Fukaya category for Landau-Ginzburg Orbifolds and Berglund-Hübsch homological mirror symmetry for curve singularities

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Motivation/Application

Baby example (already non-trivial): $x^2 + y^2$

Symplictic Geometry

Previous works on BH HMS
- Seidel
- Takahashi-Saito-Kajiura
- Ueda-Futaki
- Lekili-Ueda
- Harbermann-Smith, ...

Complex algebraic geometry: Matrix factorizations.

$x^2 + y^2 = (x+iy)(x-iy) \quad \rightarrow \quad 1 \times 1 \text{ matrix factorization}$

$\begin{pmatrix} 0 & x+iy \\ x-iy & 0 \end{pmatrix} \begin{pmatrix} 0 & x-iy \\ x+iy & 0 \end{pmatrix} = \begin{pmatrix} x^2+y^2 & 0 \\ 0 & x^2+y^2 \end{pmatrix} \cdot \text{Id}_{2 \times 2} = \begin{pmatrix} y \times x \\ x \times -y \end{pmatrix} \cdot \begin{pmatrix} y \times x \\ x \times -y \end{pmatrix}$
Add diagonal symmetry group: \[ W = x^2 + y^2 \quad G_w = \mathbb{Z}_2 \times \mathbb{Z}_2 \text{ on each variables} \]

Plan: "Fukaya \((W, G_w)\)" → MF\((W)\)

1. Take the Milnor fiber and its quotient.
   \[ x^2 + y^2 = \frac{1}{G_w} \]

2. Take Seidel Lagrangian in the orbi-sphere (immersed Lagrangian representing the skeleton)

3. Look at lifts of Seidel Lagrangian on the Milnor fiber.
   And count polygons with X,Y,Z corners through a generic point.

\[ W_{\text{L}} = x^2 + y^2 + xyz \]
(4) Set $z = 0$ to obtain the dual polynomial

\[ x^2 + y^2 \xrightarrow{z=0} x^2 + y^2 + xyz = W_{\mathcal{L}} \]

(5) Apply localized mirror functor (w/ Hong-Lau) to prove HMS for Milnor fiber

\[
\text{Wrapped Fukaya } \left( \frac{1}{x^2 + y^2 = 1} \right) \xrightarrow{\text{Canonical } A_{\infty}-\text{functor}} \text{MF}(W_{\mathcal{L}})
\]

\[
\text{G}_w\text{-equivariant } - \text{WF}(x^2 + y^2 = 1)
\]

Any Lagrangian $K$ decorated Floer Complex

\[
\text{CW}_{\text{even}}(K, \mathbb{L}) \leftrightarrow \text{CW}_{\text{odd}}(K, \mathbb{L})
\]

\[
m_1^{a,b} \text{ counts } \\
\text{x, y, z corners allowed on } \mathbb{L}
\]

Consider test Lagrangian $K$

\[
(y, x + yz) \cdot (y, x + yz) = (x^2 + y^2 + xyz) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
Thm 1. (1)~(6) works for invertible curve singularities: \( \text{WF}(\mathcal{W}^\circ(\Gamma\backslash G_w)) \to \text{MF}(\mathcal{W}^\circ) \)

(Fermat) \( x^p + y^q = 1 \)

\[ G_w \cong \mathbb{Z}/p \times \mathbb{Z}/q \]

(Chain) \( x^p + xy^q = 1 \)

\[ G_w \cong \mathbb{Z}/p^2 \times \mathbb{Z}/q^2 \]

\[ G_w \cong \mathbb{Z}/(p^2q) \]

\[ G_w \cong \mathbb{Z}/(pq-1) \]

(Loop) \( x^p y + xy^q = 1 \)

\[ G_w \cong \mathbb{Z}/(pq-1) \]

The localized mirror functor as \( \mathbb{Z}/2 \)-graded \( A_{\infty} \)-cat.

\[ \text{WF}(x^2 + y^2 + x y z) \rightarrow \text{MF}(x^2 + y^2 ) \]

\[ \text{MF}(x y z) \]

\[ (x-y)(x-y) \]
Aim:

1. Define the new category using monodromy information.
2. Show that this operation is mirror to the restriction to $z=g(x,y)$ hypersurface.
3. Prove HMS for invertible curve singularities.

Variation operator:

$\text{var}: \, H_i(M,\partial M) \rightarrow H_i(M)$

$[K] \rightarrow (\text{id} - \phi_k)[K] = V_K$
In the quotient, we get a Reeb orbit \( \Gamma \) corresponding to the monodromy.

Quantum cap action by monodromy orbit on wrapped Fukaya category

\[
\begin{array}{ccc}
\text{in} & \Rightarrow & \text{out} \\
\end{array}
\]

\[
WF(X)_{\Delta} \overset{\Lambda\Gamma}{\longrightarrow} WF(X)_{\Delta}
\]

- A\(_{\infty}\)-bimodule map over \( WF(X) \)
- \( WF(X)_{\Delta} \): diagonal bimodule

In our baby example,

Cap action by \( \Gamma \sim \) multiplication by \( \alpha + \beta \)

on \( CW (K,K) \)

Recall that in Lagrangian Floer theory, cone of a morphism may be regarded as a surgery.

\[
\begin{array}{ccc}
\psi(K) & \overset{\Delta}{\longrightarrow} & K \\
\end{array}
\]

\[
\text{Cone}(K \overset{\alpha}{\longrightarrow} \psi(K)) \approx \xrightarrow{\sim} \xrightarrow{\sim}
\]

Analogue of variation operator: We take a cone of quantum cap action by \( \Gamma \)

\[
WF(X)_{\Delta} \overset{\Lambda\Gamma}{\longrightarrow} WF(X)_{\Delta} \rightarrow \mathcal{C}_\Gamma
\]

1. This gives only A\(_{\infty}\)-bimodule, not an A\(_{\infty}\)-category.
   We construct new A\(_{\infty}\)-operations on \( \mathcal{C}_\Gamma \)

2. Morphisms in \( \mathcal{C}_\Gamma \) are different from morphisms of category of such cones (it is roughly half of it)

3. The image of cap action vanishes, most of wrapped generators are killed to give finite dim. hom space.
Variation operator

**Monodromy around the origin (fixing boundary).**

\[ \phi : H^*_r(M,\partial M) \rightarrow H^*_r(M,\partial M) \]

Variation operator

\[ \text{var} = \phi_x - \text{id} \]

Lagrangian Floer theory

**Reeb orbit** \( \gamma \) of the quotient

\[ \gamma \]

(not fixing boundary)

**Quantum cap action**

\[ \gamma : WF(x) \rightarrow WF(x) \]

**New A-infinity category with distinguished exact triangle**

\[ E \gamma(x) \]

\[ \text{var} \]

\[ \text{var} \quad \text{var} \quad \text{var} \]

To define the new category, we first consider the manifold case.

**Thm 2.** Given a Louville manifold \( M \) and an element \( \varphi \in \mathcal{H}^0(M) \), there exist a new \( A_\infty \)-category \( \mathcal{C}_\gamma(M) \) with such a distinguished triangle as \( A_\infty \)-bimodules

**Construction**

- **Objects:** Same set of objects as \( WF(M) \) : (wrapped Fukaya category)

- **Morphisms:**

\[ \text{Hom}_{\mathcal{C}_\gamma}^\bullet(L_1, L_2) = \text{Hom}_{WF}^\bullet(L_1, L_2) \oplus \text{Hom}_{WF}^\bullet(L_1, L_2) \]

- **To define** \( A_\infty \)-operations:

we use popsicle structures in the domain, developed by Abouzaid-Seidel and Seidel

Geometric setting is very different, but algebraic structures as well as desired algebraic effects are similar to the recent work of Seidel (Lefschetz fibration 6)

Popsicles are discs with interior marked points lying on hyperbolic geodesics
- geodesic between $z_0$ and $\{z_{i_1}, \ldots, z_{i_k}\}$ boundary marking
- interior marked points $\{z^+_1, \ldots, z^+_j\}$
  - $z^+_j$ on $l_{ij}$
- use $z^+_j$ as an input for $\Gamma$
- only one interior marking

\[ \Rightarrow \text{Quantum Cap-action} D \]

Remark: $z^+_1$ was not used as an input in Abouzaid-Seidel, Seidel.

\begin{align*}
\text{Abouzaid-Seidel:} & \text{ subclosed one-form } \to \text{Wrapped Fukaya with linear Hamiltonian.} \\
\text{Seidel:} & \text{Continuation map to kill compact part or intersect } \\
& \text{CF}(L_0, L_1; H) \to \text{CF}(L_0, L_1; 0) \quad \text{fiber at } \infty.
\end{align*}

\[ M_n(a_1, a_2, \ldots, e^{\lambda_{i_1}}, a, \ldots, e^{\lambda_{i_k}}, \ldots, a_n) = M^a_n + M^b_n e \]

1. $M^a_n$:
   - put $\Gamma$-insertion for each $e^{\lambda_{i_j}}$
   - on a geodesic between $z_0$ and $z^+_j$
   - for all $i_1, \ldots, i_k$

2. $M^b_n$:
   - put $\Gamma$-insertion for each $e^{\lambda_{i_j}}$
   - for all $i_1, \ldots, i_k$ except one
Fukaya category for weighted homogeneous polynomial $W$ with diagonal symmetry group $G_W$:

$$W(t^{w_1}z_1 \cdots , t^{w_n}z_n) = t^hW(z_1, \ldots , z_n)$$

(Milnor). Characteristic homeomorphism for monodromy of WHS:

$$[z_1, \ldots , z_n] \mapsto \left( e^{\frac{2\pi i W_1}{h}}z_1, \ldots , e^{\frac{2\pi i W_n}{h}}z_n \right) \sim \text{fund. class of } [\text{Link}/G_W]$$

**Thm 3.** For a weighted homogeneous polynomial $W$, and its maximal diagonal symmetry group $G_W$, take the Milnor fiber quotient $[M_W/G_W] = X$.

Then (1) we can define the Monodromy Reeb orbit $\Gamma_W$ in $X$.

(2) New A-category $C^\Gamma_W(X)$ is well-defined.

**Def**

For any subgroup $G \subset G_W$,

$$\mathcal{C}(W, G) := C^\Gamma_W(X) \times G^T, \quad G^T := \text{Hom}(G_W/G, \mathbb{C}^\times)$$

Monodromy is mirror to the restricting hypersurface.

Kodaira-Spencer map

$$\text{Jac}(W_{\text{IL}}): \text{output } \gamma \text{ is multiple of unit } \mapsto \text{take coefficient}$$

- **Fermat**
  - $\Gamma_W = \gamma_3$
  - $\mathbb{Z}$

- **Chain**
  - $\Gamma_W = \gamma_1^p + \gamma_3$
  - $\mathbb{Z} - x^{p-1}$

- **Loop**
  - $\Gamma_W = \gamma_2 + \gamma_1^{p-1} + \gamma_2^{q-1}$
  - $\mathbb{Z} - x^{p-1} - y^{q-1}$
**Thm 4.** Let $W$ be an invertible curve singularity.

1. Cap action for monodromy orbit is mirror to the hypersurface restriction. (up to homotopy)

2. There exist an $A_\infty$-functor which proves Berglund-Hübsch HMS for curve singularities.

\[ \begin{array}{c}
WF( M_W / G_w ) \xrightarrow{F_{\mu}} MF( W_{\mu} ) \\
\downarrow \cong \quad \downarrow \text{multiplication by } \overline{\text{ks}(T_W)} = \mathbb{Z} - h(x,y) \\
WF( M_W / G_w ) \quad \longrightarrow \quad MF( W_{\mu} ) \\
\downarrow \\
\text{BH HMS} : \quad E_{T_W}( M_W / G_w ) \quad \longrightarrow \quad MF( W^t ) \\
\end{array} \]

**Proof**) Combine popsicles and localized mirror functor.

**Cor** BH HMS in general form: $F(M, G) \cong MF(W^t, G^t)$ for any $G < G_w$.

**Thm 5.** For an ADE curve singularity $W^t$,

1. we find all Lagrangians in $[W^t(0) / G_w]$ corresponding to indecomposable MF,

2. we realize Auslander-Reiten exact sequence for MF as Lagrangian surgeries.

We illustrate this for E7 singularity:
$E_7$ Milnor Fiber

$x^3 + xy^3 = 1$, genus 3 surface.
Indecomposable MF for $E_7$ singularity $x^3 + xy^3$

- Yuji Yoshino's textbook (1990):

*Maximal Cohen-Macaulay modules over Cohen-Macaulay rings*

\[ \begin{array}{ccccccccc}
A & \rightarrow & M_2 & \rightarrow & Y_2 & \rightarrow & Y_3 & \rightarrow & Y_1 & \rightarrow & M_1 & \rightarrow & R \\
B & \rightarrow & N_2 & \rightarrow & X_2 & \rightarrow & X_3 & \rightarrow & X_1 & \rightarrow & N_1 &
\end{array} \]

- $A$ (resp. $B$): $1 \times 1$ factorization $x \cdot (x^2 + y^3)$ (resp. $(x^2 + y^3) \cdot x)$.

- $C$ (resp. $D$): $2 \times 2$ MF $(\gamma, \delta)$ (resp. $(\delta, \gamma)$)

\[ \gamma = \begin{pmatrix} x^2 & xy \\ xy^2 & -x^2 \end{pmatrix}, \delta = \begin{pmatrix} x & y \\ y^2 & -x \end{pmatrix} \]

- $M_j$ (resp. $N_j$): $2 \times 2$ MF $(\phi_j, \psi_j)$ (resp. $(\psi_j, \phi_j)$) for $j = 1, 2$.

\[ \phi_1 = \begin{pmatrix} x & y \\ xy & -x^2 \end{pmatrix}, \psi_1 = \begin{pmatrix} x^2 & y \\ xy^2 & -x \end{pmatrix}, \phi_2 = \begin{pmatrix} x & y^2 \\ xy & -x \end{pmatrix}, \psi_2 = \begin{pmatrix} x^2 & y^2 \\ xy & -x \end{pmatrix} \]

- $X_j$ (resp. $Y_j$): $3 \times 3$ MF $(\xi_j, \eta_j)$ (resp. $(\eta_j, \xi_j)$) for $j = 1, 2$.

\[ \xi_1 = \begin{pmatrix} xy^2 & -x^2 & -xyy \\ xy & -y^2 & x^2 \\ x^2 & xy & xy^2 \end{pmatrix}, \eta_1 = \begin{pmatrix} y & 0 & x \\ -x & xy & 0 \\ 0 & -x & y \end{pmatrix} \]

\[ \xi_2 = \begin{pmatrix} x^2 & -y^2 & -x^2 y \\ xy & x & -y^2 \\ xy^2 & x & x^2 \end{pmatrix}, \eta_2 = \begin{pmatrix} x & 0 & y \\ -xy & x^2 & 0 \\ 0 & -xy & x \end{pmatrix} \]

- $X_3$ (resp. $Y_3$): $4 \times 4$ MF $(\xi_3, \eta_3)$ (resp. $(\eta_3, \xi_3)$).

\[ \xi_3 = \begin{pmatrix} \gamma & \epsilon \\ 0 & \delta \end{pmatrix}, \eta_3 = \begin{pmatrix} \delta & -\epsilon \\ 0 & \gamma \end{pmatrix} \text{ with } \epsilon = \begin{pmatrix} y^2 & 0 \\ 0 & y \end{pmatrix} \]

**Remark:** In fact, in Yoshino's book, $X_3, Y_3$ is defined with

\[ \epsilon = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} \]

We think it is a typo/mistake.
Lemma 0.1. The following AR exact sequences (and their AR transpose) for $E_7$ singularity can be realized as Lagrangian surgeries.

(0.1) \[ 0 \to A \to M_2 \to B \to 0 \]
(0.2) \[ 0 \to D \to Y_3 \to C \to 0 \]
(0.3) \[ 0 \to M_1 \to X_1 \to N_1 \to 0 \]
(0.4) \[ 0 \to M_2 \to B \oplus Y_2 \to N_2 \to 0 \]
(0.5) \[ 0 \to Y_1 \to M_1 \oplus X_3 \to X_1 \to 0 \]
(0.6) \[ 0 \to Y_2 \to N_2 \oplus Y_3 \to X_2 \to 0 \]
(0.7) \[ 0 \to Y_3 \to X_2 \oplus C \oplus Y_1 \to X_3 \to 0 \]
(A) Lagrangian surgery for the exact sequence (0.1)

(B) for the exact sequence (0.2)

(C) Lagrangian surgery for the exact sequence (0.3)

(D) for the exact sequence (0.4)

(E) Lagrangian surgery for the exact sequence (0.5)

(F) for the exact sequence (0.6)

(G) Lagrangian surgery for the exact sequence (0.7)