Homogeneous quasimorphisms, $C^0$-topology and Lagrangian intersection

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Plan

1. Introduction - basics and motivations
2. Main result - on the existence of certain quasimorphisms
3. Key idea - which might be useful elsewhere as well
Introduction

Notations

- $(M, \omega)$: a closed monotone symplectic manifold.
- $H : [0, 1] \times M \to \mathbb{R}$: a (time-dependent) Hamiltonian on $(M, \omega)$.
- $Ham(M, \omega)$: the group of Hamiltonian diffeomorphisms of $(M, \omega)$.

Major theme in symplectic topology

What can we say about the algebraic and topological properties of $Ham(M, \omega)$?
Topological side of $\text{Ham}(M, \omega)$

▶ What topology do we consider? Let $\phi, \phi' \in \text{Ham}(M, \omega)$.

1. Hofer metric:

   \[ \mathcal{E}(H) := \int_0^1 (\sup_{x \in M} H_t(x) - \inf_{x \in M} H_t(x)) \, dt. \]

   \[ d_{\text{Hof}}(id, \phi) := \inf \{ \mathcal{E}(H) : \phi_H = \phi \}. \]

   \[ d_{\text{Hof}}(\phi, \phi') := d_{\text{Hof}}(id, \phi^{-1} \phi'). \]

2. $C^0$-topology:

   \[ d_{C^0}(\phi, \phi') := \max_{x \in M} d_M(\phi(x), \phi'(x)) \]

   where $d_M$ is the natural distance on $M$. 

Why do we care about $C^0$-topology?

- It seems that $C^0$-topology DOES have something to do with the symplectic structure even though symplectic geometry is smooth geometry.
  
e.g.

**Theorem (Eliashberg-Gromov)**

Let $\phi_n \in \text{Symp}(M, \omega)$ be a sequence of symplectomorphisms. Assume

$$\phi_n \xrightarrow{C^0} \phi \in \text{Diffeo}(M).$$

Then, $\phi \in \text{Symp}(M, \omega)$.

- The relation between the Hofer metric and $C^0$-topology on $\text{Ham}(M, \omega)$ is not fully understood.
Theorem (Banyaga ’78)

$\text{Ham}(M, \omega)$ is a simple group.

Corollary

There exist no non-trivial homomorphisms

$$\text{Ham}(M, \omega) \to \mathbb{R}.$$ 

However, there exist quasimorphisms on $\text{Ham}(M, \omega)$ for some $(M, \omega)$!
Quasimorphisms ("almost homomorphisms")

- A homogeneous quasimorphism on a group $G$ is a map

$$\mu : G \rightarrow \mathbb{R}$$

which satisfies

1. $\exists C > 0$ s.t. $\forall f, g \in G$,

   $$|\mu(f \cdot g) - \mu(f) - \mu(g)| \leq C.$$

2. $\forall k \in \mathbb{Z}, \forall f \in G$,

   $$\mu(f^k) = k \cdot \mu(f).$$

- Homogeneous quasimorphisms are useful to study algebraic and topological properties (in case $G$ is a topological group) of $G$. 
Entov-Polterovich constructed homogeneous quasimorphisms

\[ \zeta_e : \text{Ham}(M, \omega) \to \mathbb{R} \]

via spectral invariants (which are certain Floer theoretic invariants) for symplectic manifolds that meet a certain condition.

Remark

- Condition posed on the structure of the quantum cohomology ring.
- Entov-Polterovich type homogeneous quasimorphisms are Hofer Lipschitz continuous but not $C^0$-continuous.
Motivating question

Question of Entov-Polterovich-Py

1. Does there exist a non-trivial homogeneous quasimorphism

\[ \mu : \text{Ham}(S^2) \to \mathbb{R} \]

that is \( C^0 \)-continuous?

2. If yes, is it Hofer Lipschitz continuous?
New Question

Does there exist a closed symplectic manifold $(M, \omega)$ which admits a non-trivial homogeneous quasimorphism

$$\mu : \text{Ham}(M, \omega) \to \mathbb{R}$$

that is $C^0$-continuous? If yes, is $\mu$ Hofer Lipschitz continuous?

- Some results related to this question:
  - For $D^{2n}(1) \subset \mathbb{R}^{2n}$, $\exists \mu$ that are $C^0$ and Hofer Lipschitz continuous. (Entov-Polterovich-Py)
  - For closed surfaces $\Sigma_g$, $g \geq 1$, $\exists \mu$ that are $C^0$-continuous but not Hofer continuous. (Gambaudo-Ghys, Khanevsky)
  - No example of a closed symplectic manifold for which there exists $\mu$ that is $C^0$ and Hofer Lipschitz continuous.
Main result

Notation
We denote the monotone $n$-quadric by $Q^n$:

$$Q^n := \{(z_0 : z_1 : \cdots : z_{n+1}) \in \mathbb{C}P^{n+1} : z_0^2 + z_1^2 + \cdots + z_{n+1}^2 = 0\}.$$

Theorem (K '20)
There exist non-trivial homogeneous quasimorphisms

$$\mu : \text{Ham}(Q^n) \rightarrow \mathbb{R}$$

where $n = 2, 4$ that are $C^0$ and Hofer Lipschitz continuous.
Key of the proof - quantum cohomology rings with different coefficient fields

- **Classical quantum cohomology ring (Floer, Oh):**

\[
QH^*(M; \mathbb{C}) := H^*(M; \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[t^{-1}, t] 
\]

where \(\mathbb{C}[t^{-1}, t]\) is the field of Laurent series

\[
\mathbb{C}[t^{-1}, t] := \left\{ \sum_{k \geq k_0} a_k t^k : k_0 \in \mathbb{Z}, a_k \in \mathbb{C} \right\}
\]

\((t \text{ satisfies } \omega(t) = \lambda_0, \ c_1(t) = N_M.\))

- **Modern quantum cohomology ring (Fukaya-Oh-Ohta-Ono):**

\[
QH^*(M; \Lambda) := H^*(M; \mathbb{C}) \otimes_{\mathbb{C}} \Lambda
\]

where \(\Lambda\) is the universal Novikov field

\[
\Lambda := \left\{ \sum_{k=1}^{\infty} b_k T^{\lambda_k} : b_k \in \mathbb{C}, \lambda_k \in \mathbb{R}, \lim_{k \to +\infty} \lambda_k = +\infty \right\}.
\]
Classical and modern quantum cohomology rings: Difference 1

Classical and modern QH have different algebraic structures!

Example

- $QH^*(\mathbb{C}P^2; \mathbb{C})$ is a field.
- $QH^*(\mathbb{C}P^2; \Lambda)$ is semi-simple and splits into a direct sum of three fields:

$$QH^*(\mathbb{C}P^2; \Lambda) = Q_1 \oplus Q_2 \oplus Q_3$$

where $Q_1, Q_2, Q_3$ are fields.
They have different advantages!

▶ With the **classical QH**, we can do more with **spectral invariants**!
  e.g. The $\mathbb{Z}$-grading of the classical QH brings the info of both the action and the index to spectral invariants.

▶ With the **modern QH**, we can do more with **Lagrangian Floer theory**!
  e.g. With $\Lambda$-coefficients, we have a very rich Lagrangian Floer theory (FOOO). Especially, **superpotential** techniques are useful to find Lagrangian submanifolds with non-trivial HF.
Outline of the proof of the main theorem

Part 1
We use the advantage of classical QH $QH^*(M; \mathbb{C})$:

- For $Q^n$, there are two Entov-Polterovich type homogeneous quasimorphisms $\zeta_+, \zeta_-$. 
- Define 
  \[ \mu : \text{Ham}(Q^n) \to \mathbb{R} \]
  \[ \mu := \zeta_+ - \zeta_- \].

- Prove $\mu$ is $C^0$-continuous by using a result on the $C^0$-control of spectral invariants ([K19]) which uses the information of the action and the index of spectral invariants in the proof ($\mathbb{Z}$-grading plays an essential role).
We need to say that $\mu$ is non-trivial i.e. $\zeta_+ \neq \zeta_-$!

**Part 2**

We use the advantage of modern QH $QH^*(M; \Lambda)$:

- In the spirit of Entov-Polterovich’s (super)heavy theory, we want to find two disjoint Lagrangian submanifolds with non-trivial HF!

- Such Lagrangians are found for $Q^n, n = 2, 4$ by superpotential techniques!
  
  ($n = 2$ case, due to Fukaya-Oh-Ohta-Ono, $n = 4$ case, due to Nishinou-Nohara-Ueda.)

- We conclude that $\zeta_+ \neq \zeta_-$.

DONE!
The proof benefited from the different advantages of classical and modern QH.

This idea of combining the two has other applications. e.g. Question of Polterovich-Wu, Lagrangian intersection etc.
Thanks for your attention!!!