# Geometry of quantum uncertainty

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Zoominar, April, 2020

with Louis Ioos and David Kazhdan

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**Thesis:** Optimal quantizations correspond to compatible almost complex structures on $(M, \omega)$.

# Deformation quantization

**Star product:** Associative (non-commutative) deformation of $(C^\infty(M), f \cdot g)$

$$f * g = fg + \hbar c_1(f,g) + \hbar^2 c_2(f,g) + \dots,$$

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F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, 1977

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Table: Quantum-Classical Correspondence

| | CLASSICAL | QUANTUM |
|---|---|---|
| | Symplectic mfd $(M, \omega)$ | $\mathbb{C}$-Hilbert space $H$ |
| OBSERVABLES | $f \in C^\infty(M)$ | $A \in \mathcal{L}(H)$ |
| STATES | Probability measures on $M$ | Density ops $\rho \in \mathcal{S}$ |
| BRACKET | Poisson bracket $\{f, g\}$ | Commutator $\frac{i}{\hbar}[A, B]$ |

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**Reversibility:** $\mathcal{B}_\hbar := (n_\hbar)^{-1} T_\hbar^* T_\hbar$ - Berezin transform, composition of dequantization and quantization.

# Unsharpness cocycle

There exists a bi-differential operator
$c : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ such that

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Associativity $\Rightarrow$ $c$ - Hochschild cocycle:
$f_1\, c(f_2, f_3) - c(f_1 f_2, f_3) + c(f_1, f_2 f_3) - c(f_1, f_2)\, f_3 = 0$
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**Corollary**

*There exists a bilinear symmetric form $G$ on $TM$:*
$c_+(f, g) =: -\frac{1}{2} G(sgrad\, f, sgrad\, g)$
*where $sgrad\, f$, $sgrad\, g$ Hamiltonian vector fields of*
$f$, $g \in C^\infty(M, \mathbb{R})$

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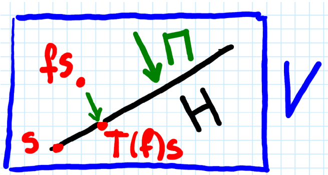

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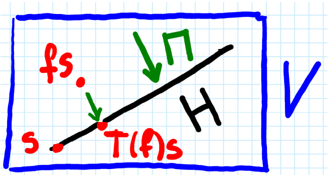


**Unsharpness tensor:** $G(\xi, \eta) = \omega(\xi, J\eta)$ (Xu)

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(II) $\mathrm{Vol}(M,G) \geq \mathrm{Vol}(M,\omega)$, *with equality* $\Leftrightarrow G=G_J$, $\rho=0$.

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(II) $\mathrm{Vol}(M, G) \geq \mathrm{Vol}(M, \omega)$, *with equality* $\Leftrightarrow G = G_J$, $\rho = 0$.

(III) *Assume $(M, \omega)$ is quantizable. Then every Riemannian metric as in (I) arises from some Berezin-Toeplitz quantization.*

# Remarks

Decomposition $G = G_J + \rho$, $\rho \geq 0$ in general not unique. But there exists unique $G$-orthogonal $\omega$-compatible almost complex structure $J$: $G(J\xi, J\eta) = G(\xi, \eta)$.

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Proof of (III): almost Kähler quantization followed by diffusion.

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Proof of (III): almost Kähler quantization followed by diffusion.

Proof of (I): unsharpness (noise) of quantum measurements.

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**Positive Operator Valued Measure (POVM)** $W$ on $(\Omega, \mathcal{C})$

$W : \mathcal{C} \to \mathcal{L}(H)$

- $W(X) \geq 0$ for all $X \in \mathcal{C}$,
- countably additive,
- $W(\Omega) = \mathbb{1}$.

# POVMs-2

**Fact:** Chiribella, D'Ariano, Schlingemann

There exists

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**Example 1.** Berezin-Toeplitz quantization is given by sequence $\mathcal{L}(H_\hbar)$-valued POVMS on $M$:

$$\boxed{T_\hbar(f) = \int f dW_\hbar}$$

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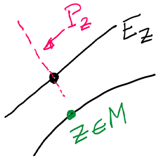

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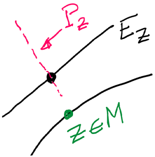


There exists Rawnsley function $R_\hbar \in C^\infty(M)$:

$$\boxed{T_\hbar(f) = \int_M f(x) R_\hbar(x) P_{x,\hbar} d\mathrm{Vol}(x)}$$

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Vanishes for projector valued POVMs

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Coefficient jumps! Based on **noise inequality:**

$(\mathrm{tr}(\Delta_W(f)\rho)^{1/2} (\mathrm{tr}(\Delta_W(g)\rho)^{1/2} \geq \frac{1}{2}|Exp([F,G],\rho)|$

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**The least unsharpness principle:**
$\mathrm{Vol}(M, G) \geq \mathrm{Vol}(M, \omega)$ , minimizers $\leftrightarrow$ compatible metrics
(cf. Gerhenstaber, 2007)

# Classification of quantizations

**Star products:** Change of variables $A : f \mapsto f + \sum_{m \geq 1} \hbar^m a_m(f)$
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**Observation:** Unsharpness metric is an invariant of 2-equivalence.

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**Tool:** Developing **approximate representations** for Lie algebras. (for groups - Grove-Karcher-Ruh; Kazhdan; Lubotzky et al.)

# Reversibility revisited

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**Example:** For almost complex quantization, $D = -\Delta/2$, where $\Delta$- Laplace-Beltrami.

# Markov chain

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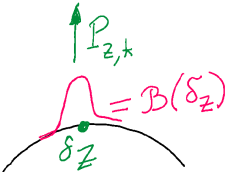

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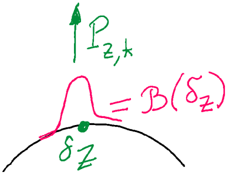


Dirac $\delta_z \to$ Coherent state proj. $P_{z,\hbar} \to$ "Gaussian" centered at $z$ concentrated in ball of radius $\sim \sqrt{\hbar}$.

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*For Berezin-Toeplitz quantization of Kähler manifolds*

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Upper bound – Karabegov-Schlichenmaier, 2001.

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*where $\lambda_1$ - first eigenvalue of Laplace-Beltrami operator.*

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# The End

THANK YOU!