# Reeb orbits that force topological entropy

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# Intuition: Topological entropy

**Theorem (Katok)**

The positivity of topological entropy for sufficiently smooth flows on a compact oriented 3-manifold is equivalent to the existence of a "horseshoe" as a flow subsystem.

**Consequence:** the number of hyperbolic periodic orbits of the flow grows exponentially with respect to the period.

# Motivation: A Denvir-Mackay theorem for Geodesic flows.

$(S,g)$ is a closed Riemannian surface.

**Theorem (Dinaburg 71)**

If the genus of $S$ greater than 1, then every geodesic flow on $S$ has positive topological entropy.

**Theorem (Denvir and MacKay 98.)**

If a Riemannian metric $g$ on $T^2$ has a closed contractible geodesic then the geodesic flow $\phi_g$ of $g$ has positive topological entropy.

# Transverse links that force topological entropy

$(Y, \xi)$ -closed contact 3-manifold.

A **transverse link** in $(Y, \xi)$ is a link $L \hookrightarrow Y$ that is everywhere transverse to $\xi$.

### Definition

A transverse link $L$ in $(Y, \xi)$ ***forces topological entropy*** if every Reeb flow on $(Y, \xi)$ which has $L$ as a set of Reeb orbits has positive topological entropy.

# Existence of the forcing orbit for any contact 3-manifold

**Theorem A (Alves-P. 20)**

On every contact 3-manifold there exist transverse knots which force topological entropy.

# First results about positive $h_{top}$ for every Reeb flows.

$T_1S$-unit tangent bundle.

There exist $\lambda_g$ on the $T_1S$ such that geodesic flow of $(S,g)$ is a Reeb flow of $\lambda_g$ and $\xi_{geo} = ker\lambda_g$ defines a ***standard contact structure*** on $T_1S$.

**Theorem (Macarini and Schlenk 11, Alves 15)**

If the genus of $S$ greater than 1, then every Reeb flow on $(T_1S, \xi_{geo})$ has positive topological entropy.

Exponential growth of Lagrangian Floer homology

Exponential **homotopical** growth of cylindrical contact homology

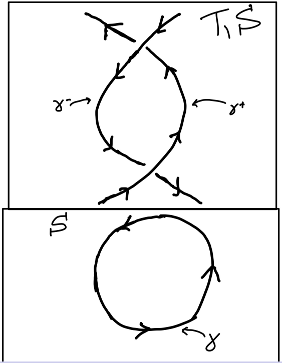
# A flat knot

## Definition

**A flat knot** in $S$ is an immersed curve with only transverse self intersections.

Let $\gamma(t) : S^1 \to S$ be a parametrised flat knot on $(S, g)$.

We have knots $\gamma_g^+(t) = (\gamma(t), \frac{\dot{\gamma}(t)}{||\dot{\gamma}(t)||_g})$ and $\gamma_g^-(t) = (\gamma(-t), -\frac{\dot{\gamma}(-t)}{||\dot{\gamma}(-t)||_g})$ on $T_1 S$.



# A Denvir-Mackay theorem for Reeb flows.

**Theorem B (Alves, Hryniewicz, Salomão and -P. work in progress)**

Let $\gamma$ be a **contractible** flat knot in the two dimensional torus $T^2$. Then the transverse link $\mathscr{L}_\gamma = \gamma_g^+ \sqcup \gamma_g^- \in (T^3, \xi_{\text{geo}})$ forces topological entropy.

**Theorem (Denvir and MacKay 98.)**

If a Riemannian metric $g$ on $T^2$ has a closed contractible geodesic then the geodesic flow $\phi_g$ of $g$ has positive topological entropy.

# Cylindrical Contact homology of $L$

**Cylindrical contact homology** is a Morse-like homology: we count pseudo holomorphic cylinders instead of negative gradient flow lines. (Eliashberg, Givental and Hofer 00)

**Cylindrical Contact homology in a *homotopy class* $\rho$ complement of a link $L$ ($CH_L^{\rho}$):** Here we count cylinders with zero *Siefring intersection number* with trivial cylinder over the link. (Momin 11, Hryniewicz, Momin, and Salomão 15)

## Theorem C (Alves-P. 20)

Let $L$ be a transverse link in $(Y, \xi)$ and $\lambda_0$ be a contact form on $(Y, \xi)$ such that

- $\lambda_0$ has $L$ as a set of periodic orbits and is hypertight on the complement of $L$,
- the cylindrical contact homology $CH_L(\lambda_0)$ has exponential homotopical growth.

Then, the transverse link $L$ forces topological entropy. **(for all $\lambda$)**

# Hypertight complement of $L$

**Definition**

Assume that $L$ is a collection of Reeb orbits of $\lambda_0$. We say that $\lambda_0$ is ***hypertight in the complement of*** $L$ if

- any disk in $Y$ whose boundary is a component of $L$ must have an interior intersection point with $L$,
- every Reeb orbit of $\lambda_0$ is non-contractible in $Y \setminus L$.

# The exponential homotopical growth rate of $CH_L(\lambda_0)$

Let $\Omega(Y \setminus L)$ be the set of ***free homotopy classes*** of loops in $Y \setminus L$.

For every positive real number $T$ we define the set $\Omega_L^T(\lambda_0) \subset \Omega(Y \setminus L)$ such that $\rho \in \Omega_L^T(\lambda_0)$ if $(\lambda_0, L, \rho)$ satisfies

- PLC condition
- every Reeb orbit of $\lambda_0$ in $\rho$ has action smaller than $T$
- $CH_L^{\rho}(\lambda_0) \neq 0$.

**Definition**

$CH_L(\lambda_0)$ has ***exponential homotopical growth***, if a number $\limsup_{T \to +\infty} \frac{\log \# \Omega_L^T(\lambda_0)}{T}$ is positive.

# Outline of the Proof of Theorem C

Fix $\lambda = f\lambda_0$. Assume $a > 0$ denotes the exponential homotopical growth rate of $CH_L(\lambda_0)$.

**Step 1**

The exponential homotopical growth rate of Reeb flow of $\lambda$ in $Y \setminus L$ is at least $\frac{a}{\max f_\lambda}$.

**Step 2**

If the exponential homotopical growth rate of non singular flow in $Y \setminus L$ is $\frac{a}{\max f_\lambda}$, then $h_{top}(\phi_\lambda) \geq \frac{a}{\max f_\lambda}$.

# Outline of the Proof of Theorem A

**Theorem A (Alves-P. 20)**

On every contact 3-manifold there exist transverse knots which force topological entropy.

**Theorem (Giroux 00, Colin and Honda 08)**

Given a contact $3$-manifold $(Y, \xi)$ there exists an open book decomposition of $Y$ supporting $(Y, \xi)$. Moreover, the open book decomposition can be chosen to have connected binding and pseudo-Anosov monodromy.

Two contact structures supported by diffeomorphic open book decompositions are diffeomorphic.

# Outline of the Proof of Theorem A

## Theorem D

Let $(S, \psi, \Psi)$ be an open book decomposition that supports $(Y, \xi)$ and satisfies

- $\partial S$ is connected
- the monodromy of the first return map $\psi$ is pseudo-Anosov map.

Then there exists an open book decomposition $(S, \psi', \Psi')$ diffeomorphic to $(S, \psi, \Psi)$ that also supports $(Y, \xi)$ and whose binding $\mathscr{B}'$ forces topological entropy.

# Outline of the Proof of Theorem D

## Step 1

Constructing the special contact form $\lambda_0$.

The set $\Omega^T_{\mathscr{B}'}(\lambda_0)$ of free homotopy classes of loops in $Y \setminus \mathscr{B}'$ that

- contain only non-degenerate Reeb orbits with action $\leq T$,
- contain an odd number of Reeb orbits,

satisfies

$$\limsup_{T \to +\infty} \frac{\log \# \Omega^T_{\mathscr{B}'}(\lambda_0)}{T} > 0. \tag{1}$$

## Step 2

Computation of the exponential homotopical growth rate of $CH_{\mathscr{B}'}(\lambda_0)$.

Thank you for your attention!
Any questions?

# PLC(Proper link class)

## Definition

Let $(Y, \xi)$ be a contact 3-manifold, $L$ be a transverse link in $(Y, \xi)$, $\lambda_0$ be a contact form on $(Y, \xi)$ and $\rho$ be a non-trivial free homotopy class of loops in $Y \setminus L$. Assume that $L$ is a collection of Reeb orbits of $\lambda_0$ and that $\lambda_0$ is hypertight in the complement of $L$. We say that $(\lambda_0, L, \rho)$ satisfy the "proper link class" condition (PLC) if

- for any connected component $x$ of $L$, no Reeb orbit $\gamma$ in $\rho$ can be homotoped to $x$ in $Y \setminus L$, i.e. there is no homotopy $I : [0,1] \times S^1 \to Y$ with $I(0, \cdot) = \gamma$ and $I(1, \cdot) = x$ such that $I([0,1) \times S^1) \subset Y \setminus L$.
- every Reeb orbit of $\lambda_0$ belonging to the class $\rho$ is non-degenerate and simply covered.

# Topological entropy

A set $E \subset M$ is said to be $(t,\varepsilon)$-separated, if for every $x, y \subset E$ with $x \neq y$ there is $t_0 \subset (0,t)$ such that $d(\phi_{t_0}x, \phi_{t_0}y) \geq \varepsilon$. Let $s(t,\varepsilon)$ be the maximal cardinality of an $(t,\varepsilon)$-separated set in $M$. The **topological entropy** is obtained as

$$h_{top}(T) = \lim_{\varepsilon \to 0} \lim sup_{t \to \infty} \frac{\log s(t,\varepsilon)}{t}$$