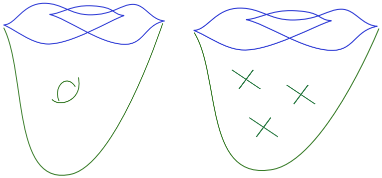
The Geography of Immersed Lagrangian Fillings of Legendrian Submanifolds

Lisa Traynor

Bryn Mawr College



April 2020

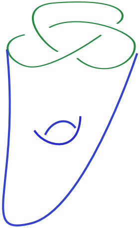
Joint Work

Joint work with **Samantha Pezzimenti** (PhD Bryn Mawr '18), Assistant Teaching Professor at Penn State Brandywine.

Outline

Filling Smooth Knots

Topological Problem: Given a smooth knot $K \subset S^3$, find smooth "filling" surfaces.



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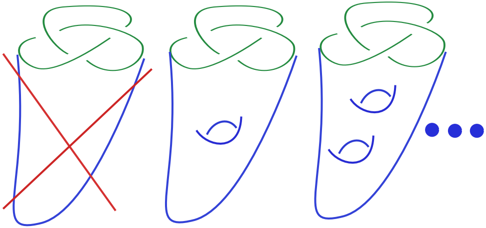

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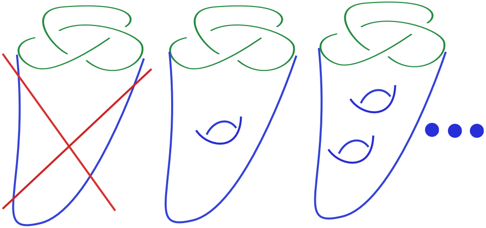


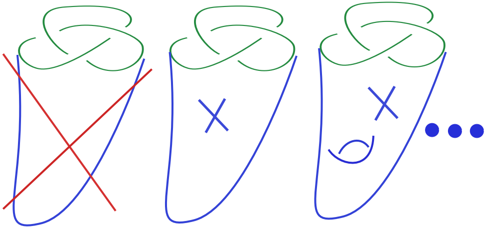
$$F \subset B^4 \text{ with } \partial F = K.$$

There are many options! Find **minimal genus**:

$$g_4(K) := \min \{\text{genus}(F) : \partial F = K\}.$$

Immersed Fillings of Smooth Knots

One can also try to find **immersed fillings** with transverse double points.



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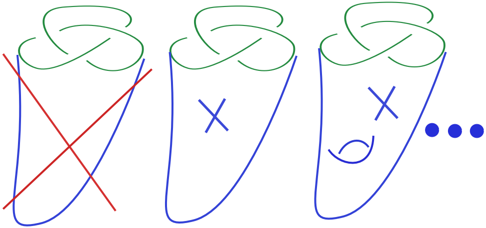

4-ball (genus 0) **crossing number:**

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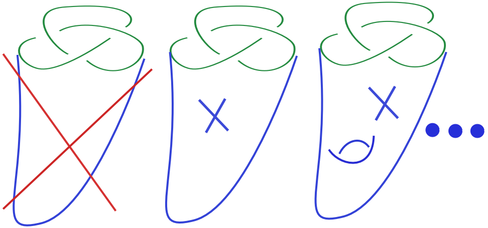

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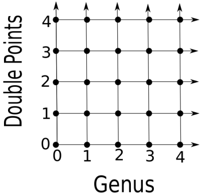
Example: $c_4^0\,(m(5_2)) = 1$; $c_4^0(7_4) = 2$. [Strle-Owens ('15)]

Geography of Smooth Fillings of Smooth Knots

Smooth Geography Question: Given a smooth knot $K \subset S^3$, what combinations of genus and double points can be realized by smooth fillings?



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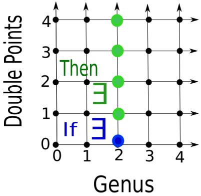

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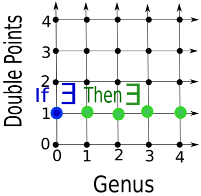

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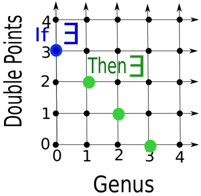

It is always possible to:

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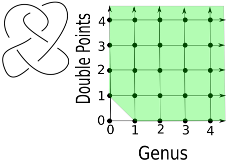
- fix number of double points and increase the genus by 1;
- eliminate a double point at the cost of increasing the genus.

Examples of Smooth Geography for Knots



Smooth Geography of the knot $m(5_2)$.

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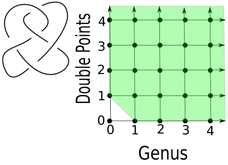

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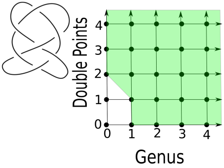

Smooth Geography of the knot 7_4.

Higher-Dimensional Geography

Smooth Geography Question: Given a smooth m-dimensional submanifold $K^m \subset S^{2m+1}$, what combinations of Betti numbers and double points can be realized by a smooth filling $F^{m+1} \subset B^{2m+2}$?

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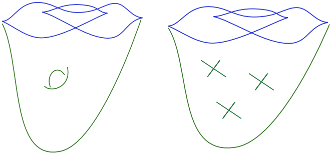
I would like to know if *others* know anything about this problem!

Lagrangian Geography

Symplectic Problem: Given a **Legendrian knot** in a 3-dimensional space, try to find **immersed Lagrangian surface fillings** in a 4-dimensional space.



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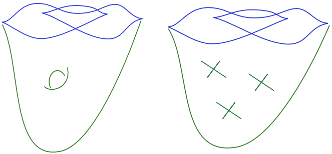

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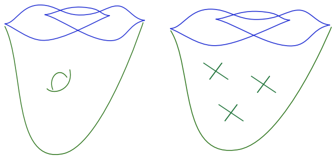

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How does Lagrangian Geography compare to Smooth Geography?

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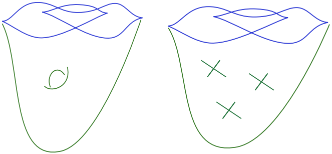


Question: How flexible/rigid are Lagrangian fillings?

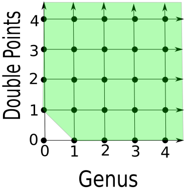
How does Lagrangian Geography compare to Smooth Geography?

Higher-Dimensional version is also interesting.

Preview

Will consider “GF-compatible” fillings. Find much more rigidity!

Smooth Geography



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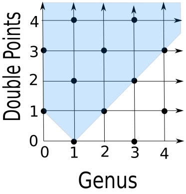

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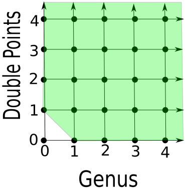

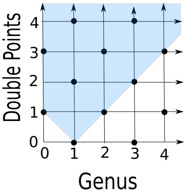

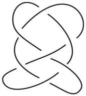

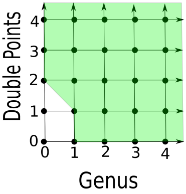

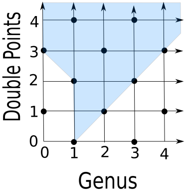

Outline

Contact and Symplectic Setting

Standard Contact Manifold: $(\mathbb{R}^{2n+1}, \xi = \ker \alpha)$

$$J^1(\mathbb{R}^n) = T^*\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{2n+1}, \quad \alpha = dz - \sum_i y_i dx_i$$

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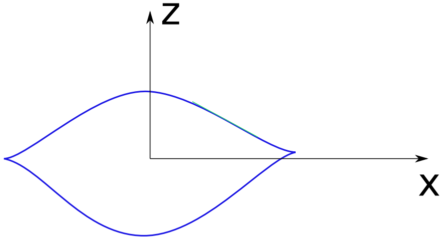

$y = \frac{dz}{dx}$

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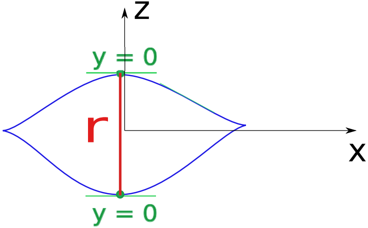

Reeb Chord

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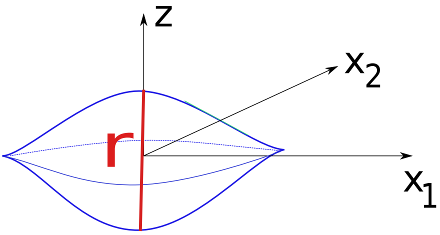

$y_1 = \frac{\partial z}{\partial x_1}, \quad y_2 = \frac{\partial z}{\partial x_2}$

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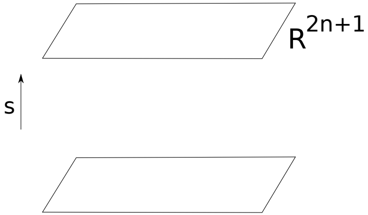

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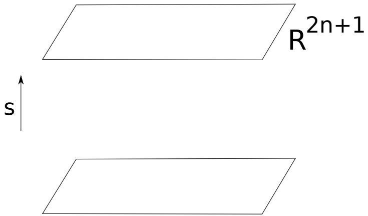

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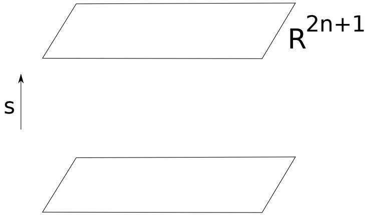

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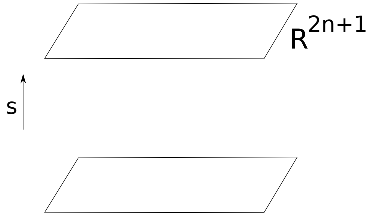

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There are no *closed*, exact Lagrangians (Gromov).

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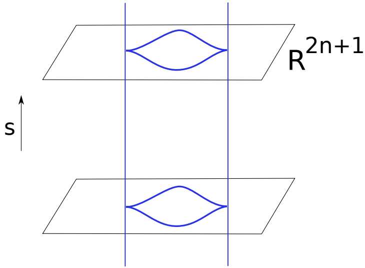

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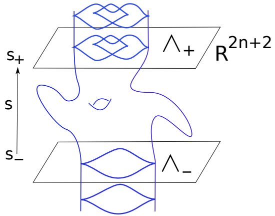
- For a Legendrian Λ, the **cylinder** $\mathbb{R} \times \Lambda$ **is an exact Lagrangian.**

Lagrangian Cobordism between Legendrians

A **Lagrangian cobordism** from Λ_- to Λ_+:



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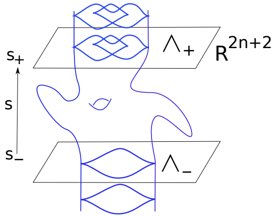

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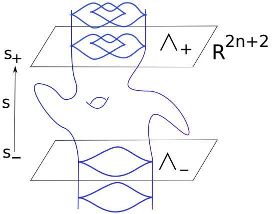

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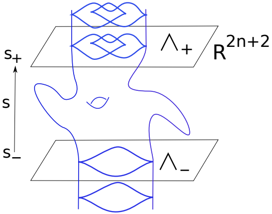

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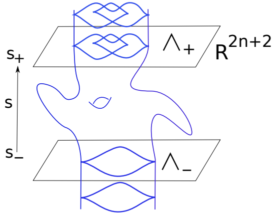

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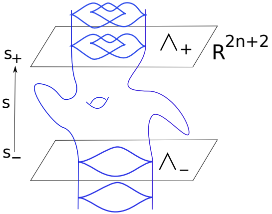

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Arise in relative SFT (Eliashberg-Givental-Hofer)

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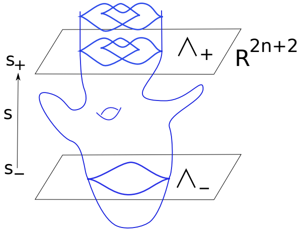

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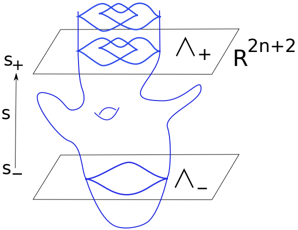

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GF-Compatible Lagrangian Fillings

Geography via technique of "generating families of functions".

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- Lagrangian fillings that can be “generated” by an extension, F, of this function.

  Maybe corresponds to fillings that induce specified augmentation ε.

Generating Families

Classic technique;
Modernized by Laudenbach, Sikorav, Chaperon, Viterbo

Generating Families

Classic technique;
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For Legendrians $\Lambda \subset J^1M$, want to study Lagrangian fillings $L \subset \mathbb{R} \times J^1M$ via generating families.

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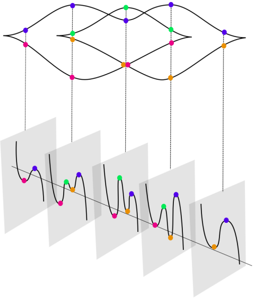
Strategy: Apply analysis/Morse theoretic arguments to these functions to obtain **invariants of** and **relationships between** the Lagrangian and Legendrian submanifolds.

Generating Family of Functions Example

Idea: Construct a 1-parameter family of functions $F_x : \mathbb{R} = \{e\} \to \mathbb{R}$



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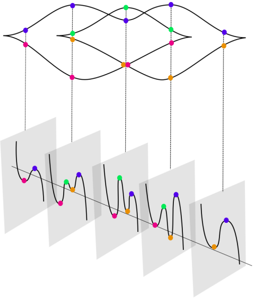

$\exists F : \mathbb{R} \times \mathbb{R}^1 \to \mathbb{R}$ so that Λ is the "1-jet of F along the fiber critical submanifold":

$$\Lambda = \left\{ \left(x, \frac{\partial F}{\partial x}(x,e), F(x,e) \right) : \frac{\partial F}{\partial e}(x,e) = 0 \right\}.$$

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For Legendrian knots: [Chekanov-Pushkar; Fuchs - Rutherford]

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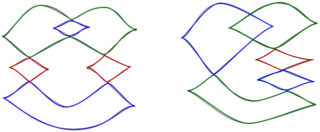

Graded normal rulings of two different Legendrian $m(5_2)$ knots.

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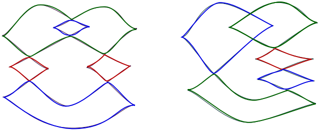

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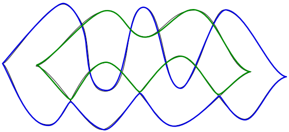

Graded normal ruling of a Legendrian 7_4 knot.

Generating Family Cohomology for Legendrians

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$$\Gamma_{\Lambda,f}(t) = \sum \dim GH^k(\Lambda, f) t^k.$$

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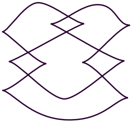

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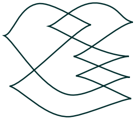

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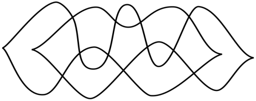

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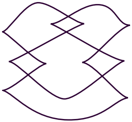

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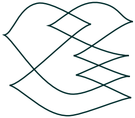

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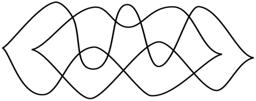

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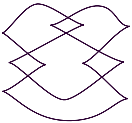

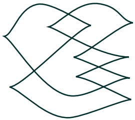

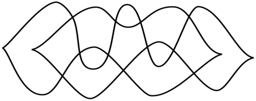

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For a connected m-dimensional Legendrian:

$$\Gamma_{(\Lambda,f)} = t^m + b_{m-1}t^{m-1} + \cdots + b_1 t + \sum_{i \geq \lfloor \frac{m-1}{2} \rfloor}^{n} c_i \left(t^i + t^{-i+(m-1)} \right),$$

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Generating Families for Lagrangian Fillings

Lagrangian cobordisms in $\mathbb{R} \times J^1M \equiv T^*(\mathbb{R}_+ \times M)$:

$$\begin{aligned} \psi : \mathbb{R} \times J^1M &\to T^*(\mathbb{R}_+ \times M) \\ (s, x, y, z) &\mapsto (e^s, x, z, e^s y). \end{aligned}$$

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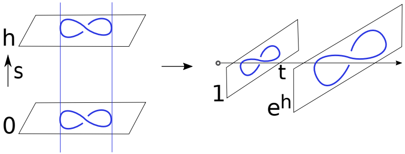

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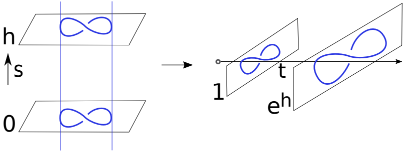

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$$F(t, x, \eta) = t\, f(x, \eta), \quad t \geq t_+.$$

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Outline

1 Geography of Fillings

2 Legendrians, Lagrangians, and Lagrangian Cobordisms
- Generating Families

3 Obstructions to Lagrangian Fillings

4 Constructions

GF Seidel Isomorphism

Theorem (Sabloff-Traynor, '13)

If Λ has a GF f, then any GF-compatible embedded Lagrangian filling L satisfies:

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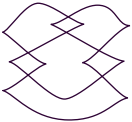

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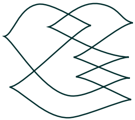

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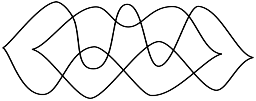

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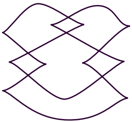

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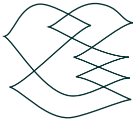

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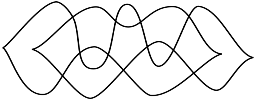

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Immersed Lagrangian Fillings

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However, **any** Λ with a GF will have an **immersed** GF-compatible filling. [Bourgeois-Sabloff-Traynor].

Question: What can the GF-polynomial $\Gamma_{(\Lambda,f)}(t)$ tell us about the Lagrangian geography problem?

Polynomial Obstructions to Immersed Fillings

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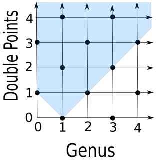
Moreover, if p is the number of double points, then

$$p + g \equiv \sum_{i=0}^{n} c_i \mod 2.$$

Potential Geography

Potential "check-mark" geography for Lagrangian fillings of Λ with $\Gamma_{\Lambda,f} = t + \sum_{i=0}^{n} c_i \left(t^i + t^{-i}\right)$.



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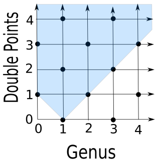

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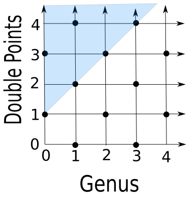

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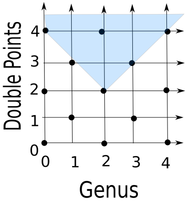

if $c_0 = 2, c_1 = 1, c_2 = 1$

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From a GF-compatible immersed Lagrangian filling L,

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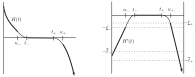

define the **sheared difference function** $\Delta : \mathbb{R}_+ \times M^m \times \mathbb{R}^{2N} \to \mathbb{R}$ by:

$$\Delta(t, x, \eta, \tilde{\eta}) = F(t, x, \tilde{\eta}) - F(t, x, \eta) + H(t).$$

Ideas underlying Immersed GF-Isomorphism

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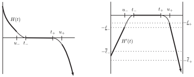

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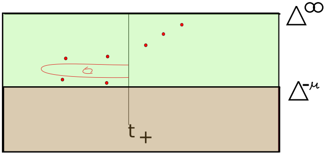
- **Key:** If L is the immersed image of Σ, then Δ has
  - a **critical submanifold** diffeomorphic to Σ of index $-1 + (N + 1)$;
  - for each **double point** of L, a **pair of critical points** $x_i^{\pm}$ with opposite values and indices $(i + \lfloor \frac{m-1}{2} \rfloor) + (N + 1)$ and $-(i + \lfloor \frac{m-1}{2} \rfloor) + (m - 1) + (N + 1)$
  - a **critical point** for each **Reeb chord** of Λ.

Sketch of Proof of Immersed GF-Isomorphism

- View $(\Delta^{\infty}, \Delta^{-\mu})$ as a Relative Mapping Cone.



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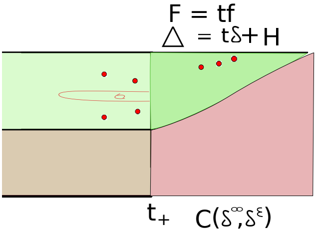

(by compatibility)

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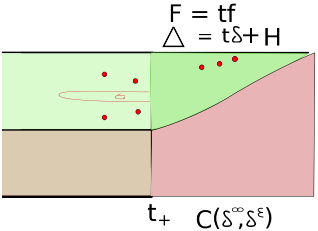

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- Get a long exact sequence:

$$\cdots \to H^{k+1}(\Delta^{\infty}, \Delta^{-\mu}) \to H_{m-k}(L : X) \to GH^{k}(\Lambda, f) \to H^{k}(\Delta^{\infty}, \Delta^{-\mu}) \to \cdots$$

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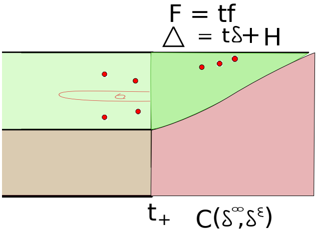

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$H^{*}(\Delta^{\infty}, \Delta^{-\mu}) = 0, \forall * \implies$

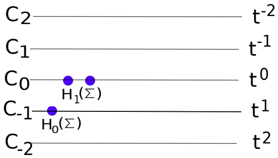
$$\cdots \to 0 \to H_{m-k}(L : X) \xrightarrow{\cong} GH^{k}(\Lambda, f) \to 0 \to \cdots . \quad \square$$

Homology Groups of Immersed Fillings

Illustration: What types of fillings can be realized if $\Gamma_{\Lambda,f} = t + 2$?

By Theorem, need:

- $|H_{-1}(L : X)| = 1$,
- $|H_0(L : X)| = 2$.



Embedded genus 1 is possible!

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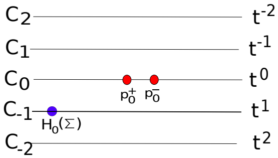

Disk with 1 double point (of index 0) is possible!

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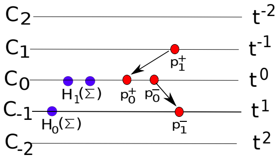

Genus 1 with 2 double points is possible!

Homology Groups of Immersed Fillings

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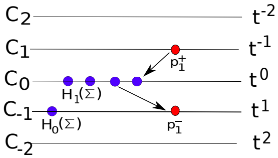

Genus 2 with 1 double point is possible!

Homology Groups of Immersed Fillings

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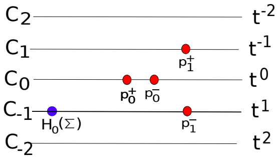
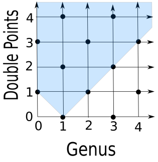
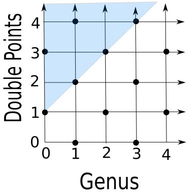


Disk with 2 double points is not possible!

Potential "check-mark" geography for Lagrangian fillings of Λ with $\Gamma_{\Lambda,f} = t + \sum_{i=0}^{n} c_i \left(t^i + t^{-i}\right)$



if $c_0 = 1$



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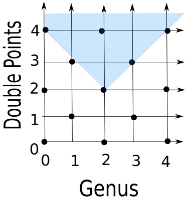

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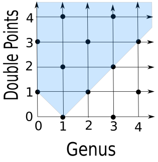

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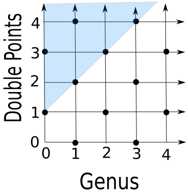

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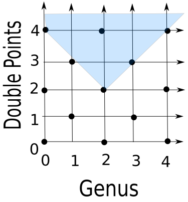

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Question: Which of these can be realized?

Outline

Embedded Moves

Theorem (Bourgeois-Sabloff-Traynor '15)

Suppose Λ_+ has a generating family. Then there exists an embedded GF-compatible Lagragian cobordism between Λ_- and Λ_+ if:

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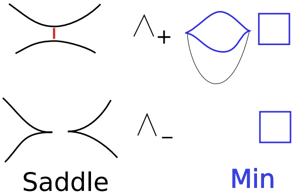
Theorem (Bourgeois-Sabloff-Traynor '15)

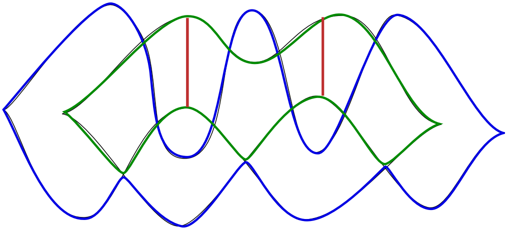
Suppose Λ_+ has a generating family. Then there exists an embedded GF-compatible Lagragian cobordism between Λ_- and Λ_+ if:

- *Λ_- is Legendrian isotopic to Λ_+;*
- *Λ_- is obtained from Λ_+ by "pinch moves" (compatible with ruling);*
- *Λ_- is obtained by "filling" a trivial unknotted component of Λ_+.*



Embedded Filling

An embedded filling of 7_4:



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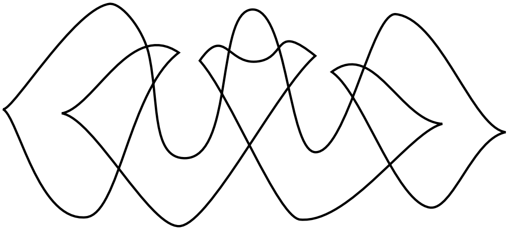

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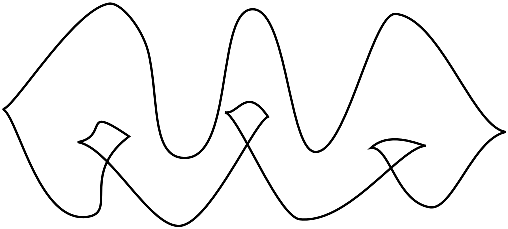

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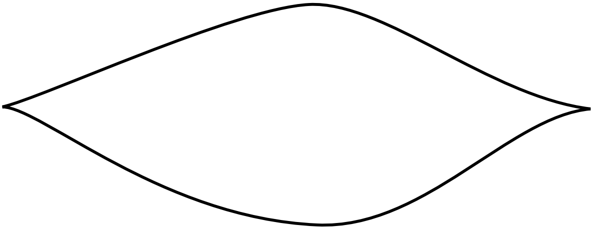

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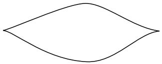

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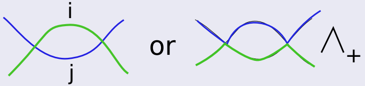
$$\emptyset$$

Construction of Immersed Lagrangian Cobordism

Theorem (Pezzimenti-Traynor)

If a Legendrian knot Λ_+ has a ruling that is well behaved with respect to a **clasp**, *and Λ_- is obtained by* **unclasping**,



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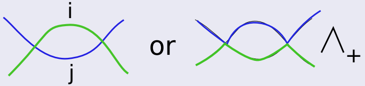
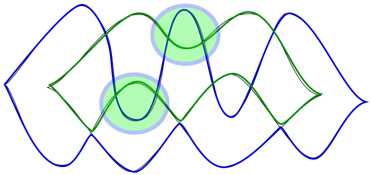


then there exist GFs $f_\pm$ for $\Lambda_\pm$ and an **immersed** *GF-compatible Lagrangian cobordism from (Λ_-, f_-) to (Λ_+, f_+) with a double point of index $|i - j|$.*

There is also a “clasping” move.

Immersed Moves

An immersed disk filling of 7_4 with 3 double points:



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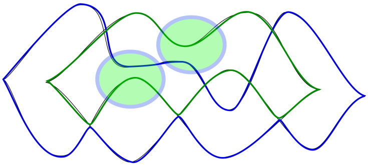

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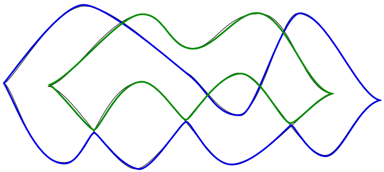

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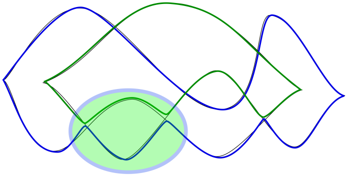

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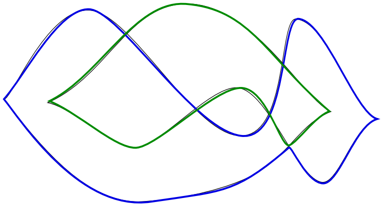

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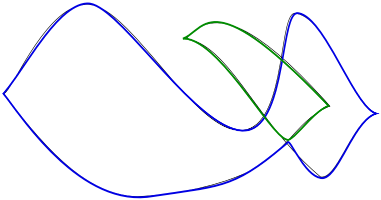

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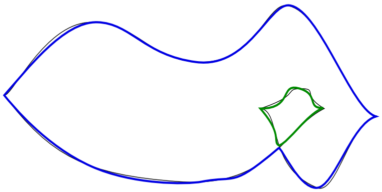

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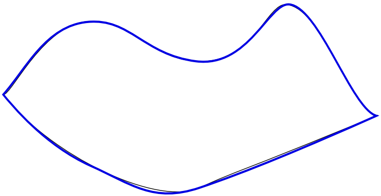
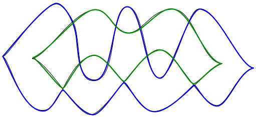
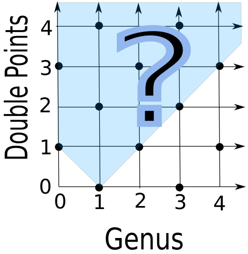


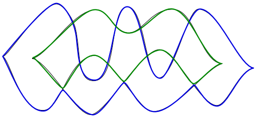
Geography of a Legendrian 7_4

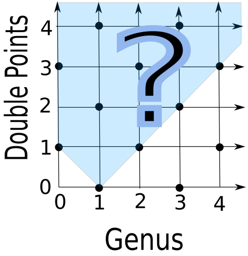


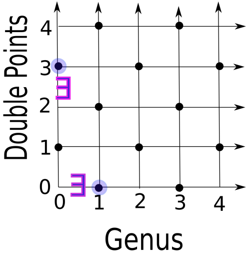


From Polynomial

Geography of a Legendrian 7_4







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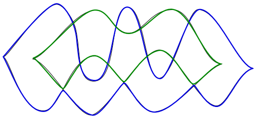
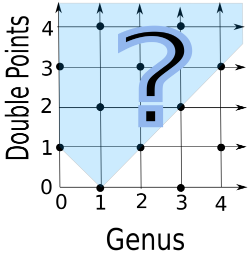
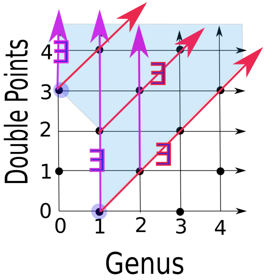






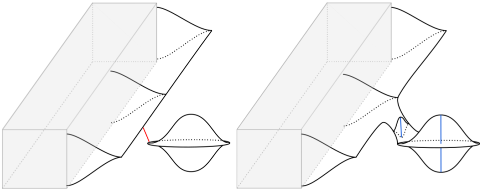
It is always possible to:

- fix genus and increase the number of double points by 2; ↑
- increase genus by 1 & increase # of double points by 1. ↗

New Fillings from Old:

Lagrangian fillings have Legendrian lifts:

Adding two double points: ↑



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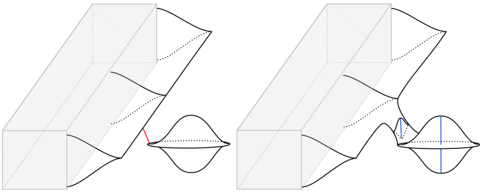
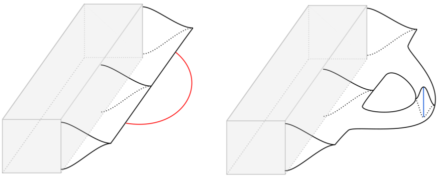
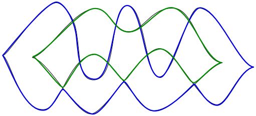
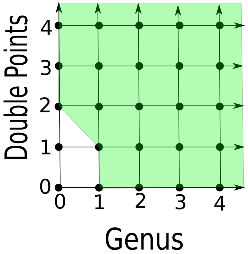


Adding genus and a double point: $\nearrow$

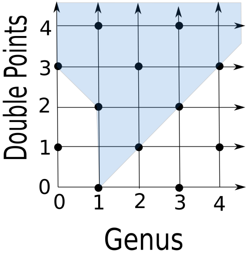


Smooth vs Lagrangian Geography: 7_4



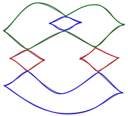


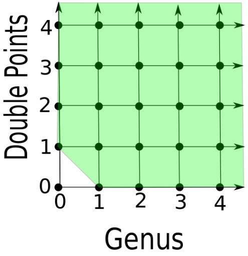
Smooth Geography

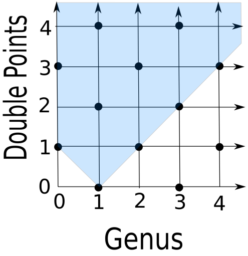


Lagrangian Geography

Smooth vs Lagrangian Geography: $m(5_2)$



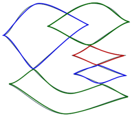


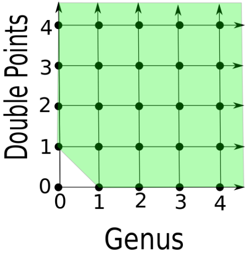


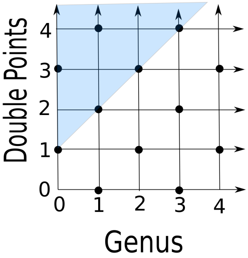
Smooth Geography

Lagrangian Geography

Smooth vs Lagrangian Geography: another $m(5_2)$







Smooth Geography

Lagrangian Geography

Further Questions: Geography

- **[Geography]**

Q: For fixed (Λ, f), when can one *not* realize the chart determined by smooth topology and the polynomial $\Gamma_{\Lambda,f}$ (or $\Gamma_{\Lambda,\varepsilon}$)?

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Q: For fixed (Λ, f), when can one *not* realize the chart determined by smooth topology and the polynomial $\Gamma_{\Lambda,f}$ (or $\Gamma_{\Lambda,\varepsilon}$)?

Are there obstructions from product and A_∞ relations on $GH^*(\Lambda, f)$ ($LCH^*(\Lambda, \varepsilon)$)?

Further Questions: Geography

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Q: For fixed (Λ, f), when can one *not* realize the chart determined by smooth topology and the polynomial $\Gamma_{\Lambda,f}$ (or $\Gamma_{\Lambda,\varepsilon}$)?

Are there obstructions from product and A_∞ relations on $GH^*(\Lambda, f)$ ($LCH^*(\Lambda, \varepsilon)$)?

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  [Etgü '18, Ekholm-Lekili '17] $\exists(\Lambda, \varepsilon) : \Gamma(\Lambda, \varepsilon) = t + 6$,
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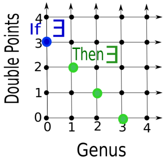
Further Questions: Mutation

- **[Mutation]** How are different fillings of (Λ, f) **related**?



Related results:

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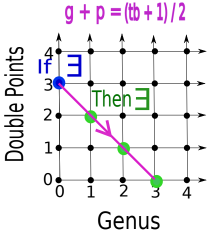

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  [Capovilla Searle - Legout - Limouzineau - Murphy - Pan - Traynor]

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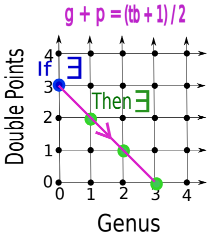

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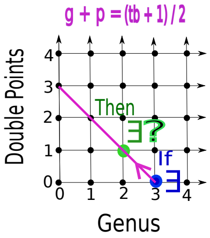

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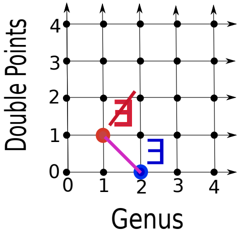

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    A: No. Augmentation counts (via A_∞ arguments) can prove this.

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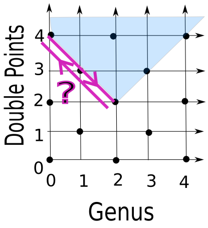

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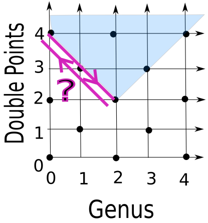

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  - **Q:** Can one interchange double points and genus in GF-fillings?

Further Questions: Botany

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Q: For fixed (Λ, f), count number of fillings with fixed topology and double points?

Thank you!