A bordered approach to link Floer homology

Western Hemisphere

Peter Ozsváth

July 10, 2020

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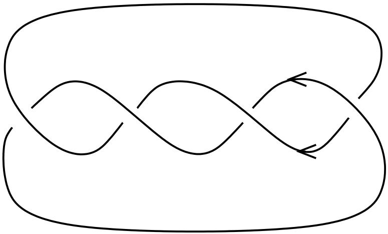
In fact, this is a generalization of the bordered knot Floer homology Szabó and I have been working on recently.

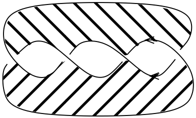
Knots and links

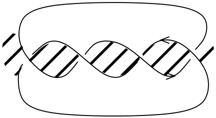
A knot can be represented by a closed curve immersed in the plane, with crossing data. A link has a similar presentation, as an ℓ-component closed curve.



The Seifert genus

It is a theorem of Seifert that any oriented link $\vec{L}$ can be realized as the boundary of an embedded, oriented surface in $\mathbb{R}^3$, called a *Seifert surface*.





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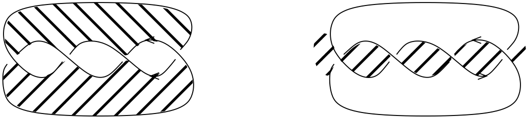

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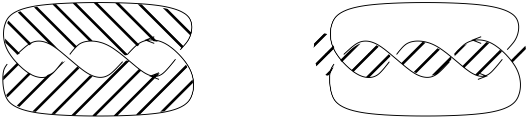

The *Seifert genus* $g(\vec{L})$ is the minimal genus of any Seifert surface for $\vec{L}$. When $\vec{L}$ is a knot, the Seifert genus is a fundamental measure of its complexity.

Elaboration of the Seifert genus

More generally, if L is an ℓ-component link, $H_2(\mathbb{R}^3, L; \mathbb{Z}) \cong \mathbb{Z}^\ell$. Each relative homology class $\xi \in H_2(\mathbb{R}^3, \vec{L}; \mathbb{Z})$ can be represented by a possibly disconnected, embedded surface $F \subset \mathbb{R}^3 \setminus \nu(L)$.

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W. P. Thurston gave a very elegant formulation of this. Define the *complexity* of a (possibly disconnected) surface F to be

$$\chi(F) = -\chi(F'),$$

where $F = F' \cup \text{Spheres}$. The *Thurston (semi-)norm of L* is the function

$$x\colon H_2(\mathbb{R}^3, L; \mathbb{Z}) \to \mathbb{Z},$$

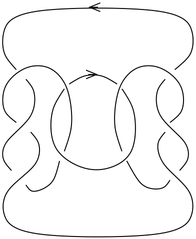
defined by

$$x(\xi) = \min\{\chi_-(F) \big| F \text{ represents} \xi\}.$$

The Thurson norm

The function x is linear on rays, so it naturally extends to a function $x\colon H_2(\mathbb{R}^3, L; \mathbb{Q}) \to \mathbb{Q}$; and by continuity to a function $x\colon H_2(\mathbb{R}^3, L; \mathbb{R}) \to \mathbb{R}$. The unit ball in $H_1(\mathbb{R}^3, L; \mathbb{R}) \cong \mathbb{R}^\ell$ is a polytope, called the *Thurston polytope*. Thus, the Thurston polytope is a polytope in Euclidean space that governs the minimal genus representatives of homology classes in a link complement. By Poincaré duality, the vector space $H_2(\mathbb{R}^3, L; \mathbb{R})$ is dual to $H_1(S^3 \setminus L; \mathbb{R})$. The *dual Thurston polytope* is the unit ball in $H_1(S^3 \setminus L; \mathbb{R})$.

An example



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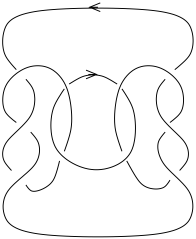
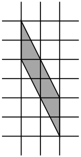
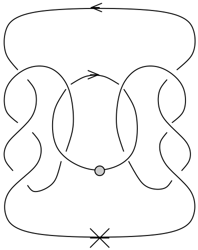




Kauffman states

Represent $\vec{L}$ by its projection. Choose also a marked edge.



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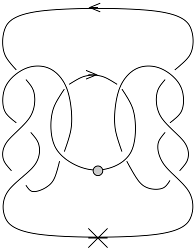

A *Kauffman state* is a map κ that assigns to each crossing in the diagram, one of the four adjacent quadrants (represented by a dot), subject to the following constraints:

- No two dots lie in the same region.
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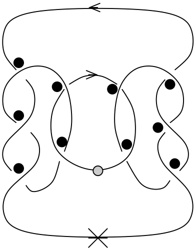

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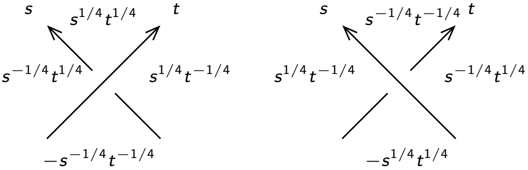
Kauffman states and the multi-variable Alexander polynomial

To each Kauffman state, we can associate a monomial in $t_1, \ldots, t_\ell$, with local contributions as follows:



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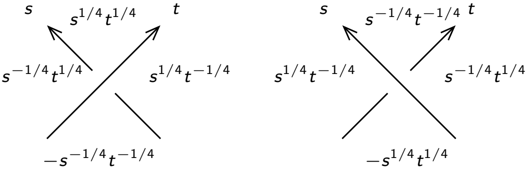

The *Alexander polynomial of* $\vec{L}$ is the polynomial $\mathbb{Z}[t_1, \ldots, t_\ell]$ which is the sum of monomials associated to each Kauffman state. More invariantly, the Alexander polynomial Δ_L can be viewed as an element

$$\Delta_L \in \mathbb{Z}[H_1(S^3 \setminus L; \mathbb{Z})].$$

Link Floer homology

Link Floer homology is a variant of Lagrangian Floer homology in the symmetric product of a Riemann surface. For a link with ℓ-components, this gives a graded vector space over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ with $\ell + 1$ gradings,

$$\widehat{\mathrm{HFL}}(L) = \bigoplus_{d \in \mathbb{Z}, h \in H_2(\mathbb{R}^3, L)} \widehat{\mathrm{HFL}}_d(L, h).$$

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Its Euler characteristic is the Alexander polynomial:

$$\sum_{d \in \mathbb{Z}} (-1)^d \dim \widehat{\mathrm{HFL}}_d(L, h)[h].$$

Link Floer homology determines the Thurston norm

THEOREM (O.-Szabó, 2006) The convex hull of all $h \in H_1(\mathbb{R}^3 \setminus L)$ so that $\widehat{\mathrm{HFL}}_*(L,h) \neq 0$ is the sum of the dual Thurston polytope and a hypercube. Equivalently, for $h \in H^1(S^3 \setminus L; \mathbb{R})$,

$$x(PD[h]) + \sum_{i=1}^{\ell} |\langle h, \mu_i \rangle| = 2 \max_{\{s \in H_1(L,\mathbb{R}) \,\big|\, \widehat{\mathrm{HFL}}(L,s) \neq 0} \langle s, h \rangle.$$

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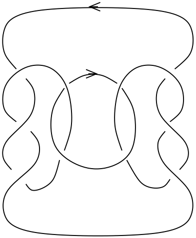
- This is an analogue of a gauge-theoretic theorem of Kronheimer and Mrowka.
- A very elegant proof of this was given shortly afterward by András Juhász, using his *sutured Floer homology*.

An example



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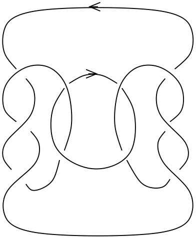

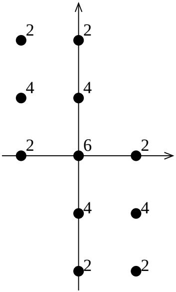

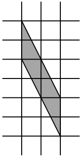

Computational approaches

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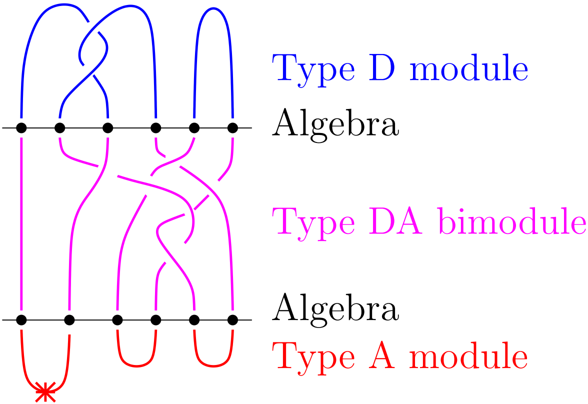
Slicing link diagrams



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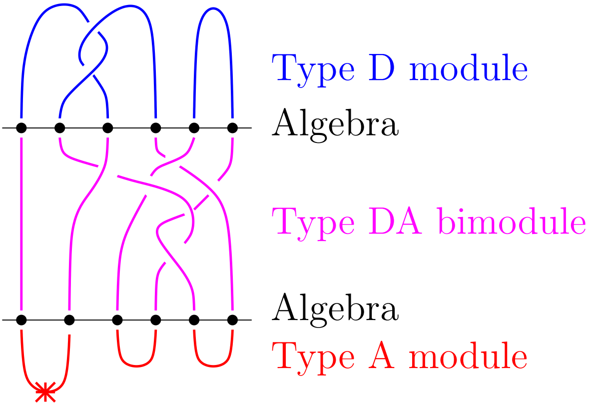

Inspired by Bordered Floer homology, work of Robert Lipshitz, Dylan Thurston, and me from 2008.

Pairing Theorem

$\widehat{\mathrm{HFL}}(\vec{L})$ can be computed by a suitable successive tensor product of bimodules over an algebra.

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I will now sketch the ingredients that go into the "successive tensor product".

Curved algebra

An associative algebra, equipped with a preferred central element.

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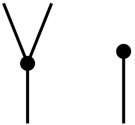
Pictorially:

Operations: Relations:



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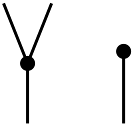

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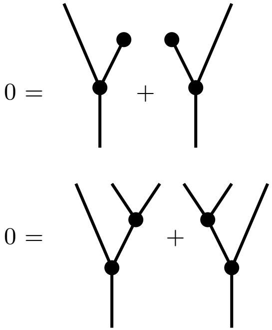

Curved type D modules

Vector space X and a map $X \to A \otimes X$ If we think of the generating set as $\{x_i\}_{i=1}^{n}$, the we obtain a matrix $A = (a_{i,j})_{i=1}^{n}$ with

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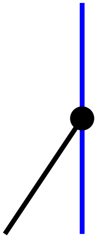
Curved type D structures (graphically)

Operation:



Relation:

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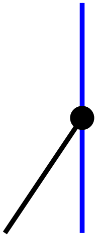

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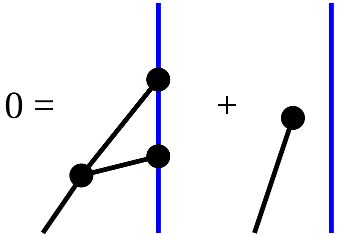

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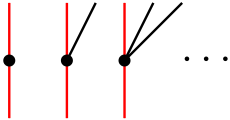

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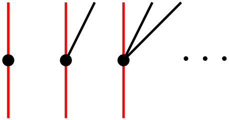

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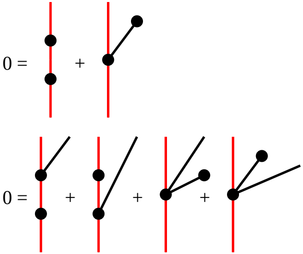

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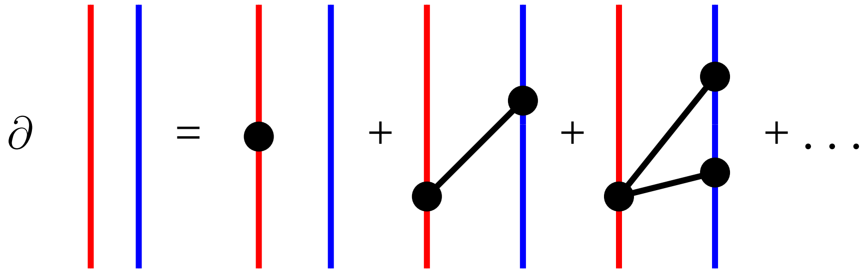
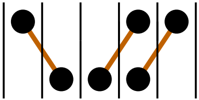
Tensor product



Our algebra

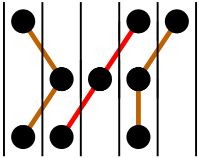
Idempotent 

Algebra element
$R_2 L_4 L_5$



U_3

Relations



Curvature

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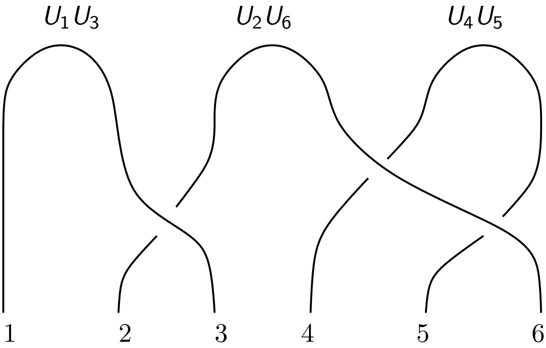
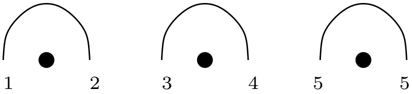


Module associated to the very top

One generator and no differential, in the indicated idempotent.



$dX = 0$. Note that

$$d^2X = (U_1U_2 + U_3U_4 + U_5U_6) \otimes X = 0.$$

Generators of a marked minimum

Six generator types X_0, X_1, X_2, Y_0, Y_1, Y_2.

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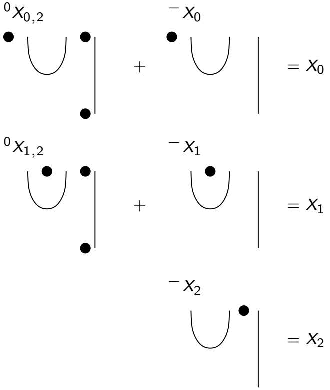
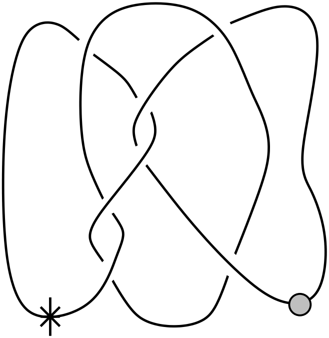
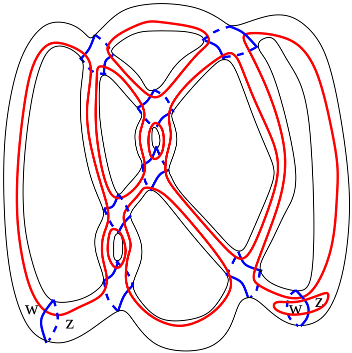


Actions on a marked minimum

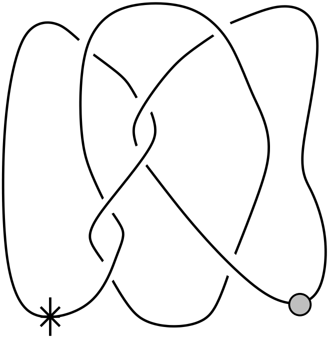
$$
\begin{tikzcd}
 & & X_1 \arrow[dll, "R_2"'] \arrow[drr, "L_1"] \arrow[ddl, "{(U_2, L_1)}"'] \arrow[ddd, "{(U_1, U_2) + (U_2, U_1)}"] \arrow[ddr, "{(U_1, R_2)}"] & & \\
X_2 \arrow[dr, "{(L_2, L_1)}"] \arrow[ddrr, bend right, "{(L_2, U_1)}"'] & & & & X_0 \arrow[dl, "{(R_1, R_2)}"'] \arrow[ddll, bend left, "{(R_1, U_2)}"] \\
 & Y_0 \arrow[dr, "R_1"] & & Y_2 \arrow[dl, "L_2"'] & \\
 & & Y_1 & &
\end{tikzcd}
$$

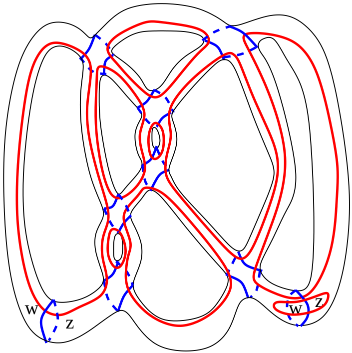
From link projections to Heegaard diagrams





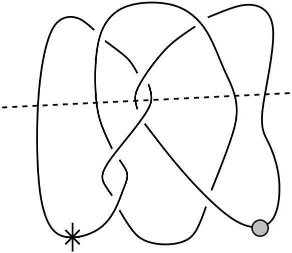
From link projections to Heegaard diagrams

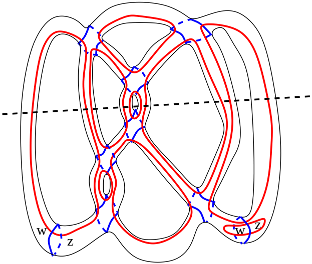




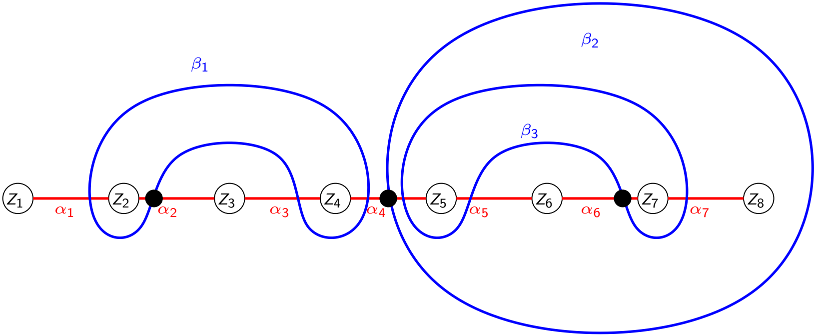
For a knot, Heegaard Floer generators correspond to Kauffman states. (This diagram was considered by us back in 2003.)

Degenerating link diagrams

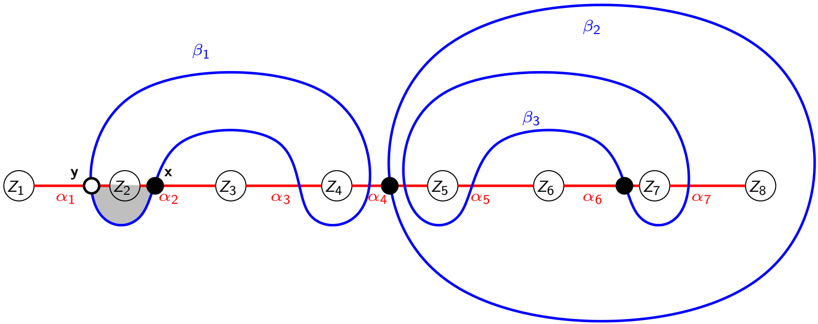




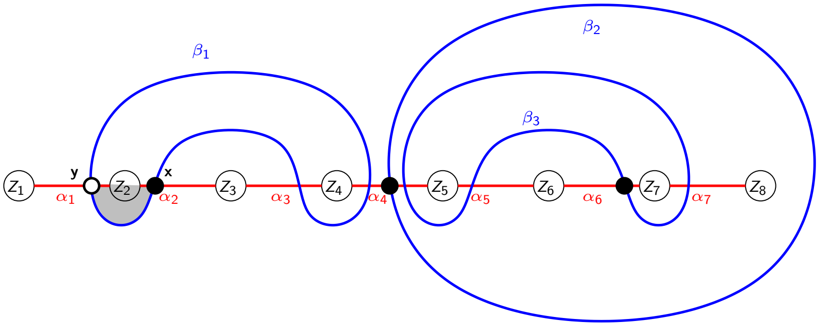
Upper Heegaard diagrams



Upper Heegaard diagrams: differentials

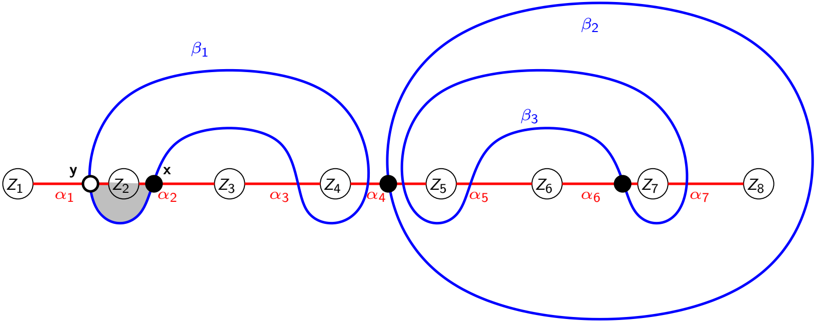


Upper Heegaard diagrams: differentials



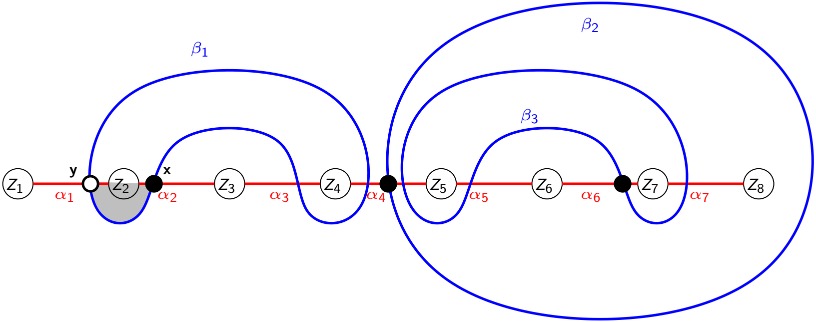
$$\delta^1(\mathbf{x}) = R_2 \otimes \mathbf{y} + \ldots$$

Upper Heegaard diagrams: differentials



$$\delta^1(\mathbf{x}) = R_2 \otimes \mathbf{y} + \dots \qquad \delta^1(\mathbf{y}) = L_2 U_4 \otimes \mathbf{x} + \dots$$

Upper Heegaard diagrams: differentials

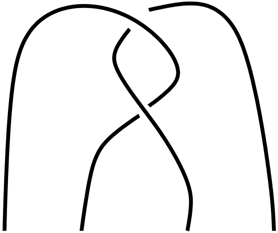


$\delta^1(\mathbf{x}) = R_2 \otimes \mathbf{y} + \dots \qquad \delta^1(\mathbf{y}) = L_2 U_4 \otimes \mathbf{x} + \dots$

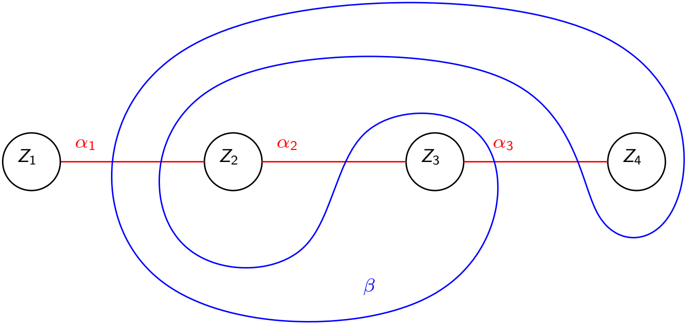
$\mu_0 = U_1 U_3 + U_2 U_4 + U_5 U_7 + U_6 U_8$

An example

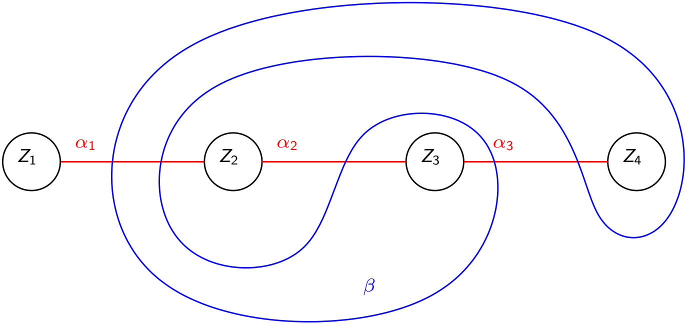
Consider the upper link diagram:



Corresponding upper Heegaard diagram

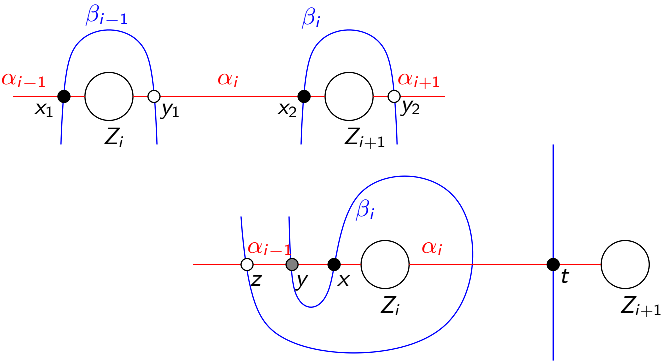


Corresponding upper Heegaard diagram

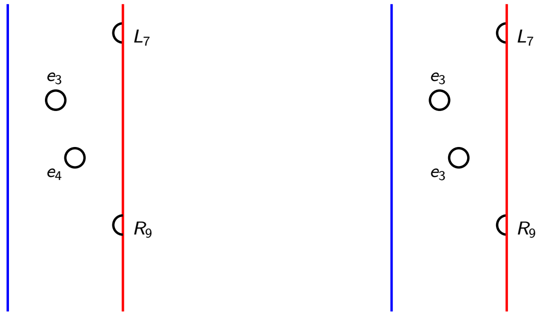


$$\begin{tikzcd}[column sep=large]
x_1 \arrow[r, shift left, "U_4"] & x_2 \arrow[l, shift left, "U_3"] \arrow[r, shift left, "L_2 U_1"] \arrow[rrr, bend left, "L_2 L_3"] & t \arrow[l, shift left, "R_2"] \arrow[r, shift left, "L_3"] & y_1 \arrow[l, shift left, "R_3 U_4"] \arrow[r, shift left, "U_2"] \arrow[lll, bend left, "R_3 R_2"] & y_2 \arrow[l, shift left, "U_1"]
\end{tikzcd}$$

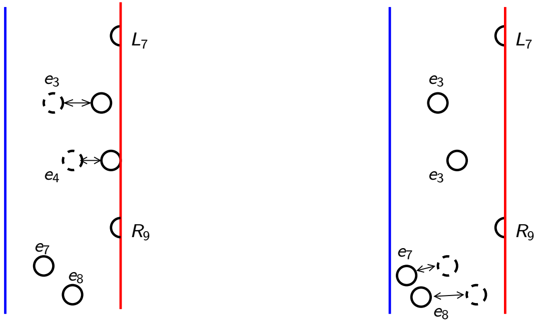
Relations in the algebra



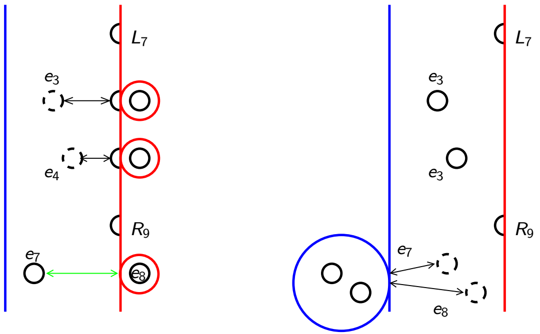
Step 1: Fiber product description of moduli spaces



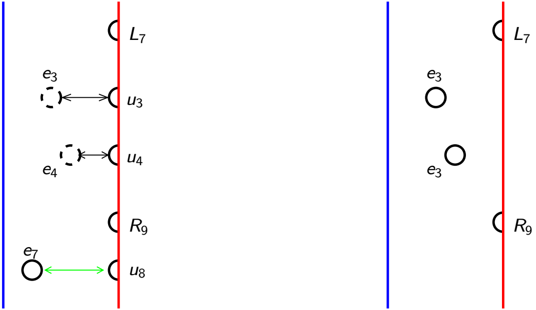
Step 2: Moving orbits



Step 3: Limiting orbits



Step 4: Prune the curve



Step 5: Time dilation