

Complexity of Lagrangian Submanifolds

(work in progress)

Octav Cornea

(part of a larger project with Giovanni Ambrosioni and Paul Biran)

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Introduction

Aim of talk: discuss some properties of the metric space

$$(\text{Lag}^*(M), d_\gamma)$$

where $\text{Lag}^*(M)$ is a certain class of Lagrangian submanifolds in the symplectic manifold (M, ω) and d_γ is the spectral metric (assumed to be defined in our setting – to be reviewed later – together with relevant Floer machinery).

- This space is not locally compact, nor complete.
- We are interested here in small scale questions.
- We will see that this space is locally non-precompact (there is no pre-compact neighbourhood around any given point).

I. Main result

We will provide, in appropriate settings, a characterization of pre-compact subsets

$$S \subset (\text{Lag}^*(M), d_\gamma) .$$

Recall: S is pre-compact if its Cauchy completion is compact, or, equivalently, if it is totally bounded (which means that for $\forall \epsilon > 0$ it admits a finite ϵ -net).

Will use a *complexity function*

$$c : \text{Lag}^*(M) \times (0, \infty] \times [1, \infty)^2 \rightarrow [0, \infty) .$$

This will be written as $c(L; \epsilon)(X, Y) \in [0, \infty)$ with the variables:

$$(\epsilon, X, Y) \in (0, \infty] \times [1, \infty) \times [1, \infty) .$$

Theorem

$S \subset (\text{Lag}_{\text{ex}}^{cl}(D^*N), d_\gamma)$ is pre-compact if and only if for all $\epsilon > 0$ and $X, Y \in [1, \infty)$, we have

$$\sup_{L \in S} c(L; \epsilon)(X, Y) < \infty .$$

$\text{Lag}_{\text{ex}}^{cl}(D^*N)$ is the class of closed, exact, graded Lagrangians in the cotangent disk bundle D^*N with some choice of metric.

In the class $\text{Lag}_{\text{ex}}^{cl}(D^*N)$ we fix a primitive and a grading for each element L , but d_γ is independent of these choices.

Remark

- The definition of $c(-)$ is based on persistence homological algebra applied to the Fukaya category; $c(-; -)(1, 1)$ was introduced in ABC (2026).
- $c(-; \infty)(1, Y) =$ (basically) a notion of categorical complexity introduced by Dimitrov-Haiden-Katzarkov-Kontsevich (2014).
- $c(-; \infty)(1, -1) \sim$ Euler characteristic (the definition makes sense formally also for $Y < 0$).
- $c(-; \infty)(1, 1) \sim$ cone-length in topology if the construction is applied to spaces.
- if $S \subset \text{Lag}^*(M)$ is pre-compact, then it is bounded. The estimate in the Theorem is coherent with previous results concerning diameter w.r. to d_γ .

Basic properties of the complexity function.

- For ϵ, Y fixed, $c(L; \epsilon)(-, Y)$ is monotone increasing for $X \in [1, \infty)$ and continuous at each point. Similarly, with respect to Y .
- For X, Y fixed, $c(L; -)(X, Y)$ is monotone decreasing for $\epsilon \in (0, \infty)$.
- If $d_\gamma(L, L') \leq \delta$, then

$$c(L; \epsilon + \delta) \leq c(L', \epsilon)$$

as functions in X, Y . In particular,

$$c(L'; \epsilon - d_\gamma(L, L')) \geq c(L; \epsilon) \geq c(L'; \epsilon + d_\gamma(L, L'))$$

whenever $d_\gamma(L, L') < \epsilon$.

- There are various estimates of $c(-)$ from above and from below that involve certain barcode calculations in HF .

A geometric corollary.

Some background on d_γ .

- For $L \in \text{Lag}_{\text{ex}}^{cl}(D^*N)$, denote by $f_L : L \rightarrow \mathbb{R}$ a choice of primitive. Consider L' in the same class, $L \pitchfork L'$. The complex

$$CF(L, L') = \mathbb{Z}_2 \langle L \cap L' \rangle$$

is filtered by $f_{L'}(x) - f_L(x)$, $x \in L \cap L'$. Thus $HF(L, L')$ is a persistence module.

- Let P be a persistence module. Denote by $\mathcal{B}_P$ its barcode.
- Let $\sigma(\mathcal{B}_P)$ be the spectrum of $\mathcal{B}_P = \text{subset} \subset \mathbb{R}$ of all the ends of the bars in $\mathcal{B}_P$.
- For $\epsilon \in [0, \infty)$, let $\mathcal{B}_P^\epsilon$ be the barcode obtained from $\mathcal{B}_P$ by removing all bars of length $\leq \epsilon$.

- Similarly, $\mathcal{B}_P^\infty$ is the barcode associated to the persistence module P after eliminating all finite bars.
- d_γ is given by:

$$d_\gamma(L, L') = \text{diam}[\sigma(\mathcal{B}_{HF(L, L')}^\infty)].$$

For $\epsilon > 0$ and F a fibre of D^*N consider

$$\Delta(L, F; \epsilon) = \#\mathcal{B}_{HF(L, F)}^\epsilon + \text{diam}[\sigma(\mathcal{B}_{HF(L, F)}^\epsilon)]$$

Corollary

- Let $S \subset \text{Lag}_{\text{ex}}^{cl}(D^*N)$ be such that:

$$\sup_{L \in S, F \pitchfork L} \#(F \cap L) < \infty \quad , \quad \sup_{L \in S} \text{Var}(f_L) < \infty .$$

Then S is pre-compact.

- Conversely, if $\exists \epsilon, F$ such that $\sup_{L \in S} \Delta(L, F; \epsilon) = \infty$, then S is not pre-compact.*

Some examples.

Point ii. implies non pre-compactness in many examples, as follows.

Pick $\phi \in \text{Ham}(M)$ such that ϕ^k gets complicated when $k \rightarrow \infty$ (in the sense of entropy, for instance).

One can then expect following work of Cineli-Ginzburg-Gurel (2020) and later work of theirs and others:

$$\limsup_{k \rightarrow \infty} \# \mathcal{B}_{HF}^{\epsilon}(\phi^k(L), F) = \infty$$

for some well chosen L and F and, if so,

$$S = \{\phi^k(L)\}_{k \in \mathbb{N}} \subset \text{Lag}_{\text{ex}}^{cl}(D^*N)$$

is not pre-compact. Examples of this type:

- a. Hamiltonian flows based on the geodesic flows for hyperbolic manifolds N , ABC (2026).

- b. Local examples based on suspensions of Smale horseshoes based on the work of Meiwes (2024) (the relation to complexity is an observation in the current work).
- c. Other examples are due to Guillermou-Viterbo-Zhang (2026).

In all these examples $c(\phi^n(L); \infty)(1, 1)$ is constant. However, in the first two cases (and most likely also in the third) we have, for small enough ϵ :

$$0 < \limsup_{n \rightarrow \infty} \frac{1}{n} \log(\#\mathcal{B}_{HF(\phi^n(L), F)}^\epsilon) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log c(\phi^n(L); \epsilon)(1, 1) \leq h_{top}(\phi)$$

From the local examples at b. above it immediately follows:

Corollary

$(Lag_{ex}^{cl}(D^*N), d_\gamma)$ is locally non pre-compact.

II. Definition of the complexity

Background on triangulated persistence categories, BCZ (2024).

- A persistence category $\mathcal{P}$ is a category enriched in persistence modules. Specifically, $\text{hom}(A, B)$ is a persistence module (meaning a family $(\text{hom}^\alpha(A, B))_{\alpha \in \mathbb{R}}$, and maps $i_{\alpha, \beta} : \text{hom}^\alpha(A, B) \rightarrow \text{hom}^\beta(A, B)$ for $\alpha \leq \beta$). These maps satisfy the usual persistence axioms and are compatible with composition.
- There are two associated categories with the same objects: the 0-level category $\mathcal{P}^0$ with morphisms $\text{hom}^0(\cdot, \cdot)$, and the ∞ -level category $\mathcal{P}^\infty$ with morphisms $\text{hom}^\infty(\cdot, \cdot) = \varinjlim \text{hom}^\alpha(\cdot, \cdot)$.

- Given a persistence category $\mathcal{P}$, there is a pseudo-metric (called interleaving) on $Ob(\mathcal{P})$. It is defined as:

$$d_{int}(X, Y) = \inf\{r \geq 0 \mid \exists \phi \in \text{hom}^r(X, Y), \psi \in \text{hom}^r(Y, X) \\ \psi \circ \phi = i_{0,2r}(id_X), \phi \circ \psi = i_{0,2r}(id_Y)\}.$$

- A triangulated persistence category (TPC) $\mathcal{C}$ is a persistence category such that $\mathcal{C}^0$ is *triangulated*. Additionally, there are exact shift functors Σ^s satisfying

$$\text{hom}^\alpha(\Sigma^r A, \Sigma^s B) \cong \text{hom}^{\alpha-s+r}(A, B) .$$

These isomorphisms are induced by natural transformations $\eta_{r,s} : \Sigma^r \rightarrow \Sigma^s$ that are part of the structure.

- If $\mathcal{D}$ is a triangulated category and $\mathcal{C}$ is a TPC such that $\mathcal{D} \cong \mathcal{C}^\infty$, we say that $\mathcal{C}$ is a TPC refinement of $\mathcal{D}$.

Background on categorical approximability (ABC '26)

A metric space (Z, d) is ϵ -approximable in a TPC $\mathcal{C}$ if $Z \subset \text{Ob}(\mathcal{C})$ and there exists a *finite* family $\mathcal{F}_\epsilon \subset \text{Ob}(\mathcal{C})$ such that, for all $A \in Z$, there exists $C_A \in \text{Ob}(\mathcal{C})$ with two properties:

- i. $d_{\text{int}}(A, C_A) < \epsilon$
- ii C_A fits in a sequence Δ of iterated exact triangles in $\mathcal{C}_0$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_1 & \longrightarrow & C_2 & \longrightarrow & \cdots C_{n-1} \longrightarrow C_n = C_A \\ & \nwarrow & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow \\ & & \Sigma^{s_0} T^{l_0} F_{i_0} & & \Sigma^{s_1} T^{l_1} F_{i_1} & & \Sigma^{s_{n-1}} T^{l_{n-1}} F_{i_{n-1}} \end{array}$$

where T is the translation in $\mathcal{C}_0$, $l_i \in \mathbb{Z}$, $s_i \in \mathbb{R}$, and $F_i \in \mathcal{F}_\epsilon$.

If for each $\epsilon > 0$, (Z, d) is ϵ -approximable in some $\mathcal{C}_\epsilon$ we say that it is *approximable* in the (family) $\bar{\mathcal{C}} = \{\mathcal{C}_\epsilon\}$ provided that, in addition:

- iii. For $\epsilon' \leq \epsilon$ there is a TPC functor $\mathcal{C}_\epsilon \rightarrow \mathcal{C}_{\epsilon'}$ that maps $\mathcal{F}_\epsilon$ to $\mathcal{F}_{\epsilon'}$.

Remark

We skip here the subtleties of dealing with a family $\bar{\mathcal{C}} = \{\mathcal{C}_\epsilon\}$ instead of a single TPC. We write $\mathcal{C}$ instead of any of the $\mathcal{C}_\epsilon$ and assume $\mathcal{F}_\epsilon \subset \mathcal{F}_{\epsilon'}$ whenever $\epsilon' \leq \epsilon$.

For a sequence of exact triangles Δ as at ii. above we set

$$P(\Delta)(X, Y) = \sum_{i=0}^{n-1} X^{s_i} Y^{l_i} \in \Lambda_0[Y]$$

(X controls the “action”; Y the “degree”).

Definition (of complexity)

Assume (Z, d) is approximable in $\mathcal{C}$ with ϵ -approximating families $\mathcal{F}_\epsilon$. For $A \in Z$, and $X, Y \in [1, \infty)$, let:

$$c(A; \epsilon)(X, Y) = \inf \{ P(\Delta)(X, Y) \mid \Delta \text{ ending with } C_A \text{ as before,} \\ \text{with } s_i, l_i \geq 0, \text{ and } d_{int}(\Sigma^s T^l A, C_A) < \epsilon \text{ for some } s, l \}.$$

Approximability implies that $c(-; -)$ is well defined.

Remark

- $c(A; \epsilon)(1, 1) \in \mathbb{N}$ equals the minimal number of triangles needed to get ϵ -close to A using shifts and translations of elements in $\mathcal{F}_\epsilon$.
- The condition $s_i, l_i \geq 0$ implies $c(A; \epsilon)(X, Y)$ is monotone increasing in X as well as in Y .
- The coherence condition iii. relating the $\mathcal{C}_\epsilon$ implies that $c(A; \epsilon)(X, Y)$ is decreasing in ϵ .

The geometric example of interest here:

For each $\epsilon > 0$, the space $(\text{Lag}_{\text{ex}}^{cl}(D^*N), d_\gamma)$ is ϵ -approximable in a TPC

$$\mathcal{C}_\epsilon = \mathcal{P}_\epsilon \text{DFuk}(\text{Lag}_{\text{ex}}^{cl}(D^*N))$$

which is an enlargement of a refinement of $\text{DFuk}(\text{Lag}_{\text{ex}}^{cl}(D^*N))$ (ABC, 2026).

- The ϵ -approximating families $\mathcal{F}_\epsilon$ consist of fibers of D^*N .

- The objects of $\mathcal{C}_\epsilon$ are modules over the filtered A_∞ -category $Fuk^{fil}(\text{Lag}_{ex}^{cl}(D^*N))$ with objects the closed, exact, graded Lagrangians in D^*N .
- All Yoneda modules of these Lagrangians are objects in $\mathcal{C}_\epsilon$. Additionally, each one of the fibers $F_x \in \mathcal{F}_\epsilon$ is identified with an A_∞ -module, $\bar{F}_x$, (over $Fuk^{fil}(\text{Lag}_{ex}^{cl}(D^*N))$) and is viewed as an object of $\mathcal{C}_\epsilon$.
- From a technical standpoint, the filtered A_∞ -category $Fuk^{fil}(\text{Lag}_{ex}^{cl}(D^*N))$ is set up using Ambrosioni's work (2024).

In summary: *for $L \in \text{Lag}_{ex}^{cl}(D^*N)$, we have a well defined complexity $c(L; \epsilon)(X, Y) \in [0, \infty)$ for each $\epsilon \geq 0$, $X, Y \in [1, \infty)$ that satisfies the properties mentioned earlier.*

III. Proof of the main theorem.

Here it is the main statement again:

Theorem (restated)

We fix the base ring to be $\mathbb{Z}_2$ and $M = D^*N$. In this setting $S \subset \text{Lag}_{\text{ex}}^{cl}(D^*N)$ is pre-compact if and only if for all $\epsilon > 0$ and $X, Y \in [1, \infty)$, we have

$$\sup_{L \in S} c(L; \epsilon)(X, Y) < \infty .$$

To explain the argument we fix a TPC refinement of $DFuk(\text{Lag}_{\text{ex}}^{cl}(D^*N))$

$$\mathcal{C} = \mathcal{PDFuk}(\text{Lag}_{\text{ex}}^{cl}(D^*N))$$

such that the metric space $(Z, d) = (\text{Lag}_{\text{ex}}^{cl}(D^*N), d_\gamma)$ is ϵ -approximable in $\mathcal{C}$ with approximating family $\mathcal{F}_\epsilon$, for each $\epsilon > 0$.

Remark

We neglect here that, in reality, for each $\epsilon > 0$ we have a different category $\mathcal{C}_\epsilon$ (also dependent on some choices of perturbations !).

The simple implication.

If the complexity $c(s; \epsilon)(X, Y)$ is unbounded over $s \in S$ for some fixed $\epsilon > 0$ and $X, Y \in [1, \infty)$, then S is not pre-compact by a simple argument from ABC (2026) as follows.

Suppose that S is pre-compact. Then, for each $\epsilon' > 0$, there is a finite ϵ' -net, $\mathcal{A}(\epsilon') \subset \text{Ob}(\mathcal{C})$ for S : for each $s \in S$ there is $A_s \in \mathcal{A}(\epsilon')$ such that $d_{\text{int}}(s, A_s) < \epsilon'$.

For $\delta > 0$ let $K(\epsilon'; \delta)(X, Y) = \max_{A \in \mathcal{A}(\epsilon')} c(A; \delta)(X, Y)$.

Therefore, $\forall s \in S, \delta, \epsilon' > 0$:

$$c(s; \delta + \epsilon') \leq c(A_s; \delta) \leq K(\epsilon'; \delta) < \infty .$$

We take $\epsilon' + \delta \leq \epsilon$ and thus for all s ,
 $K(\epsilon', \delta) > c(s; \delta + \epsilon') \geq c(s; \epsilon)$ which contradicts the assumption.

The hard implication.

Preparatory remarks.

- In $\mathcal{C} = \mathcal{PDFuk}(\text{Lag}_{\text{ex}}^{cl}(D^*N))$ the objects are filtered A_∞ modules over $Fuk^{fil}(\text{Lag}_{\text{ex}}^{cl}(D^*N))$ together with some additional modules A_∞ -module, $\bar{F}_x$, representing the fibers $F_x \in \mathcal{F}_\epsilon$.
- The modules $\bar{F}_x$ are the pull-back of Yoneda modules of certain Lagrangian spheres S_x that are included in the total space of a Lefschetz fibration $\pi : E_\epsilon \rightarrow \mathbb{C}$ such that $D^*N \subset E_\epsilon$.

- These Lefschetz fibrations (depending on ϵ but this aspect will be omitted here) are used to prove approximability by an argument using ideas of Fukaya-Seidel-Smith (2008), some cobordism work of BC (2017), and a recent construction due to Giroux (2025).
- This argument implies that the category $\mathcal{C}$ can be taken small enough such that the morphisms between $\bar{F}_x$ and $\bar{F}_y$ are identified with $CF(S_x, S_y)$. In particular, these are finite dimensional vector spaces.
- In summary, because our coefficients are in $\mathbb{Z}_2$, $\text{hom}(\bar{F}_x, \bar{F}_y)$ is a finite set for all pairs of elements F_x and F_y of the approximating families $\mathcal{F}_\epsilon$.

Assume that S is not pre-compact. Then there exists $\delta > 0$ and a sequence $A_i \in S$ such that $d_\gamma(A_i, A_j) \geq \delta$ whenever $i \neq j$.

Fix $X, Y > 1$, and $\epsilon > 0$. By hypothesis, there is some K such that for all $A \in S$ we have $c(A; \epsilon)(X, Y) \leq K$.

The definition of the complexity implies that there exists an iterated sequence of triangles Δ_A :

$$\begin{array}{ccccccc}
 0 & \longrightarrow & C_1 & \longrightarrow & C_2 & \longrightarrow & \cdots C_{n-1} \longrightarrow C_n = C_A \\
 & \nwarrow & \downarrow & \nwarrow & \downarrow & & \nwarrow \downarrow \\
 & & \Sigma^{s_0} T^{l_0} F_{i_0} & & \Sigma^{s_1} T^{l_1} F_{i_1} & & \Sigma^{s_{n-1}} T^{l_{n-1}} F_{i_{n-1}}
 \end{array}$$

with $d_\gamma(\Sigma^h T^l A, C_A) < \epsilon$, $s_i, l_i, h, l \geq 0$, and such that for a small δ we have

$$\sum_{0}^{n-1} X^{s_i} Y^{l_i} \leq K + \delta .$$

Given that $X, Y > 1$ we deduce from

$$\sum_0^{n-1} X^{s_i} Y^{l_i} \leq K + \delta .$$

that the total number of terms in the iterated sequence is bounded, for all $A \in S$, by some $N \in \mathbb{N}$.

Similarly, the maximal power s_i appearing above, is bounded by some $B > 0$, for each $A \in S$. As well, the maximal power l_i , for all $A \in S$, is bounded by $L > 0$.

For an element $A \in S$, denote by $\bar{s}_A$ the vector $(s_i) \in [0, B]^N$ and by $\bar{l}_A$ the vector $(l_i) \in [0, L]^N$.

Three crucial remarks are next:

- Because $\text{hom}(\bar{F}_x, \bar{F}_y)$ is finite $\forall x, y$ there are only finitely many iterated cones of length $< N$ that can be built out of the elements in the family $\mathcal{F}_\epsilon$, if filtrations and the grading are omitted.
- Coming back to our sequence $\{A_i\}$, by passing to a subsequence, we may assume that all the iterated triangles Δ_{A_i} are the same, when forgetting all filtration and grading data.
- By again passing to a subsequence, we may also assume that

$$(\bar{s}_{A_i}, \bar{l}_{A_i}) \in [0, B)^N \times [0, L)^N$$

is a Cauchy sequence.

The last point implies that for i, j large enough we have $d_{int}(C_{A_i}, C_{A_j}) \leq \epsilon$.

From the triangle inequality, we deduce $d_\gamma(A_i, A_j) \leq 3\epsilon$. It remains to pick $\epsilon < \frac{\delta}{3}$ to end the proof. $\square$

Remark

- a. The implication: *bounded complexity* $\Rightarrow$ *pre-compactness* is reminiscent of the Arzela-Ascoli theorem.
- b. There is a less restrictive notion of complexity,

$$c^r(L; \epsilon)(X, Y)$$

where r stands for retract. This is defined just as c except that $d_{int}(L, C_L) < \epsilon$ is replaced by $\exists G$ such that $d_{int}(L \oplus G, C_L) < \epsilon$.

- c. $c^r(-)$ is easier to estimate compared to $c(-)$ (using split generation, rather than actual generation) just as in topology the LS-category of X , that can be defined as the minimal cone-length of a wedge $X \vee G$, is easier to calculate than cone-length itself. However, c^r can not replace c in the main statement.
- d. The difference between d_γ and d_{Hofer} is an interesting topic. It is very likely that there are many examples of d_γ - pre-compact sets that are not pre-compact in the Hofer metric.