Homogeneous quasimorphisms, C^0-topology and Lagrangian intersection

Yusuke Kawamoto

École Normale Supérieure, Paris

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Symplectic Zoominar
(CRM-Montreal, Princeton/IAS, Tel Aviv and Paris)

Plan

1. Introduction - basics and motivations
2. Main result - on the existence of certain quasimorphisms
3. Key idea - which might be useful elsewhere as well

Introduction

Notations

- (M, ω): a closed monotone symplectic manifold.
- $H : [0,1] \times M \to \mathbb{R}$: a (time-dependent) Hamiltonian on (M, ω).
- $Ham(M, \omega)$: the group of Hamiltonian diffeomorphisms of (M, ω).

Major theme in symplectic topology

What can we say about the algebraic and topological properties of $Ham(M, \omega)$?

Topological side of $Ham(M, \omega)$

- What topology do we consider? Let $\phi, \phi' \in Ham(M, \omega)$.
  1. Hofer metric:
  $$\mathcal{E}(H) := \int_0^1 (\sup_{x \in M} H_t(x) - \inf_{x \in M} H_t(x))dt.$$
  $$d_{Hof}(id, \phi) := \inf\{\mathcal{E}(H) : \phi_H = \phi\}.$$
  $$d_{Hof}(\phi, \phi') := d_{Hof}(id, \phi^{-1}\phi').$$
  2. C^0-topology:
  $$d_{C^0}(\phi, \phi') := \max_{x \in M} d_M(\phi(x), \phi'(x))$$
  where d_M is the natural distance on M.

Why do we care about C^0-topology?

- It seems that C^0-topology DOES have something to do with the symplectic structure even though symplectic geometry is smooth geometry.
  e.g.

Theorem (Eliashberg-Gromov)

Let $\phi_n \in Symp(M, \omega)$ be a sequence of symplectomorphisms. Assume

$$\phi_n \xrightarrow{C^0} \phi \in Diffeo(M).$$

Then, $\phi \in Symp(M, \omega)$.

- The relation between the Hofer metric and C^0-topology on $Ham(M, \omega)$ is not fully understood.

Algebraic side of $Ham(M, \omega)$

Theorem (Banyaga '78)

$Ham(M, \omega)$ is a simple group.

Corollary

There exist no non-trivial homomorphisms

$$Ham(M, \omega) \to \mathbb{R}.$$

- However, there exist quasimorphisms on $Ham(M, \omega)$ for some (M, ω)!

Quasimorphisms ("almost homomorphisms")

- A homogeneous quasimorphism on a group G is a map

$$\mu : G \to \mathbb{R}$$

which satisfies

1. $\exists C > 0$ s.t. $\forall f, g \in G$,

$$|\mu(f \cdot g) - \mu(f) - \mu(g)| \leqslant C.$$

2. $\forall k \in \mathbb{Z}, \forall f \in G$,

$$\mu(f^k) = k \cdot \mu(f).$$

- Homogeneous quasimorphisms are useful to study algebraic and topological properties (in case G is a topological group) of G.

Quasimorphisms in symplectic topology

- Entov-Polterovich constructed homogeneous quasimorphisms

$$\zeta_e : Ham(M, \omega) \to \mathbb{R}$$

via spectral invariants (which are certain Floer theoretic invariants) for symplectic manifolds that meet a certain condition.

Remark

- Condition posed on the structure of the quantum cohomology ring.
- Entov-Polterovich type homogeneous quasimorphisms are Hofer Lipschitz continuous but not C^0-continuous.

Motivating question

Question of Entov-Polterovich-Py

1. Does there exist a non-trivial homogeneous quasimorphism

$$\mu : Ham(S^2) \to \mathbb{R}$$

   that is C^0-continuous?
2. If yes, is it Hofer Lipschitz continuous?

New Question

Does there exist a closed symplectic manifold (M, ω) which admits a non-trivial homogeneous quasimorphism

$$\mu : Ham(M, \omega) \to \mathbb{R}$$

that is C^0-continuous? If yes, is μ Hofer Lipschitz continuous?

- Some results related to this question:
  - For $D^{2n}(1) \subset \mathbb{R}^{2n}$, $\exists \mu$ that are C^0 and Hofer Lipschitz continuous. (Entov-Polterovich-Py)
  - For closed surfaces $\Sigma_g,\ g \geqslant 1,\ \exists \mu$ that are C^0-continuous but not Hofer continuous. (Gambaudo-Ghys, Khanevsky)
- No example of a closed symplectic manifold for which there exists μ that is C^0 and Hofer Lipschitz continuous.

Main result

Notation

We denote the monotone n-quadric by Q^n:
$Q^n := \{(z_0 : z_1 : \cdots : z_{n+1}) \in \mathbb{C}P^{n+1} : z_0^2 + z_1^2 + \cdots + z_{n+1}^2 = 0\}$.

Theorem (K '20)

There exist non-trivial homogeneous quasimorphisms

$$\mu : Ham(Q^n) \to \mathbb{R}$$

where $n = 2, 4$ that are C^0 and Hofer Lipschitz continuous.

Key of the proof - quantum cohomology rings with different coefficient fields

- classical quantum cohomology ring (Floer, Oh):

$$QH^*(M;\mathbb{C}) := H^*(M;\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[t^{-1}, t||]$$

  where $\mathbb{C}[t^{-1}, t||]$ is the field of Laurent series

$$\mathbb{C}[t^{-1}, t||] := \{ \sum_{k \geqslant k_0} a_k t^k : k_0 \in \mathbb{Z}, a_k \in \mathbb{C} \}$$

  (t satisfies $\omega(t) = \lambda_0, \ c_1(t) = N_M$.)

- modern quantum cohomology ring (Fukaya-Oh-Ohta-Ono):

$$QH^*(M;\Lambda) := H^*(M;\mathbb{C}) \otimes_{\mathbb{C}} \Lambda$$

  where Λ is the universal Novikov field

$$\Lambda := \{ \sum_{k=1}^{\infty} b_k T^{\lambda_k} : b_k \in \mathbb{C}, \lambda_k \in \mathbb{R}, \lim_{k \to +\infty} \lambda_k = +\infty \}.$$

Classical and modern quantum cohomology rings: Difference 1

Classical and modern QH have different algebraic structures!

Example

- $QH^*(\mathbb{C}P^2; \mathbb{C})$ is a field.
- $QH^*(\mathbb{C}P^2; \Lambda)$ is semi-simple and splits into a direct sum of three fields:

$$QH^*(\mathbb{C}P^2; \Lambda) = Q_1 \oplus Q_2 \oplus Q_3$$

where Q_1, Q_2, Q_3 are fields.

Classical and modern quantum cohomology rings: Difference 2

They have different advantages!

- With the classical QH, we can do more with spectral invariants!
  e.g. The $\mathbb{Z}$-grading of the classical QH brings the info of both the action and the index to spectral invariants.
- With the modern QH, we can do more with Lagrangian Floer theory!
  e.g. With Λ-coefficients, we have a very rich Lagrangian Floer theory (FOOO). Especially, superpotential techniques are useful to find Lagrangian submanifolds with non-trivial HF.

Outline of the proof of the main theorem

Part 1

We use the advantage of classical QH $QH^*(M;\mathbb{C})$:

- For Q^n, there are two Entov-Polterovich type homogeneous quasimorphisms ζ_+, ζ_-.
- Define
$$\mu : Ham(Q^n) \to \mathbb{R}$$
$$\mu := \zeta_+ - \zeta_-.$$
- Prove μ is C^0-continuous by using a result on the C^0-control of spectral invariants ([K19]) which uses the information of the action and the index of spectral invariants in the proof ($\mathbb{Z}$-grading plays an essential role).

We need to say that μ is non-trivial i.e. $\zeta_+ \neq \zeta_-$!

Part 2

We use the advantage of modern QH $QH^*(M;\Lambda)$:

- In the sprit of Entov-Polterovich's (super)heavy theory, we want to find two disjoint Lagrangian submanifolds with non-trivial HF!
- Such Lagrangians are found for $Q^n, n = 2, 4$ by superpotential techniques!
  ($n = 2$ case, due to Fukaya-Oh-Ohta-Ono, $n = 4$ case, due to Nishinou-Nohara-Ueda.)
- We conclude that $\zeta_+ \neq \zeta_-$.

DONE!

Extra Remark

- The proof benefited from the different advantages of classical and modern QH.
- This idea of combining the two has other applications.
  e.g. Question of Polterovich-Wu, Lagrangian intersection etc.

Thanks for your attention!!!