Hamiltonian classification and unlinkedness of fibres in cotangent bundles of Riemann surfaces

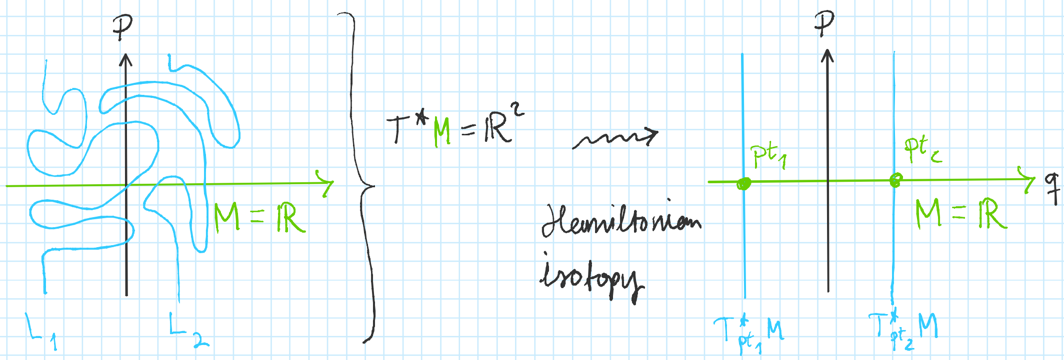
Friday, September 4, 2020 7:49 AM

joint with L. Côté [arXiv: 2004.04233]

Plan: 0. Introduction

1. Results
2. Relations to differential topology
3. Proof

0. Introduction



$H^1(L_i, \mathbb{R}) = 0$: any Lagrangian isotopy is gen. by a Hamiltonian

$H^1_c(L_i, \mathbb{R}) \neq 0$: not necessarily compactly supported

but $H^1_c(\mathbb{R}^n, \mathbb{R}) = 0$ if $n > 1$

Definitions For M a smooth n-dim$^{\underline{l}}$ manifold

T^*M is a symplectic $2n$-dim$^{\underline{l}}$ mfd. with the tautological symplectic form $\omega = d(pdq)$

In local coordinates $q_1, \dots, q_n \in M \rightsquigarrow$ conjugate coordinates $p_i \in T^*M$

$$\omega = d\left(\sum_{i=1}^{n} p_i \, dq_i\right)$$

Ex $T^*\mathbb{R}^n = \mathbb{R}^{2n}$ is the std. symplectic vector space

$0_M, T^*_{pt}M \subseteq T^*M$ examples of Lagrangian submanifolds

i.e. • n-dimensional (half-dimensional)
• $p_i\,dq_i$ pulls back to a closed 1-form

Fact A compactly supported isotopy $L_t \subseteq (X,\omega)$ through Lagrangian submanifolds (i.e. a Lagrangian isotopy) is generated by an ambient Hamiltonian iso. if $H^1(L_t,\mathbb{R})=0$ (comp. supp. if $H^1_c(L_t,\mathbb{R})=0$).

1. Results

Thm If M is an $\boxed{\text{open}}$ (important!) Riemann surface and $L \subseteq T^*M$ is a Lagrangian submanifold which is $\boxed{\text{diffeomorphic to } \mathbb{R}^2}$ (can be replaced by the condition to be exact Lagr.) & coincides with a fibre $T^*_{pt}M$ outside of a compact subset, then L is compactly supported Ham. iso to $T^*_{pt}M$.

Rem Since T^*M is subcrit. when M is open, an exact Lagr. which coincides w. $T^*_{pt}M$ outside of a compact subset can be shown to be contractible. (Floer homology with local coeff.) When $n \gg 0$, the diffeom. induced on its ideal bdy S^{n-1} is isotopic to $\mathrm{id}_{S^{n-1}}$, as shown for $M=\mathbb{R}^n$ by [Ekholm-Smith]

(M being open is crucial: consider Lag. surgery on $0_{\mathbb{P}^2} \cup T^*_{pt}\mathbb{P}^2 \subseteq T^*\mathbb{P}^2$)

Previous results

- $M=\mathbb{R}^2$ was established by [Eliashberg-Polterovich '96]
(in fact: space of such Lag's was shown to be weakly contractible)

Both our & their proofs use pseudoholomorphic foliations, the main difference is how to control embeddedness.

they: pseudoconvex hypersurfaces; we: SFT-techniques

- $M = \mathbb{R}^n \setminus \{0\}$ homotopy version of the statement was proven by [Ekholm-Smith] $\left(\pi_n(S^{n-1}) = \mathbb{Z}_2 \quad n \geq 3\right)$

<u>Unlinkedness</u> (our original interest & aim)

Let $L_1 \amalg \ldots \amalg L_N \subseteq T^*M$ be a Lagrangian <u>link</u>
with
- $L_i = T^*_{pt_i} M$ outside of a cpct subset
- $L_i \cong \mathbb{R}^2$

Cor If M is a (possibly closed) surface & L_1 is comp. supp. Ham iso. to $T^*_{pt_1} M$, then $L_1 \amalg \ldots \amalg L_N$ is comp. supp. Ham. iso. to $T^*_{pt_1} M \amalg \ldots \amalg T^*_{pt_N} M$.

<u>Pf</u> $T^*M \setminus T^*_{pt_1} M = T^*(\underbrace{M \setminus pt_1}_{\text{open.}})$

Apply the theorem inductively to $L_2, L_3, \ldots,$ etc. $\square$

2. Discussion

Relations to questions in differential topology
(what to expect for families & in higher dim$^{\underline{\ell}}$ $T^*(\mathbb{R}^n \setminus \{0\})$

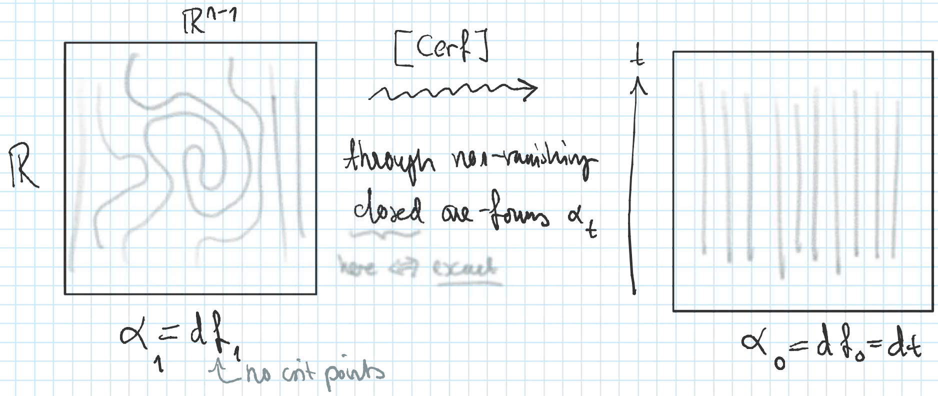
Fix a linear one-form on $\mathbb{R}^n \ni \bar{p}$

$$\alpha_0 = q_1^0 dp_1 + \ldots + q_n^0 dp_n, \quad \bar{q}^0 \in \mathbb{R}^n \setminus \{0\}$$

The graph of a one-form α which coincides w. α_0 outside of a compact subset is a plane $\subseteq T^* \mathbb{R}^n$ which
- coincides w. $T^*_{\bar{q}^0} \mathbb{R}^n$ outside of a compact subset
- is Lagr. $\Leftrightarrow$ α is closed
  (use $d(p\,dq) = -d(q\,dp)$)
- is disjoint from $T^*_0 \mathbb{R}^n$ $\Leftrightarrow$ α nonvanishing

For graphical $L \subseteq T^*\mathbb{R}^n$ the conclusions of the theorem is a consequence of:

- Smale's result $\pi_k(\mathrm{Diff}^\partial(D^2)) = 0$ (the case when $n=2$)
- Cerf's pseudoisotopy thm. when $n \geq 6$



Rem The space of pseudoisotopies is not contractible for $\mathbb{R}^n$, $n \gg 0$

consequently (since torsions Wh_k can be defined for Floer homology for exact Lag.s)

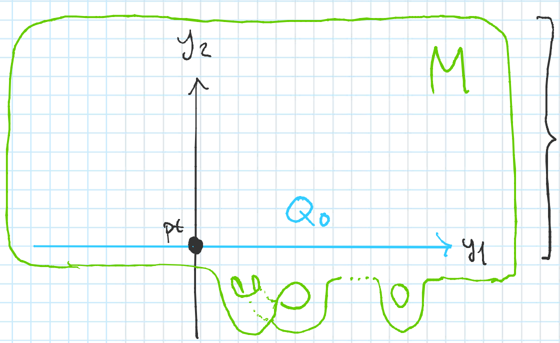
[Eliashberg–Gromov] there are non-triv. families of Lagrangian planes in $T^*(\mathbb{R}^n \setminus \{0\})$ also after dropping the graphical condition

based on the thy. of generating families

the "classical case"

3. Proof

Take $L \subseteq T^*M$ a Lag. plane which coincides with $T^*_{pt}M$ outside of a compact subset.



$T^*(M \cap \{y_2 > -1\}) = \underset{y_1}{\mathbb{R}} \times \underset{y_2}{(-1,+\infty)} \times \underset{(x_1,x_2)}{\mathbb{R}^2}$

i.e. "$dp \wedge dq = dx \wedge dy$"

p's q's

$Q_0 = \{y_2 = 0\} = \underset{y_1}{\mathbb{R}} \times \underset{y_2}{\{0\}} \times \underset{(x_1,x_2)}{\mathbb{R}^2} \subseteq T^*M$ hypersurface

- contains $L \subseteq Q_0$
- foliated by symplectic planes $\Sigma^0_\ell = \underset{y_1}{\mathbb{R}} \times \underset{y_2}{\{0\}} \times \underset{x_1}{\mathbb{R}} \times \underset{x_2}{\{\ell\}}$

Goal: Construct "such a hypersurface" for L as well.

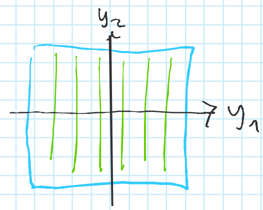
More precisely, we want to find a hypersurface $Q \subseteq T^*M$ such that

(P1): Q is foliated by <u>symplectic</u> planes $\Sigma_\ell \subseteq Q$

- Σ_ℓ coincides w. $\Sigma_\ell^0 = \{y_2 = 0, x_2 = \ell\}$ outside of cpct subset
- $\Sigma_\ell = \Sigma_\ell^0$ for all $|\ell| \gg 0$

(P_2): $L \subseteq Q$

(P_3): the parallel transport $\Sigma_{-N} \xrightarrow{\text{(sympl)}} \Sigma_N$ induced by the char. dist. preserves the foliation $\{y_1 = s\} \subseteq \Sigma_{\pm N}$



Fact. (P3) can be achieved by soft techniques; the symplectic suspension of a suitable Ham. can be used to deform the hypersurface in order to correct the above monodromy map

- (P1) – (P3) implies that Q is <u>foliated by Lagr. planes</u> parametrized by $\ell \in \mathbb{R}$, such that
  - all planes are "standard" planes $\{x_2 = 0, y_1 = s\}$ for $|s| \gg 0$
  - all planes are "standard" outside of a compact subset
  - L coincides with one of the leaves

$\Rightarrow$ L is comp. supp. Lagr. isotopic to $T^*_{pt}M$

elementary techniques

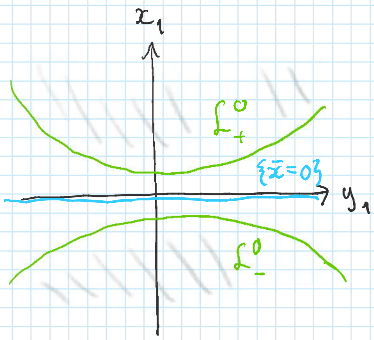
what remains:

<u>How to produce Q satisfying (P1) – (P2)</u>

sympl. coordinates

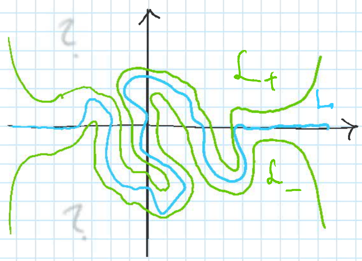
$Q_0 = \underset{y_1}{\mathbb{R}} \times \{0\} \times \underset{x_1}{\mathbb{R}} \times \underset{x_2}{\mathbb{R}}$ contains two Lagrangian cylinders

$\mathcal{L}^0_\pm = S^1_R(c_\pm) \times \underset{x_2}{\mathbb{R}}$ centred at $c_\pm = \{x_1 = \pm(R + \varepsilon), y_1 = 0\}$



(1) Remove $B^2_R(c_\pm)\times[-R,R]$ for $R>>0$

from $Q_0 \rightsquigarrow \tilde{Q}_0 = Q_0 \setminus \dots$



(2) Weinstein nbhd. thm: L & $\{\bar{x}=0\}$ have symplectomorphic neighbourhoods. We can thus find a symplectomorphism defined in a nbhd of Q^0 which

- is the identity outside some bdd subset
- maps $\{\bar{x}=0\}$ to L.

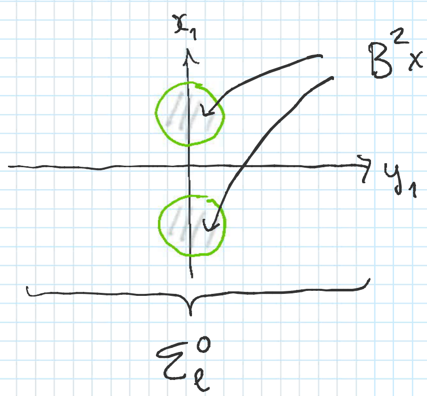
We get a hypersurface $\tilde{Q}$ containing L and the Lag^n cyl^s $\mathcal{L}_\pm$

What remains: fill the "holes" by putting back "$B^2_R(c_\pm)\times\mathbb{R}$",

i.e. find a suitable $B^2\times\mathbb{R} \hookrightarrow T^*M\setminus(\mathcal{L}_+\cup\mathcal{L}_-)$ foliated by sympl. B^2's.

Foliations by finite energy pshol. planes

$T^*M\setminus(\mathcal{L}_+\cup\mathcal{L}_-)$ is a sympl. mfd w. cylindrical concave ends



$B^2\times\{\ell\}$ for $|\ell|>>0$ are finite energy pshol. planes $\mathbb{C}$ for an appropriate cylindrical a.c.s.

Since these planes live in a 1-dim moduli space which locally foliates a hypersurface

- automatic transversality [Wendl] &
- asymptotic intersection thy [Siefring] &
- SFT compactness [Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder]

gives us the sought embedding

$$\underset{\substack{\uparrow \\ \text{finite energy pshol. planes}}}{\mathbb{C} \times \mathbb{R}} \hookrightarrow T^*M \setminus (\mathcal{L}_+ \cup \mathcal{L}_-)$$

that "fills the hole" of $\tilde{Q}$.

After smoothing, we get the sought Q.

$\square$