Embedding Ellipsoids in the one-point blow up of $\mathbb{C}P^2$

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(I): The Embedding Capacity Function

Consider symplectic embeddings of the open ellipsoid

$$E(1,z) := \left\{(\zeta_1,\zeta_2) \in \mathbb{C}^2 \;\middle|\; \pi(|\zeta_1|^2 + \tfrac{1}{z}|\zeta_2|^2) < 1\right\}$$

into a scaling $\lambda X := (X, \lambda\omega)$ of a sympl. 4-manifold (X,ω). Define

$$c_X(z) := \inf\left\{\lambda \;\middle|\; E(1,z) \text{ sympl. embeds in } \lambda X\right\}.$$

- $c_X(z) \geq \sqrt{\frac{z}{\mathrm{Vol}\,(\mathrm{X},\omega)}}$ (note: we need $\mathrm{Vol}\,\mathrm{E}(1,\mathrm{z}) = \mathrm{z}\mathrm{Vol}\,\mathrm{E}(1,1) < \mathrm{Vol}\,(\mathrm{X},\lambda\omega)) = \lambda^2\mathrm{Vol}\,(\mathrm{X},\omega)$, and hence $\lambda^2 \geq z/\mathrm{Vol}\,(\mathrm{X},\omega)$ if we normalize $\mathrm{Vol}\,\mathrm{E}(1,1) := 1$;
- if X is closed or a convex toric domain, $c_X(z) = \sqrt{\frac{z}{\mathrm{Vol}\,(\mathrm{X},\omega)}}$ for sufficiently large z.
- **(scaling)** for $\lambda \geq 1$, $c_X(\lambda z) \leq \lambda c_X(z)$ because

$$E(1,z) \overset{s}{\hookrightarrow} X \Longrightarrow E(1,\lambda z) \subset \lambda E(1,z) \overset{s}{\hookrightarrow} \lambda X$$

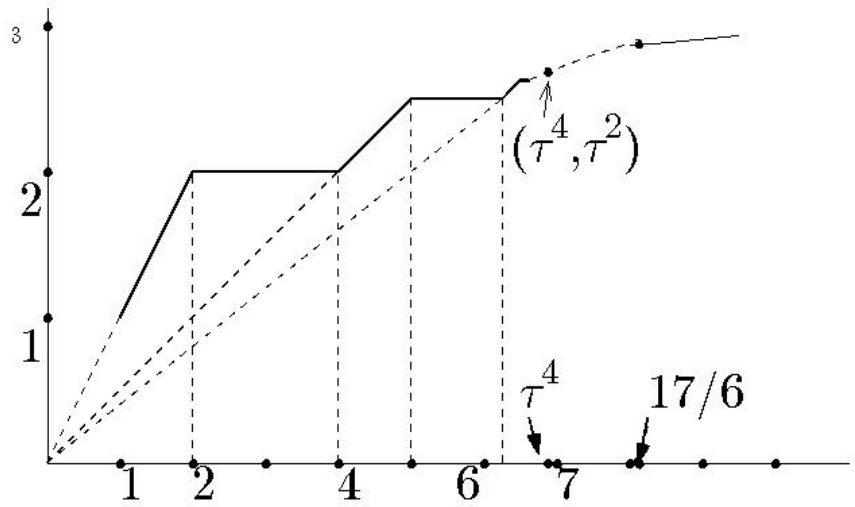
Problem: Compute $c_X(z)$ when $X = H_b := \mathbb{C}P^2(1)\#\overline{\mathbb{C}P}^2(b)$.

The case $b = 0$ (the ball, or $\mathbb{C}P^2$)

For $a \geq 1$ define $c(z) := c_{H_0}(z) = \inf\{\lambda : E^4(1,z) \text{ embeds in } B^4(\lambda)\}$.

- for $z \geq 8\frac{1}{36} = \left(\frac{17}{6}\right)^2$, $c(z) = \sqrt{z}$

– no obstruction except for volume, i.e. full fillings



- $\tau^4 < z < 8\frac{1}{36}$ is transitional region (where $\tau = \frac{1+\sqrt{5}}{2}$);
- For $z < \tau^4 \approx 6.7$

there is an infinite staircase with numerics based on the odd placed Fibonacci numbers $F_k := 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

- there are obstructions at the outer corners $\frac{2}{1}, \frac{5}{1}, \frac{13}{2}, \frac{34}{5}, \ldots, \frac{F_{2k+5}}{F_{2k+1}}, \ldots$ and full fillings at the inner corners $\frac{2^2}{1^2}, \frac{5^2}{2^2}, \frac{13^2}{5^2}, \ldots, \frac{F_{2k+3}^2}{F_{2k+1}^2}, \ldots$ Because of rescaling, need only check these values of $c(z)$ to prove $\exists$ staircase.

Other targets

Many papers deal with other targets, cf. Ellipsoid Day, June 2020. Our refs:

McSch = McDuff–Schlenk (2012) for the ball.

Usher = arXiv:1801.06762 (AGT 2019) Embedding ellipsoids into the irrational polydisc $B^2(1) \times B^2(b), b > 1$ – Usher found a doubly infinitely family of staircases $\mathcal{S}_{i,n}, i.n \geq 1$, each at a different irrational value of b, related by numerical moves — most impressively by 'Brahmagupta' moves (dating back to 7th century) that relate different solutions of the Diophantine equation $x^2 - 2\delta^2 = C$ – gave us inspiration, and a useful estimate.

C-GHMP = Cristofaro-Gardiner, Holm, Mandini, Pires; arXiv:2004.13062; Staircases in rational toric manifolds contains this theorem:

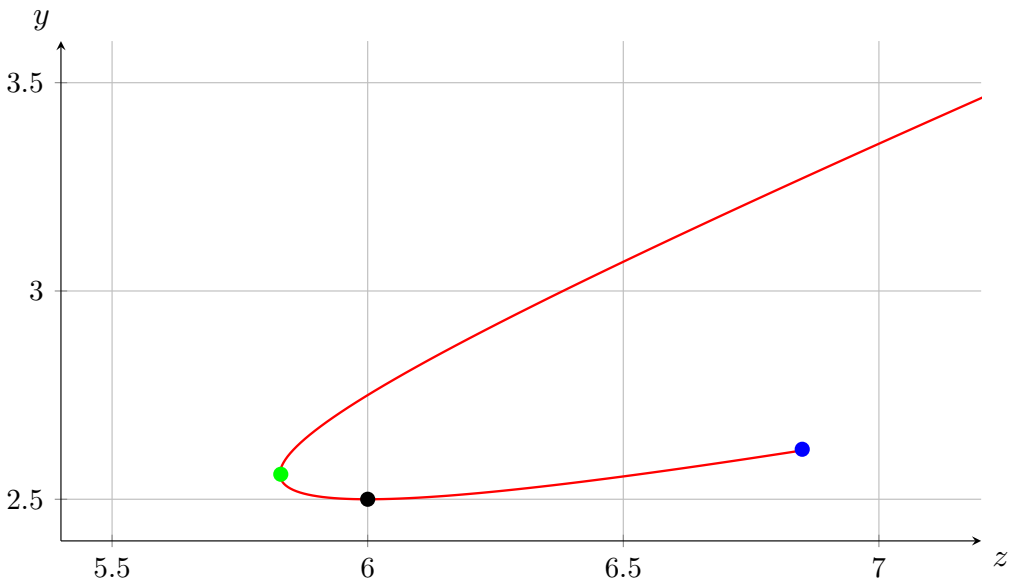
Thm [C-GHMP] *If $H_b = \mathbb{C}P^2(1)\#\overline{\mathbb{C}P}^2(b)$ has a staircase $\mathcal{S}$, then $\mathcal{S}$ must accumulate at the point $z = \mathrm{acc}(b)$ which is the unique solution > 1 of the equation*

$$z^2 - \Big(\frac{(3-b)^2}{1-b^2} - 2\Big)z + 1 = 0.$$

Moreover $c_{H_b}(\mathrm{acc}(b)) = V_{H_b}(\mathrm{acc}(b))$. In the toric model (moment polytope) of X_b, $3 - b$ is the affine length of the boundary, while $1 - b^2$ is twice its area.

The accumulation curve for H_b

The function $b \mapsto z = \mathrm{acc}(b)$ (where $z^2 - \left(\frac{(3-b)^2}{1-b^2} - 2\right)z + 1 = 0$), decreases for $0 \leq b \leq 1/3$ from $\tau^4 \approx 6.8$ to $3 + 2\sqrt{2} \approx 5.8$ then increases to ∞ as $b \to 1$.



This shows the accumulation point $z = \mathrm{acc}(b)$, in a diagram where the y-coordinate of each point on the red curve records the volume constraint $V_b(\mathrm{acc}(b))$. The blue point with $b = 0$ is at (τ^4, τ^2) and is the accumulation point for the Fibonacci stairs. The green point with $b = 1/3$ is the accumulation point for the stairs in $H_{1/3}$ (discussed in [C-GHMP]) and is the minimum of the function $z \mapsto \mathrm{acc}(z)$. The **black point at** $(z, b) = (6, 1/5)$ is the place where $V_b(\mathrm{acc}(b))$ takes its minimum.

New staircases for H_b

We found three new (double) families of staircases — where staircases labelled ℓ (= lower) ascend to the accum point, and those labelled u (= upper) descend.

- $\left(\mathcal{S}^U_{\ell,n}\right)_{n\geq 1}$ and $\left(\mathcal{S}^U_{u,n}\right)_{n\geq 0}$ for $b_n \in (5/11, 1)$ with accumulation points $a^U_{\ell,n,\infty} < a^U_{u,n,\infty}$ in $(7, \infty)$ and $b_n \to 1$;
- $\left(\mathcal{S}^L_{\ell,n}\right)_{n\geq 0}$ and $\left(\mathcal{S}^L_{u,n}\right)_{n\geq 1}$ for $b_n \in [0, 1/5)$ with $a^L_{\ell,n,\infty} < a^L_{u,n,\infty}$ in $(6, \tau^4]$ and $b_n \to 1/5$: note $\mathcal{S}^L_{\ell,0}$ is the Fibonacci staircase at $b = 0$.
- $\left(\mathcal{S}^E_{\ell,n}\right)_{n\geq 1}$ and $\left(\mathcal{S}^E_{u,n}\right)_{n\geq 0}$ for $b \in (1/5, 19/61)$ and $a^U_{\ell,n,\infty} < a^U_{u,n,\infty}$ in $(35/6, 6)$ and $b_n \to 1/5$.

• These staircase families are related by symmetries, e.g. there is a symmetry operation (the reflection $\Psi : z \mapsto \frac{6z-35}{z-6}$ with fixed point $z = 7$) that for each n takes $\mathcal{S}^U_{\ell,n}$ to $\mathcal{S}^L_{u,n}$, and $\mathcal{S}^U_{u,n}$ to $\mathcal{S}^L_{\ell,n}$.

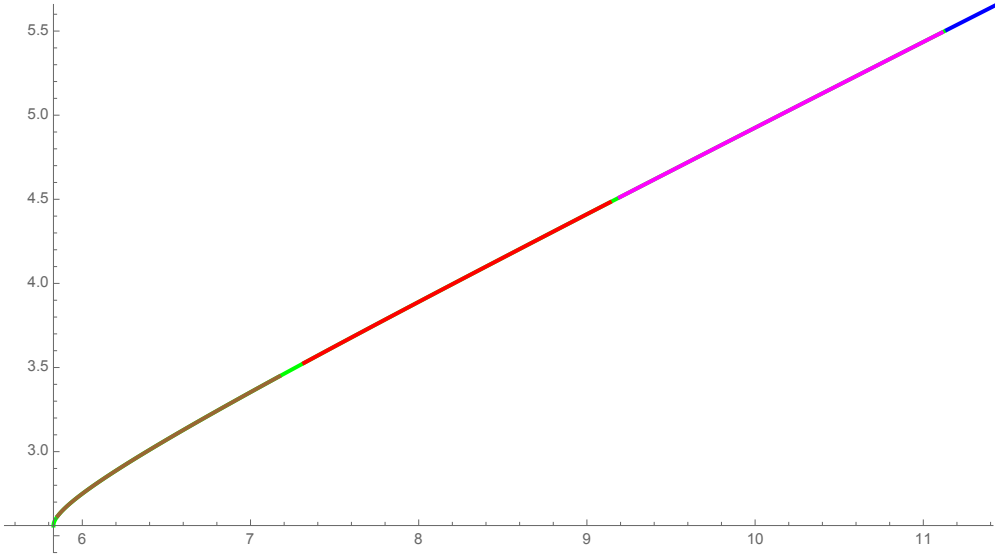
• These staircases are 2-periodic, i.e their steps have z-coordinates of the form $[2n+7, \{2n+5, 2n+1\}^k, 2n+4]$. We conjecture that all 2-periodic staircases are images of the staircases $\mathcal{S}^U_{\bullet,n}$ under some symmetry operation.

• We have found other potential staircases that are 4-, or 6-periodic etc, but have not yet embarked on the proof that these satisfy all requisite conditions.

Blocked b-intervals

Let $\star = U, L$, or E, and denote by $b^\star_{\ell,n}, b^\star_{u,n}$ the b-value that supports the corresponding staircase $\mathcal{S}^\star_{\ell,n}, \mathcal{S}^\star_{u,n}$. Then we found:

Theorem *For $\star = U, L$, or E and each n the set $J^\star_n$ of b values between $b^\star_{\ell,n}, b^\star_{u,n}$ is obstructed, i.e. $c_{H_b}(\mathrm{acc}(b)) > V_b(\mathrm{acc}(b))$ for all $b \in J^\star_n$.*



This figure shows J^U_0 (in brown), J^U_1 (red), J^U_2 (purple), and part of J^U_3 (blue) mapped onto the accumulation curve so you can see the corresponding z values. The interval J^U_n contains the point $2n+6$ and has length converging to 2; thus 'most' $z > 6$ values are not the limit points of a staircase — such points all lie in the short green intervals.

The general picture

Define: $\mathit{Block} := \{b \in [0,1) \mid c_{H_b}(\mathrm{acc}(b)) > V_b(\mathrm{acc}(b))\}$ – an open set
$\mathit{Stair} := \{b \in [0,1) \mid H_b \text{ has a staircase}\}$

We know: $\mathit{Stair} \subsetneq \mathit{Unobstr} := [0,1) \backslash \mathit{Block}$. For example, $1/5$ is unobstructed, but we prove it has no staircase.

- **Conjecture:** [C-GHMP] The only rational points in *Stair* are $0, 1/3$.
- **Conjecture:** The irrational $b \in \mathit{Stair}$ are precisely the endpoints of the components of the open set *Block*.
- **Conjecture:** Denote

$$\mathit{Block}_z^+ := \mathrm{acc}\big((1/3,1) \cap \mathit{Block}\big) \subset (3+2\sqrt{2}, \infty)$$

$$\mathit{Block}_z^- := \mathrm{acc}\big((0,1/3) \cap \mathit{Block}\big) \subset (3+2\sqrt{2}, \infty)$$

Then the intervals in Block_z^+ are invariant under the reflection

$$\Phi : (35/6, \infty) \to (35/6, \infty), \quad z \mapsto \frac{35z - 204}{6z - 35}, \quad \Phi(6) = 6,$$

and iterates of the shift $Sh^2 : z \mapsto \frac{35z-1}{6z-1}$. (and similarly for Block_z^-).
There also are symmetries (e.g. $\Psi : \mathcal{S}_\bullet^U \to \mathcal{S}_\bullet^L$) that interchange Block_z^+ and Block_z^-.

(II): Key technical points of the proof

- (I): Relation to blowing up: $\exists B^4(w) \overset{s}{\hookrightarrow} \lambda H_b = \mathbb{C}P^2(\lambda)\#\overline{\mathbb{C}P}^2(\lambda b)$ iff there is a symplectic form on the blow up $H_b\#\overline{\mathbb{C}P}^2 = \mathbb{C}P^2\#2\overline{\mathbb{C}P}^2$ in the class dual to $\lambda L - \lambda b E_0 - wE_1$ (where E_1 is the new exceptional divisor)
- (II): Decomposition into balls Every rational number $p/q > 1$ has a rational weight decomposition $w(p/q) := (w_1, w_2, \ldots, w_N)$ (where $w_1 = 1, w_N = 1/q$) s.t.

$$E(1, \tfrac{p}{q}) \overset{s}{\hookrightarrow} \lambda H_b \iff \sqcup_i B^4(w_i) \overset{s}{\hookrightarrow} \lambda H_b$$

- (Key result = I + II) $\exists E(1, \frac{p}{q}) \overset{s}{\hookrightarrow} \lambda H_b$ iff $\exists$ a symplectic form ω on $X_{N+1} :=$ the N-fold blow up of H_b in class α dual to $\lambda L - \lambda b E_0 - \sum_i w_i E_i$.
- (III): Symplectic cone of X_{N+1} (Li–Li + ...): $\exists \omega : [\omega] = \alpha$ iff
  - $\alpha^2 > 0$ (volume constraint) and
  - $\int_{\mathbf{E}} \alpha > 0$ for all classes $\mathbf{E}$ represented by exceptional spheres.

  The class $\mathbf{E} = dL - mE_0 - \sum_{i\geq 1} m_i E_i =: (d, m, \mathbf{m})$ of such a curve satisfies $c_1(\mathbf{E}) = 3d - m - \sum m_i = 1, \quad \mathbf{E}\cdot\mathbf{E} = d^2 - m^2 - \sum m_i^2 = -1.$
  (the Diophantine conditions). $\mathbf{E}$ must also transform correctly under Cremona moves (a geometric condition).

The embedding obstructions $\mu_{\mathbf{E},b}$

$$E(1,z) \overset{s}{\hookrightarrow} \lambda H_b \iff \sqcup_i B^4(w_i(z)) \overset{s}{\hookrightarrow} \lambda H_b$$
$$\iff \lambda^2(d^2-b^2) > \sum w_i(z)^2, \text{ and}$$
$$\lambda d - \lambda mb - \textstyle\sum_{i\geq 1} m_i w_i(z) > 0 \ \ \forall \ \ \mathbf{E}$$

Thus $c_{H_b}(z) \geq \frac{\sum_{i\geq 1} m_i w_i(z)}{d-mb} =: \mu_{\mathbf{E},b}(z)$, where $\mathbf{E} = (d, m, \mathbf{m})$.

An exceptional class $\mathbf{E} = (d, m, \mathbf{m})$ is perfect if $\mathbf{m} = \mathbf{qw}(\mathbf{p}/\mathbf{q})$ for some center point $a = p/q$. In this case $\sum m_i w_i(a)$ is as large as possible (since the two vectors are parallel) and one gets a particularly nice obstruction for z near a:

$$\mu_{\mathbf{E},b}(z) = \frac{p}{d-mb} \quad \text{if } a \leq z < a + \varepsilon - \textit{constant}$$
$$= \frac{qz}{d-mb} \quad \text{if } a - \varepsilon < z \leq a - \textit{line through } 0.$$

Proposition: *If* $\mathbf{E}$ *is perfect and* $b = m/d$ *then* $c_{H_b}(a) = \mu_{\mathbf{E},b}(a)$.

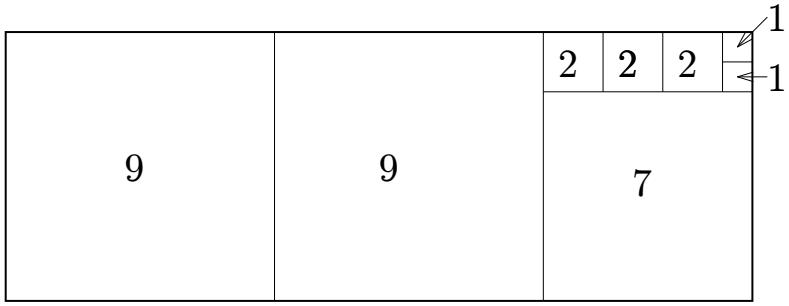
- if $c_{H_{b_0}}(a) = \mu_{\mathbf{E},b_0}(a)$ we say that $\mu_{\mathbf{E},b}$ is live at a for $b = b_0$.
- The way we calculate $c_{H_{b_0}}(z)$ is to show that some $\mu_{\mathbf{E},b}$ is live at z for that b_0. For this we need $\mathbf{E}$ to be exceptional, i.e. besides satisfying the (numerical) Diophantine conditions it must reduce correctly under Cremona moves (which is hard to check).

(you can also define obstructions and do calculations using ECH – but this is not so relevant in this context)

Continued fractions and weight expansions

Recall: an except. divisor $\mathbf{E} = (d, m, \mathbf{m}) = \mathbf{dL} - \mathbf{mE_0} - \sum \mathbf{m_i E_i}$ is perfect if $\mathbf{m} = q\mathbf{w}(p/q)$.

Here, the weight expansion $q\mathbf{w}(p/q)$ of a fraction $\frac{p}{q}$ is given by decomposing a rectangle into squares:
eg $9\mathbf{w}(\frac{25}{9}) = (\underbrace{9, 9}, 7, \underbrace{2, 2, 2}, \underbrace{1, 1})$



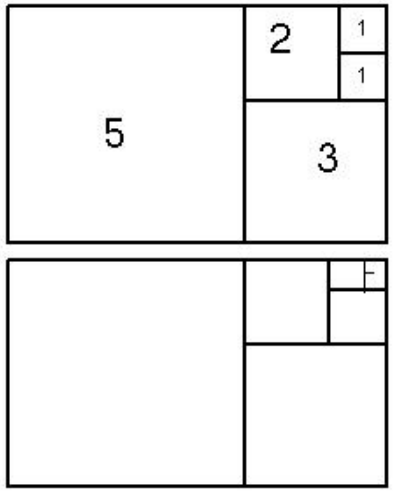
corresponds to

$$\frac{25}{9} = [2; 1, 3, 2] = 2 + \frac{1}{1 + \frac{1}{3 + \frac{1}{2}}} = 2 + \frac{1}{1 + \frac{2}{7}} = 2 + \frac{7}{9}.$$

The entries of the continued fraction are the multiplicities of the weights

Obstructions: Fibonacci numbers and continued fractions 11

Ratios of Fibonacci numbers give rise to very special continued fractions:
The Fibonacci numbers $1, 1, 2, 3, 5, 8, 13, 21, \ldots, F_k, F_{k+1}, \ldots$ have ratios $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{8}{5}, \frac{13}{8}, \ldots$ with weight expansions $W(\frac{8}{5}) = (5, 3, 2, 1, 1)$. – all entries (except the last) of multiplicity 1 so $\frac{8}{5} = [1; 1, 1, 2] = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{2}}}$.

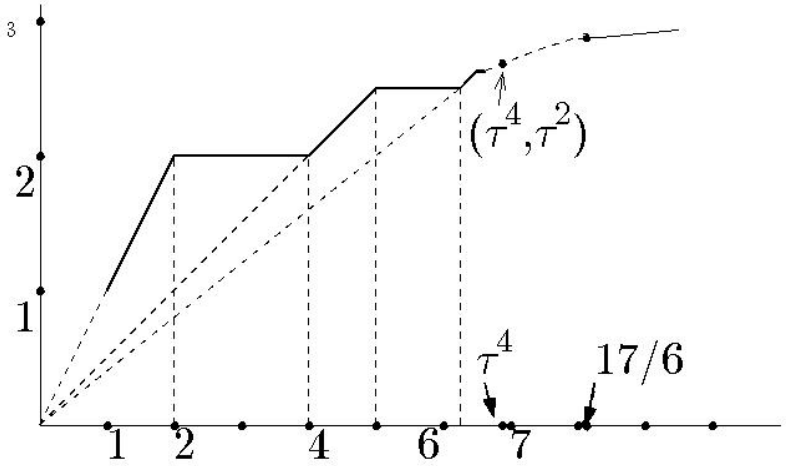


In general, $\frac{F_{k+1}}{F_k} = [1; 1, 1, 1, \ldots, 2]$.
In the limit we get
$\lim_{k\to\infty} \frac{F_{k+1}}{F_k} = \tau = \frac{1+\sqrt{5}}{2}$ – the golden ratio.
τ is irrational, with infinite cont. fract expansion $[1; 1, 1, 1, \ldots]$

Hence the ratios $\frac{F_{4k+5}}{F_{4k+1}}$ with cont. fract $= [6; 1, \{5, 1\}^{k-1}, 4]$ converge to $\tau^4 = [6; 1, \{5, 1\}^{\infty}]$. These are some of the centers of the classes of the Fibonacci stairs.

Fact: the continued fraction of a number x is eventually periodic exactly if x is a quadratic irrational (such as $\tau^4 = \frac{7+3\sqrt{5}}{2}$).

The Fibonacci stairs again



This has steps given by the perfect (exceptional) classes

$$\big(d_k, 0; q_k \mathbf{w}(p_k/q_k)\big) = \big(F_{2k+3}, 0; F_{2k+1}\mathbf{w}(F_{2k+5}/F_{2k+1})\big) \quad \text{where}$$

$$\mathbf{E}_k := \mathbf{w}(F_{4k+5}/F_{4k+1}) = [6; 1, \{5, 1\}^{k-1}, 4],$$

$$\mathbf{w}(F_{4k+7}/F_{4k+3}) = [6; 1, \{5, 1\}^{k-1}, 5, 2],$$

$$F_1 = 1, F_3 = 2, F_5 = 5, F_7 = 13, F_9 = 34, F_{11} = 89, F_{13} = 233, \ldots$$

e.g. $233/34 = [6; 1, 5, 1, 4]$ has weight exp. $\big(34^{\times 6}, 29, 5^{\times 5}, 4, 1^{\times 4}\big)$.

The inner corners of these stairs lie on the volume constraint. This is a double staircase – i.e. the steps have one of two possible endings that alternate as you go up the steps, so that there is an intertwined pair of staircases

(III): New features in $H_b = \mathbb{C}P^2(1)\#\overline{\mathbb{C}P}^2(b)$.

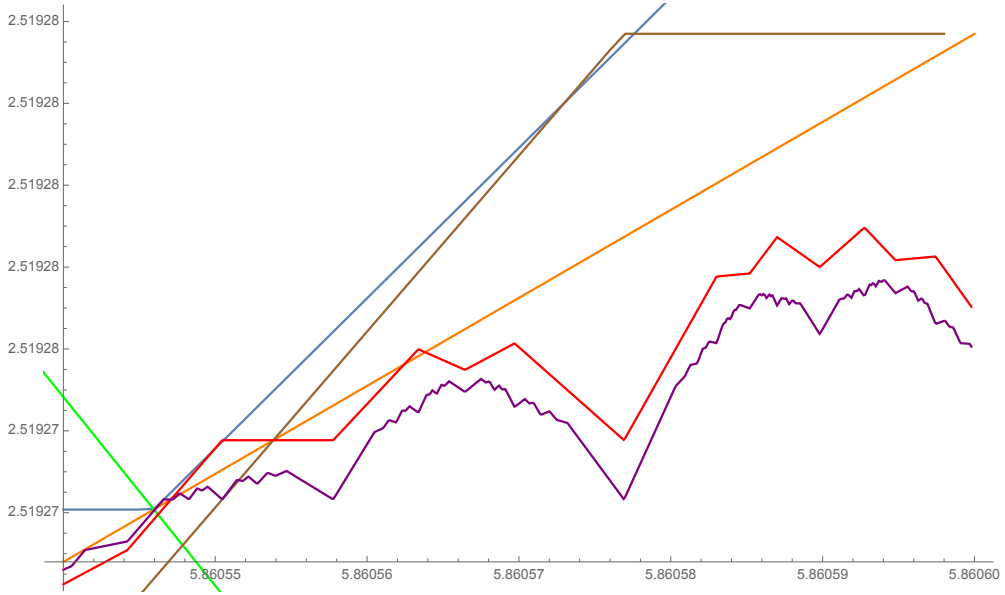
- An exceptional class $\mathbf{B} = \big(d, m; q\mathbf{w}(p/q)\big)$ is called a center-blocking class if for $b : \mathrm{acc}(b) = p/q$ we have $\mu_{\mathbf{B},b}(p/q) > V_b(p/q)$. (where we choose b in the same component of $[0, 1/3) \cup (1/3, 1)$ as m/d)
- In this case, there can be no staircase for $b = \mathrm{acc}^{-1}(p/q)$ and, by continuity, for all b in an interval $J_{\mathbf{B}}$. There is a corresponding blocked z-interval $I_{\mathbf{B}} = \mathrm{acc}(J_{\mathbf{B}})$ consisting of points that are not accumulation points of staircases for $b > 1/3$ (resp. $< 1/3$.)
- In all cases we have investigated, there are staircases accumulating at the end points of $I_{\mathbf{B}}$. (more details to follow)
- Example: the staircases $\mathcal{S}^U_{\bullet,n}$ lie at the endpoints of the intervals given by the blocking classes

$$\mathbf{B}^U_n = \big(n+3, n+2; \mathbf{w}(2n+6)\big),\ n \geq 0 \text{ with centers } 6, 8, 10, \ldots$$

- when the staircases descend they may be almost overshadowed i.e. sometimes there is a large obstruction that prevents the steps from reaching the volume curve.

The descending staircase $\mathcal{S}^E_{u,0}$

This has accumulation point $a_\infty = [5; 1, 6, \{5, 1\}^\infty] \approx 5.86$ with $b_\infty \approx 0.28$.



- the orange line is the volume constraint *Vol*
- the green line is the parametrized accumulation curve $b \mapsto (\mathrm{acc}(b), V_b(\mathrm{acc}(b))$ – meeting the orange *Vol* when $b = b_\infty$.
- the dark blue line is the max. of all obstructions with degree ≤ 200; it is given for $z < a_\infty$ by the blocking class $\mathbf{B}^E_0$, and for $z > a_\infty$ by the class $\mathbf{E} = (3, 1; 2, 1^{\times 5})$. It goes through the accum. point $(a_\infty, V_{b_\infty}(a_\infty))$.
- the brown, red and purple curves are graphs of $z \mapsto \mu_{\mathbf{E}, b_\infty}(z)$ for three different classes $\mathbf{E}$ in the staircase.

Pre-staircases

A sequence $\mathcal{S} = \left(\mathbf{E}_k\right)_{k\geq 0}$ of perfect classes $\mathbf{E}_k := \left(d_k, m_k; q_k\mathbf{w}(p_k/q_k)\right)$ is a pre-staircase if it has the following properties:

- (Recursion) There is an integer $\sigma \geq 0$ such that $\sigma + 4$ is a perfect square, and each of the sequences $x_k := d_k, m_k, p_k, q_k$ satisfies the recursion $x_{k+1} = (\sigma + 2)x_k - x_{k-1}, k \geq 0,$
- (Relation) there are integers R_0, R_1, R_2 such that $R_0 d_k = R_1 p_k + R_2 q_k, \ \forall k \geq 0.$

Note: If we only know p_k, q_k we may define d_k by (Relation) and set $m_k := 3d_k - p_k - q_k$.

Example: $\mathcal{S}^U_{u,n}$ has limit pt $a^U_{u,n,\infty} = [2n+7; \{2n+5, 2n+1\}^\infty]$; and

(Centers)	$[2n+7; \{2n+5, 2n+1\}^k, \mathrm{end_n}]$
	$\mathrm{end}_n = 2n+4 \quad \text{or } (2n+5, 2n+2)$
(Recursion)	$x_{n,k+1} = (\sigma_n + 2)x_{n,k} - x_{n,k-1}, \quad \sigma_n := (2n+1)(2n+5)$
(Relation)	$(2n+3)d_{n,k} = (n+2)p_{n,k} - (n+4)q_{n,k}$

This is a "double staircase" because there are two ends. All the new staircases have this form. Also, they all have the same recursion (coming from $\{2n+5, 2n+1\}^k$) but have different Relations (coming from the blocking classes).

When is a pre-staircase an actual staircase?

A pre-staircase is a (special kind of) sequence $\mathcal{S} = \big(\mathbf{E}_k\big)_{k\geq 0}$ of perfect classes $\mathbf{E}_k := \big(d_k, m_k; q_k\mathbf{w}(p_k/q_k)\big)$. The Recursion implies that the ratios p_k/q_k and m_k/d_k both converge. We write

$$\lim_k p_k/q_k =: a_\infty, \qquad \lim_k m_k/d_k =: b_\infty.$$

- Because each class $\mathbf{E}_k$ is obstructive at p_k/q_k for $b_k := m_k/d_k$, the argument in [C-GHMP] implies that $\mathrm{acc}(b_\infty) = a_\infty$. (For this we only need the classes $\mathbf{E}_k$ to satisfy the Diophantine conditions – they needn't even be perfect!)
- If the $\mathbf{E}_k$ are perfect, then $\mu_{\mathbf{E}_k, b_k}$ is live at p_k/q_k – and it then follows from the continuity of c_{H_b} as a function of b that b_∞ **is unobstructed**. i.e. $c_{H_{b_\infty}}(a_\infty) = \mathrm{Vol}_{b_\infty}(a_\infty)$.
- To have a staircase at b_∞ you need the $\mathbf{E}_k$ to be live at b_∞ — which involves further estimates.

Pre-staircases and blocking classes

The following theorem says that coefficients of the blocking class $\mathbf{B}$ determine the relations of its associated pre-staircases.

Theorem: Let $\mathbf{B} = \big(d, m; q\mathbf{w}(p/q)\big)$ have associated blocked z-interval $I_{\mathbf{B}}$.

(i) If the ascending pre-staircase $\mathcal{S}_\ell$ accumulates at the lower end point $a_{\ell,\infty}$ of $I_{\mathbf{B}}$ then its Relation is

$$(3m - d)d_k = (m - q)p_k + mq_k.$$

(ii) If the descending pre-staircase $\mathcal{S}_u$ accumulates at the upper end point $a_{u,\infty}$ of $I_{\mathbf{B}}$ where $\mathbf{B} = \big(d, m; q\mathbf{w}(p/q)\big)$, then its Relation is

$$(3m - d)d_k = mp_k + (m - p)q_k$$

Example: $\mathcal{S}^U_{u,n}$ has Relation $(2n+3)d_{n,k} = (n+2)p_{n,k} - (n+4)q_{n,k}$ and converges to the upper end of $I_{\mathbf{B}^U_n}$, where $\mathbf{B}^U_n$ has $d = n+3, m = n+2, p = 2n+6, q = 1$.

Summing up:

- We found three sets of blocking classes and associated 2-periodic staircases:
  - $(\mathbf{B}_n^U)_{n\geq 0}$ that blocks b-intervals in $(5/11, 1)$ and z-intervals in $(6, \infty)$
  - $(\mathbf{B}_n^L)_{n\geq 0}$ that blocks b-intervals in $[0, 1/5)$ and z-intervals in $(6, \tau^4)$
  - $(\mathbf{B}_n^E)_{n\geq 0}$ that blocks b-intervals in $(1/5, 19/61)$ and z-int. in $(35/6, 6)$.

- We conjecture these are the only 2-periodic staircases with z, b in these intervals (no idea how to prove that)

- There are more blocking classes (and staircases) blocking b-intervals in $(11/31, 5/11)$ and z-intervals in $(35/6, 6)$ that are the image of the family $(\mathbf{B}_n^U)_{n\geq 0}$ under the reflection $[6, \infty) \to (35/6, 6]$ that acts on z coordinates by $z \mapsto \frac{35z-204}{6z-35}$ with fixed point 6. This reflection fixes $\mathbf{B}_0^U$, and interchanges its two staircases $\mathcal{S}_{u,0}^U, \mathcal{S}_{\ell,0}^U$. (not fully proved yet)

- there are further symmetries of the problem connected to the sequence $(y_1, y_2, y_3, \dots) = (1, 6, 35, 204, 1189, \dots)$ (half the even Pell numbers).

Summing up: Further symmetries

Let $(y_0, y_1, y_2, y_3, \dots) := (0, 1, 6, 35, 204, 1189, \dots)$

- This sequence gives the values of y for which the Diophantine equation $x^2 - 8y^2 = 1$ has a solution.
- for all i, $b_i = \mathrm{acc}^{-1}(y_{i+1}/y_i)$ is rational, and $\lim y_{i+1}/y_i = 3 + 2\sqrt{2} = \mathrm{acc}^{-1}(1/3)$
- the shift $Sh^2 : z \mapsto \frac{35z-6}{6z-1}$ takes the interval $I_1 := (35/6, \infty)$ to $I_3 = (1189/204, 35/6)$ and we conjecture it takes $\mathbf{B}^U_\bullet, \mathcal{S}^U_{\bullet,\bullet}$ to blocking classes/staircases in I_3 (and also for $b > 1/3$).
- More generally, Sh^2 takes $I_i := (y_{i+2}/y_{i+1}, y_i/y_{i-1})$ to I_{i+2} and we conjecture that it takes the blocking classes/stairs with $z \in I_i$ to the blocking classes/stairs with $z \in I_{i+2}$. (some of this is proved).

NOTE: the symmetries $z \mapsto \frac{y_i z - y_{i+1}}{y_{i-1} z - y_i}$ act on the continued fraction expansions of the z-variables p/q in a very understandable way. e.g. $z \mapsto \frac{6z-35}{z-6}$ acts by $[6; k, \ell_2, \ell_3, \dots] \mapsto [6 + k; \ell_2, \ell_3, \dots]$. So, we can see what happens to the staircase steps. But the action on the (d, m) coordinates of the perfect classes $(d, m; q\mathbf{w}(p/q))$ is much more mysterious. In contrast, Usher encoded *all* his variables in terms of the solutions to a single Diophantine equation on which the Brahmagupta moves act.

Summing up: Structure of *Block*?

We define: *Block* $:= \{b \in [0,1) \mid c_{H_b}(\mathrm{acc}(b)) > V_b(\mathrm{acc}(b))\}$ – an open set
Stair $:= \{b \in [0,1) \mid H_b$ has a staircase$\}$

and have: *Stair* $\subsetneq$ *Unobstr* $:= [0,1) \backslash$ *Block*.

- Above we described families of blocking classes $\mathbf{B}_n^\star$, each of which defines an interval in *Block*. All seem to have associated staircases.
- But we can prove that almost all the staircase classes are themselves blocking classes!
- and, whenever we look, we find what seem to be associated staircases
- Therefore *Block* may be dense in $[0,1)$.

We will post our first paper on arXiv very soon, and another should follow at some point.

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