family Cohomology for Legendrians

\( \Lambda \subset J^1 M \) with generating family \( f : M \times \mathbb{R}^N \to \mathbb{R} \).

**Difference function:** \( \delta_f : M \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \)

\[
\delta_f(x, \eta, \tilde{\eta}) = f(x, \tilde{\eta}) - f(x, \eta).
\]

Critical points with + value \( \longleftrightarrow \) Reeb chords
Critical points with 0 value \( \longleftrightarrow \) submanifold diffeo to \( \Lambda \)

**Generating Family Cohomology Groups:**

\[
GH^k(\Lambda, f) = H^{k+N+1}(\delta_f^\infty, \delta_f^\epsilon).
\]
Generating Family Cohomology for Legendrians

\( \Lambda \subset J^1 M \) with generating family \( f : M \times \mathbb{R}^N \rightarrow \mathbb{R} \).

**Difference function:** \( \delta_f : M \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R} \)

\[
\delta_f(x, \eta, \tilde{\eta}) = f(x, \tilde{\eta}) - f(x, \eta).
\]

Critical points with + value \( \longleftrightarrow \) Reeb chords

Critical points with 0 value \( \longleftrightarrow \) submanifold diffeo to \( \Lambda \)

**Generating Family Cohomology Groups:**

\[
GH^k(\Lambda, f) = H^{k+N+1}(\delta_f^\infty, \delta_f^\epsilon).
\]

**GF-Polynomials:**

\[
\Gamma_{\Lambda,f}(t) = \sum \dim GH^k(\Lambda, f)t^k.
\]
Examples of GF-Polynomials

Examples:

\[ t + 2 \]

\[ t + (t^2 + t^2) \]

\[ t + 2 \]

All polynomials are of the form

\[ \Gamma(\Lambda, f) = t + \sum_{n=0}^{\infty} c_i (t^i + t^{i-1}) ; \]

[Sabloff Duality]

Every polynomial satisfying Sabloff Duality can be realized by a Legendrian.

[Bourgeois-Sabloff-Traynor]
Examples of GF-Polynomials

Examples:

\[ t + 2 \]
\[ t + (t^2 + t^2) \]

All polynomials are of the form

\[ \Gamma_{(\Lambda, f)} = t + \sum_{i=0}^{n} c_i \left( t^i + t^{-i} \right) ; \]

[Sabloff Duality]
Examples of GF-Polynomials

Examples:

All polynomials are of the form $\Gamma_{(\Lambda, f)} = t + \sum_{i=0}^{n} c_i (t^i + t^{-i})$; [Sabloff Duality]

Every polynomial satisfying Sabloff Duality can be realized by a Legendrian. [Bourgeois-Sabloff-Traynor]
For a connected $m$-dimensional Legendrian:

$$\Gamma(\Lambda, f) = t^m + b_{m-1} t^{m-1} + \cdots + b_1 t + \sum_{i \geq \lfloor \frac{m-1}{2} \rfloor} c_i \left( t^i + t^{-i+(m-1)} \right),$$

where $b_k + b_{m-k} = \dim H_k(\Lambda^m)$.

Every polynomial satisfying this duality can be realized by a Legendrian. [Bourgeois-Sabloff-Traynor]
For a connected $m$-dimensional Legendrian:

$$\Gamma(\Lambda, f) = t^m + b_{m-1} t^{m-1} + \cdots + b_1 t + \sum_{i \geq \lfloor \frac{m-1}{2} \rfloor} c_i \left( t^i + t^{-i+(m-1)} \right),$$

where $b_k + b_{m-k} = \dim H_k(\Lambda^m)$.

**Examples:**

- If $\Lambda$ is Legendrian $S^2$:
  $$\Gamma(\Lambda, f) = t^2 + c_0(t^0 + t^1) + c_1(t^1 + t^0) + c_2(t^2 + t^{-1}) + \ldots.$$

Every polynomial satisfying this duality can be realized by a Legendrian. [Bourgeois-Sabloff-Traynor]
For a connected $m$-dimensional Legendrian:

$$
\Gamma(\Lambda, f) = t^m + b_{m-1} t^{m-1} + \cdots + b_1 t + \sum_{i \geq \left\lfloor \frac{m-1}{2} \right\rfloor} c_i \left( t^i + t^{-i+(m-1)} \right),
$$

where $b_k + b_{m-k} = \dim H_k(\Lambda^m)$.

**Examples:**

- If $\Lambda$ is Legendrian $S^2$:
  $$
  \Gamma(\Lambda, f) = t^2 + c_0(t^0 + t^1) + c_1(t^1 + t^0) + c_2(t^2 + t^{-1}) + \ldots.
  $$

- If $\Lambda$ is Legendrian $S^3$:
  $$
  \Gamma(\Lambda, f) = t^3 + c_1(t^1 + t^1) + c_2(t^2 + t^0) + c_3(t^3 + t^{-1}) + \ldots.
  $$

Every polynomial satisfying this duality can be realized by a Legendrian. [Bourgeois-Sabloff-Traynor]
Generating Families for Lagrangian Fillings

Lagrangian cobordisms in $\mathbb{R} \times J^1 M \equiv T^*(\mathbb{R}_+ \times M)$:

$$\psi : \mathbb{R} \times J^1 M \rightarrow T^*(\mathbb{R}_+ \times M)$$

$$(s, x, y, z) \mapsto (e^s, x, z, e^s y).$$
Generating Families for Lagrangian Fillings

Lagrangian cobordisms in $\mathbb{R} \times J^1M \equiv T^*(\mathbb{R}_+ \times M)$:

$$\psi : \mathbb{R} \times J^1M \to T^*(\mathbb{R}_+ \times M)$$

$$(s, x, y, z) \mapsto (e^s, x, z, e^s y).$$

Identify Lagrangian cobordism $L$ with $\psi(L) \subset T^*(\mathbb{R}_+ \times M)$:
Generating Families for Lagrangian Fillings

Lagrangian cobordisms in \( \mathbb{R} \times J^1 M \equiv T^*(\mathbb{R}_+ \times M) \):

\[
\psi : \mathbb{R} \times J^1 M \to T^*(\mathbb{R}_+ \times M) \\
(s, x, y, z) \mapsto (e^s, x, z, e^s y).
\]

Identify Lagrangian cobordism \( L \) with \( \psi(L) \subset T^*(\mathbb{R}_+ \times M) \):

\( F \) is a generating family for \( L \) means \( F : \mathbb{R}_+ \times M \times \mathbb{R}^N \to \mathbb{R} \) such that

\[
\psi(L) = \left\{ \left( x, \frac{\partial F}{\partial x}(x, e) \right) : \frac{\partial F}{\partial e}(x, e) = 0 \right\}.
\]
Assume Legendrian $\Lambda$ and Lagrangian filling $L$ can be described by compatible generating families:

$\exists$ generating families $f: M \times \mathbb{R}^N \to \mathbb{R}$ for $\Lambda$; and $F: (\mathbb{R}^+ \times M) \times \mathbb{R}^N \to \mathbb{R}$ for $\psi(L) \subset T^*(\mathbb{R}^+ \times M)$ that correlates with $f$ for large $t \in \mathbb{R}^+$:

$F(t, x, \eta) = t f(x, \eta)$,

$t \geq t^+$. 

GF $f$ for $\Lambda$ "extends" to a GF for filling $L$. 

Lisa Traynor (Bryn Mawr)
Assume Legendrian $\Lambda$ and Lagrangian filling $L$ can be described by \textit{compatible generating families}:

\[ \exists \text{ generating families} \]

\[ f : M \times \mathbb{R}^N \rightarrow \mathbb{R} \text{ for } \Lambda; \text{ and} \]

GF $f$ for $\Lambda$ “extends” to a GF for filling $L$. 
Assume Legendrian $\Lambda$ and Lagrangian filling $L$ can be described by compatible generating families:

$\exists$ generating families

- $f : M \times \mathbb{R}^N \to \mathbb{R}$ for $\Lambda$; and

- $F : (\mathbb{R}_+ \times M) \times \mathbb{R}^N \to \mathbb{R}$ for $\psi(L) \subset T^*(\mathbb{R}_+ \times M)$ that correlates with $f$ for large $t \in \mathbb{R}_+$:

$$F(t, x, \eta) = tf(x, \eta), \quad t \geq t_+.$$ 

GF $f$ for $\Lambda$ “extends” to a GF for filling $L$. 
Outline

1. Geography of Fillings

2. Legendrians, Lagrangians, and Lagrangian Cobordisms
   - Generating Families

3. Obstructions to Lagrangian Fillings

4. Constructions
GF Seidel Isomorphism

Theorem (Sabloff-Traynor, ’13)

If \( \Lambda \) has a GF \( f \), then any GF-compatible embedded Lagrangian filling \( L \) satisfies:

\[
GH^k(\Lambda, f) \cong H^{k+1}(L, \partial L).
\]
Theorem (Sabloff-Traynor, ’13)

If \( \Lambda \) has a GF \( f \), then any GF-compatible embedded Lagrangian filling \( L \) satisfies:

\[
GH^k(\Lambda, f) \cong H^{k+1}(L, \partial L).
\]

Therefore, if \( \Lambda \) admits an embedded GF-compatible Lagrangian filling,

\[
\Gamma_{\Lambda, f}(t) = t + 2c_0
\]
Theorem (Sabloff-Traynor, ’13)

If $\Lambda$ has a GF $f$, then any GF-compatible embedded Lagrangian filling $L$ satisfies:

$$GH^k(\Lambda, f) \cong H^{k+1}(L, \partial L).$$

$\iff$ If $\Lambda$ admits an embedded GF-compatible Lagrangian filling, 

$$\Gamma_{\Lambda,f}(t) = t + 2c_0$$

and any embedded filling must have genus $c_0$. 
Scarcity of Embedded Fillings

- All polynomials are of the form $\Gamma_{(\Lambda, f)} = t + \sum_{i=0}^{n} c_i (t^i + t^{-i});$
- Every polynomial satisfying Sabloff Duality can be realized by a Legendrian.
Scarcity of Embedded Fillings

All polynomials are of the form $\Gamma_{(\Lambda,f)} = t + \sum_{i=0}^{n} c_i (t^i + t^{-i})$;

Every polynomial satisfying Sabloff Duality can be realized by a Legendrian.

Only Legendrians admitting polynomial of the form $\Gamma_{(\Lambda,f)} = t + 2c_0$ have a chance of admitting an embedded filling.
Legendrians that admit \textit{embedded} Lagrangian fillings are “rare”.

\[ \text{Bourgeois-Sabloff-Traynor} \]
Legendrians that admit *embedded* Lagrangian fillings are “rare”.

However, any $\Lambda$ with a GF will have an *immersed* GF-compatible filling. [Bourgeois-Sabloff-Traynor].
Legendrians that admit *embedded* Lagrangian fillings are “rare”.

However, **any** $\Lambda$ with a GF will have an **immersed** GF-compatible filling. [Bourgeois-Sabloff-Traynor].

**Question:** What can the GF-polynomial $\Gamma_{(\Lambda,f)}(t)$ tell us about the Lagrangian geography problem?
Theorem (Pezzimenti-Traynor)

If a Legendrian knot has GF-polynomial

\[ \Gamma_{\Lambda,f}(t) = t + \sum_{i=0}^{n} c_i \left(t^i + t^{-i}\right), \]

then any GF-compatible immersed Lagrangian filling of \( \Lambda \) with genus 0 has at least \( c_0 \) double points; genus \( g \) has at least \( |g - c_0| + c_1 + \cdots + c_n \) double points.

Moreover, if \( p \) is the number of double points, then \( p + g \equiv n \sum_{i=0}^{n} c_i \mod 2 \).
Polynomial Obstructions to Immersed Fillings

Theorem (Pezzimenti-Traynor)

If a Legendrian knot has GF-polynomial

$$\Gamma_{\Lambda,f}(t) = t + \sum_{i=0}^{n} c_i \left( t^i + t^{-i} \right),$$

then any GF-compatible immersed Lagrangian filling of $\Lambda$ with

- genus 0 has at least $c_0 + c_1 + c_2 + \cdots + c_n$ double points;
Theorem (Pezzimenti-Traynor)

If a Legendrian knot has GF-polynomial

$$\Gamma_{\Lambda,f}(t) = t + \sum_{i=0}^{n} c_i \left( t^i + t^{-i} \right),$$

then any GF-compatible immersed Lagrangian filling of $\Lambda$ with

- genus 0 has at least $c_0 + c_1 + c_2 + \cdots + c_n$ double points;
- genus $g$ has at least $|g - c_0| + c_1 + \cdots + c_n$ double points.
Theorem (Pezzimenti-Traynor)

If a Legendrian knot has GF-polynomial

$$\Gamma_{\Lambda, f}(t) = t + \sum_{i=0}^{n} c_i \left( t^i + t^{-i} \right),$$

then any GF-compatible immersed Lagrangian filling of $\Lambda$ with

- genus 0 has at least $c_0 + c_1 + c_2 + \cdots + c_n$ double points;
- genus $g$ has at least $|g - c_0| + c_1 + \cdots + c_n$ double points.

Moreover, if $p$ is the number of double points, then

$$p + g \equiv \sum_{i=0}^{n} c_i \mod 2.$$
Potential “check-mark” geography for Lagrangian fillings of $\Lambda$ with $\Gamma_{\Lambda,f} = t + \sum_{i=0}^{n} c_i (t^i + t^{-i})$.

If $c_0 = 1$, then:

- Genus 0: Double Points 0
- Genus 1: Double Points 0
- Genus 2: Double Points 0
- Genus 3: Double Points 0
- Genus 4: Double Points 0
Potential “check-mark” geography for Lagrangian fillings of $\Lambda$ with $\Gamma_{\Lambda,f} = t + \sum_{i=0}^{n} c_i (t^i + t^{-i})$.

- If $c_0 = 1$

- If $c_2 = 1$

- If $c_0 = 2, c_1 = 1, c_2 = 1$
Immersed GF-Isomorphism

All dimensions:

From a GF-compatible immersed Lagrangian filling \( L \),

\[
(L, F) \simto (C_\ast(L : X), \partial),
\]

where \( C_\ast(L : X) \) records:

- the topology of the domain \( \Sigma \) of the immersion;
- the number and indices of the double points.

Homology groups of immersed filling:

\[
H_\ast(L : X).
\]

Theorem (Pezzimenti-Traynor)

If \((\Lambda, f)\) admits a GF-compatible Lagrangian filling \( L \), then

\[
GH_k(\Lambda, f) \simto H_{m-k}(L : X).
\]
Immersed GF-Isomorphism

All dimensions:

From a GF-compatible immersed Lagrangian filling $L$,

$$(L, F) \leadsto (C_*(L : X), \partial),$$

where $C_*(L : X)$ records:

- the topology of the domain $\Sigma$ of the immersion;
Immersed GF-Isomorphism

All dimensions:

From a GF-compatible immersed Lagrangian filling $L$,

$$(L, F) \mapsto (C_*(L : X), \partial),$$

where $C_*(L : X)$ records:

- the topology of the domain $\Sigma$ of the immersion;
- the number and indices of the double points.

Theorem (Pezzimenti-Traynor)

If $(\Lambda, f)$ admits a GF-compatible Lagrangian filling $L$, then

$$GH_k(\Lambda, f) \sim = H_{m-k}(L : X).$$
Immersed GF-Isomorphism

All dimensions:

From a GF-compatible immersed Lagrangian filling \( L \),

\[(L, F) \rightsquigarrow (C_*(L : X), \partial),\]

where \( C_*(L : X) \) records:
- the topology of the domain \( \Sigma \) of the immersion;
- the number and indices of the double points.

\[\implies \text{Homology groups of immersed filling: } H_*(L : X).\]
Immersion GF-Isomorphism

All dimensions:

From a GF-compatible immersed Lagrangian filling $L$,

$$(L, F) \mapsto (C_\ast(L : X), \partial),$$

where $C_\ast(L : X)$ records:

- the topology of the domain $\Sigma$ of the immersion;
- the number and indices of the double points.

$\implies \text{Homology groups of immersed filling: } H_\ast(L : X)$. 

Theorem (Pezzimenti-Traynor)

If $(\Lambda, f)$ admits a GF-compatible Lagrangian filling $L$, then

$$GH^k(\Lambda, f) \cong H_{m-k}(L : X).$$
Ideas underlying Immersed GF-Isomorphism

- Consider a “sheared difference function” for the Lagrangian filling:

  \[ \Delta(t, x, \eta, \tilde{\eta}) = F(t, x, \tilde{\eta}) - F(t, x, \eta) + H(t) \]

  - Key:
    - If \( L \) is the immersed image of \( \Sigma \), then \( \Delta \) has a critical submanifold diffeomorphic to \( \Sigma \) of index \(-1 + (N+1)\);
    - For each double point of \( L \), a pair of critical points \( x^\pm \) with opposite values and indices \((i + \lfloor (m - 1)/2 \rfloor) + (N+1)\) and \(- (i + \lfloor (m - 1)/2 \rfloor) + (m-1) + (N+1)\)
    - A critical point for each Reeb chord of \( \Lambda \).
Ideas underlying Immersed GF-Isomorphism

- Consider a “sheared difference function” for the Lagrangian filling:

Given a GF $F : \mathbb{R}_+ \times M \times \mathbb{R}^N \to \mathbb{R}$ for $\psi(L) \subset T^*(\mathbb{R}_+ \times M)$, and a function $H : \mathbb{R}_+ \to \mathbb{R}$, define the sheared difference function $\Delta : \mathbb{R}_+ \times M^m \times \mathbb{R}^{2N} \to \mathbb{R}$ by:

$$\Delta(t, x, \eta, \tilde{\eta}) = F(t, x, \tilde{\eta}) - F(t, x, \eta) + H(t).$$
Ideas underlying Immersed GF-Isomorphism

• Consider a “sheared difference function” for the Lagrangian filling:

Given a GF $F : \mathbb{R}_+ \times M \times \mathbb{R}^N \to \mathbb{R}$ for $\psi(L) \subset T^*(\mathbb{R}_+ \times M)$, and a function $H : \mathbb{R}_+ \to \mathbb{R}$,

![Graphical illustration of a sheared difference function]

define the sheared difference function $\Delta : \mathbb{R}_+ \times M^m \times \mathbb{R}^{2N} \to \mathbb{R}$ by:

$$\Delta(t, x, \eta, \tilde{\eta}) = F(t, x, \tilde{\eta}) - F(t, x, \eta) + H(t).$$

• Key: If $L$ is the immersed image of $\Sigma$, then $\Delta$ has
  
  - a critical submanifold diffeomorphic to $\Sigma$ of index $-1 + (N + 1)$;
  
  for each double point of $L$, a pair of critical points $x_i^\pm$ with opposite values and indices
    
    $$(i + \lfloor \frac{m-1}{2} \rfloor) + (N + 1) \text{ and } -(i + \lfloor \frac{m-1}{2} \rfloor) + (m - 1) + (N + 1)$$
  
  - a critical point for each Reeb chord of $\Lambda$. 

Sketch of Proof of Immersed GF-Isomorphism

- View \((\Delta^\infty, \Delta^{-\mu})\) as a Relative Mapping Cone.
Sketch of Proof of Immersed GF-Isomorphism

- View \((\Delta^\infty, \Delta^{-\mu})\) as a Relative Mapping Cone.

\[
F = tf, \quad \Delta = t\delta + H
\]

\[
t_+ \quad C(\delta^\infty, \delta^\epsilon) \quad \text{(by compatibility)}
\]
Sketch of Proof of Immersed GF-Isomorphism

- View \((\Delta^\infty, \Delta^{-\mu})\) as a Relative Mapping Cone.

\[ F = tf \]
\[ \Delta = t\delta + H \]

- Get a long exact sequence:

\[
\cdots \rightarrow H^{k+1}(\Delta^\infty, \Delta^{-\mu}) \rightarrow H_{m-k}(L : X) \rightarrow GH^k(\Lambda, f) \rightarrow H^k(\Delta^\infty, \Delta^{-\mu}) \rightarrow \cdots
\]
Sketch of Proof of Immersed GF-Isomorphism

- View \((\Delta^\infty, \Delta^{-\mu})\) as a Relative Mapping Cone.

\[
F = tf \\
\Delta = t\delta + H
\]

- Get a long exact sequence:

\[
\cdots \to H^{k+1}(\Delta^\infty, \Delta^{-\mu}) \to H_{m-k}(L : X) \to GH^k(\Lambda, f) \to H^k(\Delta^\infty, \Delta^{-\mu}) \to \cdots
\]

\[
H^*(\Delta^\infty, \Delta^{-\mu}) = 0, \forall * \implies
\]

\[
\cdots \to 0 \to H_{m-k}(L : X) \xrightarrow{\cong} GH^k(\Lambda, f) \to 0 \to \cdots
\]
Illustration: What types of fillings can be realized if $\Gamma_{\Lambda,f} = t + 2$?

By Theorem, need:

- $|H_{-1}(L : X)| = 1$,
- $|H_0(L : X)| = 2$.

Embedded genus 1 is possible!
Illustration: What types of fillings can be realized if $\Gamma_{\Lambda,f} = t + 2$?

By Theorem, need:

- $|H_{-1}(L : X)| = 1$,
- $|H_0(L : X)| = 2$.

Disk with 1 double point (of index 0) is possible!
Homology Groups of Immersed Fillings

**Illustration:** What types of fillings can be realized if $\Gamma_{\Lambda, f} = t + 2$?

By Theorem, need:

- $|H_{-1}(L : X)| = 1$,
- $|H_0(L : X)| = 2$.

Genus 1 with 2 double points is possible!
Illustration: What types of fillings can be realized if $\Gamma_{\Lambda,f} = t + 2$?

By Theorem, need:

- $|H_{-1}(L : X)| = 1$,
- $|H_0(L : X)| = 2$.

Genus 2 with 1 double point is possible!
Illustration: What types of fillings can be realized if $\Gamma_{\Lambda,f} = t + 2$?

By Theorem, need:

- $|H_{-1}(L : X)| = 1$,
- $|H_0(L : X)| = 2$.

Disk with 2 double points is not possible!
Potential “check-mark” geography for Lagrangian fillings of $\Lambda$ with $\Gamma_{\Lambda, f} = t + \sum_{i=0}^{n} c_i (t^i + t^{-i})$

If $c_0 = 1$

If $c_2 = 1$

If $c_0 = 2, c_1 = 1, c_2 = 1$
Potential “check-mark” geography for Lagrangian fillings of $\Lambda$ with $\Gamma_{\Lambda,f} = t + \sum_{i=0}^{n} c_i (t^i + t^{-i})$.

If $c_0 = 1$,

If $c_2 = 1$,

If $c_0 = 2$, $c_1 = 1$, $c_2 = 1$.

**Question:** Which of these can be realized?
1. Geography of Fillings

2. Legendrians, Lagrangians, and Lagrangian Cobordisms
   - Generating Families

3. Obstructions to Lagrangian Fillings

4. Constructions
Theorem (Bourgeois-Sabloff-Traynor ’15)

Suppose $\Lambda_+$ has a generating family. Then there exists an embedded GF-compatible Lagrangian cobordism between $\Lambda_-$ and $\Lambda_+$ if:

- $\Lambda_-$ is Legendrian isotopic to $\Lambda_+$.
- $\Lambda_-$ is obtained from $\Lambda_+$ by “pinch moves” (compatible with ruling).
- $\Lambda_-$ is obtained by “filling” a trivial unknotted component of $\Lambda_+$. 

Lisa Traynor (Bryn Mawr)  
Geography of Immersed Lagrangian Fillings  
Symplectic Zoominar  
37/51
Theorem (Bourgeois-Sabloff-Traynor ’15)

Suppose $\Lambda_+$ has a generating family. Then there exists an embedded GF-compatible Lagrangian cobordism between $\Lambda$ and $\Lambda_+$ if:

- $\Lambda$ is Legendrian isotopic to $\Lambda_+$;
- $\Lambda$ is obtained from $\Lambda_+$ by “pinch moves” (compatible with ruling);
- $\Lambda$ is obtained by “filling” a trivial unknotted component of $\Lambda_+$. 

Lisa Traynor (Bryn Mawr)  
Geography of Immersed Lagrangian Fillings  
Symplectic Zoominar  
37 / 51
Theorem (Bourgeois-Sabloff-Traynor ’15)

Suppose $\Lambda_+^{\pm}$ has a generating family. Then there exists an embedded GF-compatible Lagragian cobordism between $\Lambda_-$ and $\Lambda_+$ if:

- $\Lambda_-$ is Legendrian isotopic to $\Lambda_+$;
- $\Lambda_-$ is obtained from $\Lambda_+$ by “pinch moves” (compatible with ruling);
- $\Lambda_-$ is obtained by “filling” a trivial unknotted component of $\Lambda_+$.
An embedded filling of $7_4$:
An embedded filling of $7_4$: 

![Diagram of an embedded filling of $7_4$](image)
Embedded Filling

An embedded filling of $7_4$:
An embedded filling of $7_4$: 
An embedded filling of $7_4$:
An embedded filling of $7_4$: 

∅
Construction of Immersed Lagrangian Cobordism

**Theorem (Pezzimenti-Traynor)**

If a Legendrian knot $\Lambda_+$ has a ruling that is well behaved with respect to a clasp, and $\Lambda_-$ is obtained by unclasping,

\[
\begin{align*}
\text{or } & \quad \Lambda_+ \\
\text{or } & \quad \Lambda_-
\end{align*}
\]

There is also a "clasping" move.
Theorem (Pezzimenti-Traynor)

If a Legendrian knot $\Lambda_+$ has a ruling that is well behaved with respect to a clasp, and $\Lambda_-$ is obtained by unclasping,

then there exist GFs $f_{\pm}$ for $\Lambda_{\pm}$ and an immersed GF-compatible Lagrangian cobordism from $(\Lambda_-, f_-)$ to $(\Lambda_+, f_+)$ with a double point of index $|i - j|$.

There is also a “clasping” move.
An immersed disk filling of $7_4$ with 3 double points:
An immersed disk filling of $7_4$ with 3 double points:
An immersed disk filling of $7_4$ with 3 double points:
An immersed disk filling of $7_4$ with 3 double points:
An immersed disk filling of $7_4$ with 3 double points:
An immersed disk filling of $7_4$ with 3 double points:
An immersed disk filling of $7_4$ with 3 double points:
An immersed disk filling of $7_4$ with 3 double points:
From Polynomial

- It is always possible to:
  - fix genus and increase the number of double points by 2;
  - increase genus by 1 & increase # of double points by 1.
Geography of a Legendrian $7_4$

It is always possible to:
- Fix genus and increase the number of double points by 2.
- Increase genus by 1 and increase the number of double points by 1.

Lisa Traynor (Bryn Mawr)
Symplectic Zoominar 41 / 51
Geography of a Legendrian $7_4$

It is always possible to:
- fix genus and increase the number of double points by 2; ↑
- increase genus by 1 & increase # of double points by 1. ↗
New Fillings from Old:

Lagrangian fillings have Legendrian lifts:

Adding two double points: ↑

![Diagram of Lagrangian fillings and Legendrian lifts.](image)
New Fillings from Old:

Lagrangian fillings have Legendrian lifts:

Adding two double points: ↑

Adding genus and a double point: ↗
Smooth vs Lagrangian Geography

Smooth Geography

Lagrangian Geography
Smooth vs Lagrangian Geography: $m(5_2)$

Smooth Geography

Lagrangian Geography
Smooth vs Lagrangian Geography: another $m(5_2)$

Smooth Geography

Lagrangian Geography

Genus 0 1 2 3 4

Double Points

Genus 0 1 2 3 4

Double Points

Lisa Traynor (Bryn Mawr)

Geography of Immersed Lagrangian Fillings

Symplectic Zoominar 45/51
Further Questions: Geography

• [Geography]

Q: For fixed $(\Lambda, f)$, when can one not realize the chart determined by smooth topology and the polynomial $\Gamma_{\Lambda,f}$ (or $\Gamma_{\Lambda,\varepsilon}$)?

Yes, for $\text{LCH}$; [Etgü ’18, Ekholm-Lekili ’17] $\exists(\Lambda, \varepsilon) : \Gamma(\Lambda, \varepsilon) = t^6$, but $\not\exists$ embedded filling inducing $\varepsilon$.

Q: $\exists$ obstructions from product and $\mathcal{A}_\infty$ relations on $GH^*(\Lambda, f)$ as constructed by Ziva Myer?
Further Questions: Geography

• [Geography]

**Q:** For fixed $(\Lambda, f)$, when can one *not* realize the chart determined by smooth topology and the polynomial $\Gamma_{\Lambda,f}$ (or $\Gamma_{\Lambda,\varepsilon}$)?

Are there obstructions from product and $A_\infty$ relations on $GH^*(\Lambda, f)$ ($LCH^*(\Lambda, \varepsilon)$)?

---

Lisa Traynor (Bryn Mawr) | Geography of Immersed Lagrangian Fillings | Symplectic Zoominar | 46/51
Further Questions: Geography

[Geography]

Q: For fixed $(\Lambda, f)$, when can one not realize the chart determined by smooth topology and the polynomial $\Gamma_{\Lambda,f}$ (or $\Gamma_{\Lambda,\epsilon}$)?

Are there obstructions from product and $A_\infty$ relations on $GH^*(\Lambda, f)$ ($LCH^*(\Lambda, \epsilon)$)?

Yes, for $LCH$;

[Etgü ’18, Ekholm-Lekili ’17] \(\exists(\Lambda, \epsilon) : \Gamma(\Lambda, \epsilon) = t + 6\), but \(\nexists\) embedded filling inducing \(\epsilon\).
Further Questions: Geography

• [Geography]

Q: For fixed $(\Lambda, f)$, when can one not realize the chart determined by smooth topology and the polynomial $\Gamma_{\Lambda,f}$ (or $\Gamma_{\Lambda,\varepsilon}$)?

Are there obstructions from product and $A_\infty$ relations on $GH^*(\Lambda, f)$ ($LCH^*(\Lambda, \varepsilon)$)?

- Yes, for $LCH$;
  [Etgü ’18, Ekholm-Lekili ’17] $\exists (\Lambda, \varepsilon) : \Gamma(\Lambda, \varepsilon) = t + 6$, but $\not\exists$ embedded filling inducing $\varepsilon$.

- Q: $\exists$ obstructions from product and $A_\infty$ relations on $GH^*(\Lambda, f)$ as constructed by Ziva Myer?
Further Questions: Mutation

• [Mutation] How are different fillings of \((\Lambda, f)\) related?

Related results:

- **Smooth World**: Can always decrease number of double points by 1 at cost of increasing genus by 1.
Further Questions: Mutation

- **[Mutation]** How are different fillings of $(\Lambda, f)$ related?

![Diagram showing relationship between genus and double points with the equation $g + p = (tb + 1)/2$.]

Related results:

- **Smooth World:** Can always decrease the number of double points by 1 at the cost of increasing genus by 1.

- **Symplectic World:**
  
  [Capovilla Searle - Legout - Limouzineau - Murphy - Pan - Traynor]

  - Double pts with particular indices and actions (VIA) can be traded for genus.
Further Questions: Mutation

- **[Mutation]** How are different fillings of \((\Lambda, f)\) related?

![Diagram showing the relationship between genus and double points]

**Symplectic World:**

[Capovilla Searle - Legout - Limouzineau - Murphy - Pan - Traynor]

- Double pts with particular indices and actions (VIA) can be traded for genus.

Q: Can one trade genus for double points?

A: No. Augmentation counts (via \(A_\infty\) arguments) can prove this.
Further Questions: Mutation

- **[Mutation]** How are different fillings of \((\Lambda, f)\) related?

![Graph showing the relationship between Genus and Double Points]

**Symplectic World:**

[Capovilla Searle - Legout - Limouzineau - Murphy - Pan - Traynor]

- Double pts with particular indices and actions (VIA) can be traded for genus.
- **Q:** Can one trade genus for double points?
Further Questions: Mutation

- **[Mutation]** How are different fillings of \((\Lambda, f)\) related?

![Diagram showing the relationship between Genus and Double Points]

- **Symplectic World:**
  
  [Capovilla Searle - Legout - Limouzineau - Murphy - Pan - Traynor]

  - Double pts with particular indices and actions (VIA) can be traded for genus.
  - **Q:** Can one trade genus for double points?
    - **A:** No. Augmentation counts (via \(A_\infty\) arguments) can prove this.
Further Questions: Mutation

- **[Mutation]** How are different fillings of \((\Lambda, f)\) related?

**Symplectic World: GF-Fillings**
- NW-SE portion of “check-mark” has double points with correct indices.
Further Questions: Mutation

- **[Mutation]** How are different fillings of $(\Lambda, f)$ related?

![Graph showing the relationship between double points and genus in the Symplectic World: GF-Fillings.]

**Symplectic World: GF-Fillings**

- NW-SE portion of “check-mark” has double points with correct indices.
- **Q:** Can one interchange double points and genus in GF-fillings?
Further Questions: Botany

• [Botany]

Q: For fixed $(\Lambda, f)$, how many options for a fixed topology and number of immersion points?

Related results: [Ekholm-Honda-Kálmán ‘16, Pan ‘17, Shende-Treumann-Williams-Zaslow ‘19]: Max $\tau_b(2, n)$-torus link admits $C_n = n + 1$ embedded fillings that are smoothly isotopic but pairwise not (exact) Lagrangian isotopic.

[Casals-Gao ‘20]: “Most” max $\tau_b$ positive torus knots admit infinitely many embedded Lagrangian fillings that are smoothly isotopic but not Hamiltonian isotopic.
Further Questions: Botany

[Botany]

Q: For fixed $(\Lambda, f)$, how many options for a fixed topology and number of immersion points?

Related results:

[Ekholm-Honda-Kálmán ’16, Pan ’17, Shende-Treumann-Williams-Zaslow ’19]:

Max $tb \ (2, n)$-torus link admits $C_n = \frac{1}{n+1} \binom{2n}{n}$ embedded fillings that are smoothly isotopic but pairwise not (exact) Lagrangian isotopic.

[Casals-Gao ’20]: “Most” max $tb$ positive torus knots admit infinitely many embedded Lagrangian fillings that are smoothly isotopic but not Hamiltonian isotopic.
Q: For fixed $(\Lambda, f)$, how many options for a fixed topology and number of immersion points?

Related results:

- [Ekholm-Honda-Kálmán ’16, Pan ’17, Shende-Treumann-Williams-Zaslow ’19]:
  \[
  \text{Max } \text{tb} (2, n)\text{-torus link admits } C_n = \frac{1}{n+1} \binom{2n}{n} \text{ embedded fillings that are smoothly isotopic but pairwise not (exact) Lagrangian isotopic.}
  \]

- [Casals-Gao ’20]: “Most” max tb positive torus knots admit infinitely many embedded Lagrangian fillings that are smoothly isotopic but not Hamiltonian isotopic.
Q: For fixed \((\Lambda, f)\), how many options for a fixed topology and number of immersion points?

Related results:

- [Ekholm-Honda-Kálmán ’16, Pan ’17, Shende-Treumann-Williams-Zaslow ’19]:
  \[
  \text{Max } \text{tb} (2, n)\text{-torus link admits } C_n = \frac{1}{n+1} \binom{2n}{n} \text{ embedded fillings that are smoothly isotopic but pairwise not (exact) Lagrangian isotopic.}
  \]

- [Casals-Gao ’20]: “Most” max tb positive torus knots admit infinitely many embedded Lagrangian fillings that are smoothly isotopic but not Hamiltonian isotopic.

Q: For fixed \((\Lambda, f)\), count number of fillings with fixed topology and double points?
Thank you!