Fukaya category for Landau-Ginzburg Orbifolds and Berglund-Hübsch homological mirror symmetry for curve singularities

IAS/PU-Montreal-Paris-Tel-Aviv Symplectic Geometry Seminar

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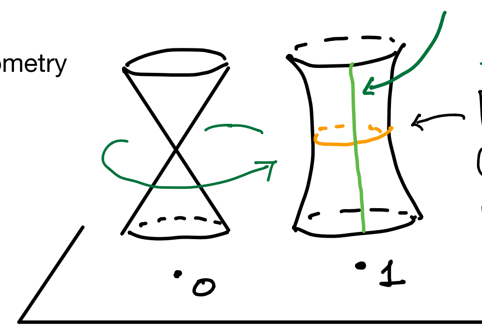
Motivation/Application

Baby example (already non-trivial) : $x^2 + y^2$

$\boxed{\text{Our plan}}$ Use this non-cpt. Lag. $\rightsquigarrow$ new Fukaya.Cat.
+ monodromy
+ symmetry group G

Symplectic Geometry

"HMS"



Fukaya-Seidel Category of Vanishing cycles $\Rightarrow$

Previous works on BH HMS
- Seidel
- Takahashi-Saito-Kajiura
- Ueda-Futaki
- Lekili-Ueda
- Harbermann-Smith, ...

$$FS^{\mathbb{Z}}(W) \cong MF^{L_W}(W^t)$$

Complex algebraic geometry : Matrix factorizations.

$x^2+y^2 = (x+iy)(x-iy) \longrightarrow$ 1 × 1 matrix factorization

or

$$\begin{pmatrix} 0 & x+iy \\ x-iy & 0 \end{pmatrix} \bullet \begin{pmatrix} 0 & x-iy \\ x+iy & 0 \end{pmatrix} = \begin{pmatrix} x^2+y^2 & 0 \\ 0 & x^2+y^2 \end{pmatrix} = (x^2+y^2)\cdot Id_{2\times 2}$$

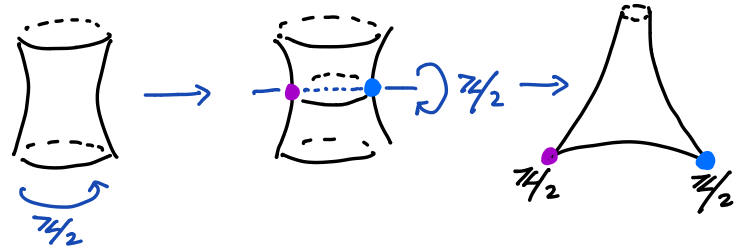
$\Leftrightarrow$ (basis change) $= \begin{pmatrix} y & x \\ x & -y \end{pmatrix} \bullet \begin{pmatrix} y & x \\ x & -y \end{pmatrix}$

Add diagonal symmetry group: $W = x^2 + y^2 \circlearrowleft G_W = \mathbb{Z}/2 \times \mathbb{Z}/2$ on each variables

Plan: "Fukaya (W, G_W)" $\longleftrightarrow$ MF(W)

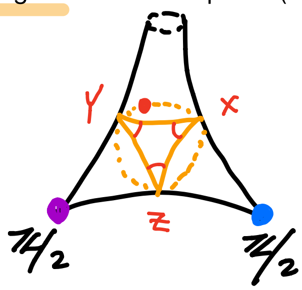
(1) Take the Milnor fiber and its quotient.

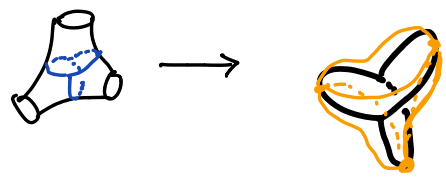
$x^2 + y^2 = 1 / G_W$



(2) Take Seidel Lagrangian in the orbi-sphere (immersed Lagrangian representing the skeleton)

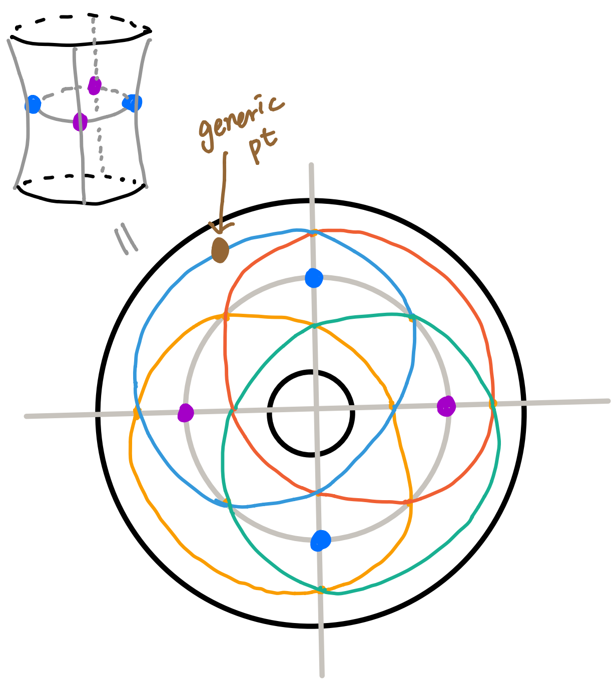
$\mathbb{L}$

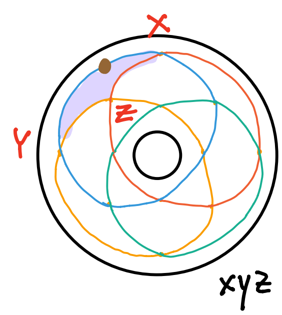


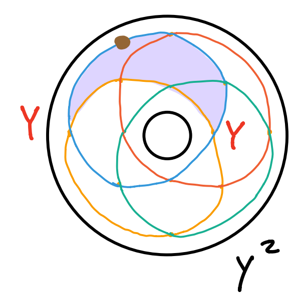


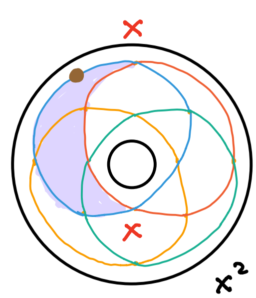
(3) Look at lifts of Seidel Lagrangian on the Milnor fiber.
And count polygons with X,Y,Z corners through a generic point.

$W_{\mathbb{L}} = x^2 + y^2 + xyz$







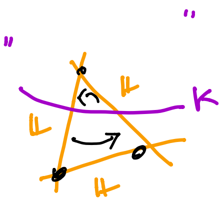


(4) Set z =0 to obtain the dual polynomial $x^2+y^2 \xleftarrow{z=0} x^2+y^2+xyz = W_{\mathbb{L}}$

(5) Apply localized mirror functor (w/ Hong-Lau) to prove HMS for Milnor fiber

$$\text{Wrapped Fukaya}\left(\{x^2+y^2=1\}/G_W\right) \xrightarrow[A_\infty\text{-functor}]{\text{Canonical}} MF(W_{\mathbb{L}})$$

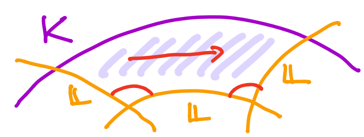
$$= \; G_W\text{-equivariant} - WF\left(x^2+y^2=1\right)$$



Any Lagrangian $K \xrightarrow{\mathcal{F}^{\mathbb{L}}}$ decorated Floer complex

$$CW^{even}(K,\mathbb{L}) \underset{-m_1^{0,b}}{\overset{-m_1^{0,b}}{\rightleftarrows}} CW^{odd}(K,\mathbb{L})$$

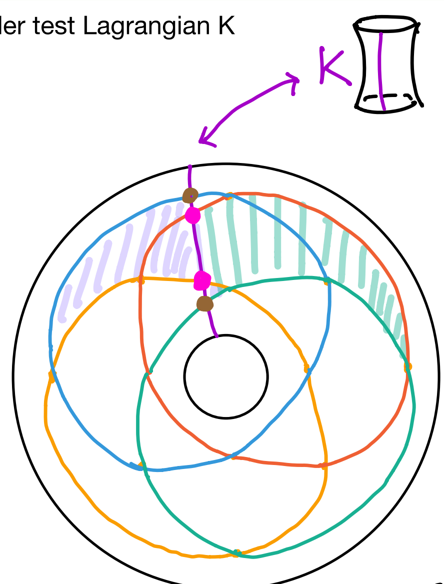
$m_1^{0,b}$:



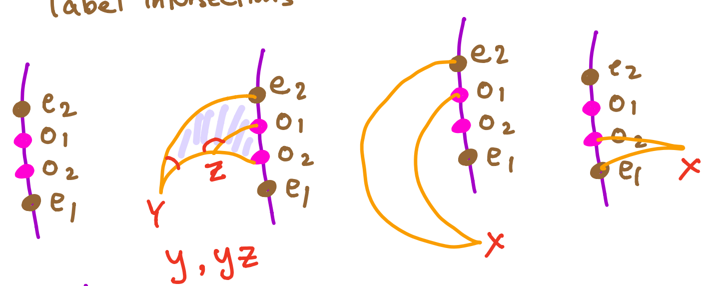
counts

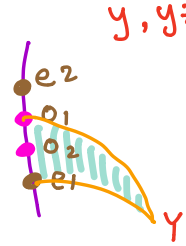
X, Y, Z corners allowed on $\mathbb{L}$

Consider test Lagrangian K



label intersections





$$\begin{array}{c|cc} & e_1 & e_2 \\ \hline o_1 & & \\ o_2 & & \end{array} \quad \begin{array}{cc} & \\ o_1 & \\ o_2 & \end{array}\begin{pmatrix} y & x+yz \\ x & -y \end{pmatrix}$$

sign rule + nontrivial spin

Same for odd → even

$$\Rightarrow \begin{pmatrix} y & x+yz \\ x & -y \end{pmatrix} \cdot \begin{pmatrix} y & x+yz \\ x & -y \end{pmatrix} = (x^2+y^2+xyz)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(6) Set z=0 to the localized mirror functor

$$\begin{tikzcd}
WF\left(x^2+y^2=1 / G_W\right) \arrow[r, "F^{\mathbb{L}}", "\cong"'] \arrow[dr, "A_\infty\text{-functor}"'] & MF(x^2+y^2+xyz) \arrow[d, "z=0"] \\
& MF(x^2+y^2)
\end{tikzcd}$$

as $\mathbb{Z}/2$-graded A_∞-cat.

$$WF \ni K \longmapsto \begin{pmatrix} y & x \\ x & -y \end{pmatrix} \cdot \begin{pmatrix} y & x \\ x & -y \end{pmatrix}$$

(c.f. Lekili-Ueda)

$\left[WF^{\mathbb{Z}}(W^{-1}(1)) \cong MF^{L_{W^t}}(W^t + x_0 \cdots x_n) \right.$ for ADE (general dim) using different ideas. $\left. \right]$

$1 \to \mathbb{C}^* \to L_{W^t} \to G_{W^t} \to 1$

Thm 1. (1)~(6) works for invertible curve singularities: $WF(W^{-1}(1)/G_W) \to MF(W^t)$

(Fermat) $t \circlearrowleft$ x^p+y^q

(Chain) $x^p + xy^q \ \cdots \begin{pmatrix} p & 0 \\ 1 & q \end{pmatrix}$

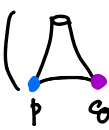
transpose

$x^py + y^q \ \cdots \begin{pmatrix} p & 1 \\ 0 & q \end{pmatrix}$

(Loop) $t \circlearrowleft$ $x^py + xy^q$

(Fermat) $x^p+y^q=1 / G_W$

$G_W = \mathbb{Z}/p \times \mathbb{Z}/q$

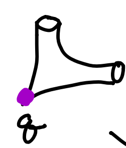
$WF\Big($  $\Big)$

$$\begin{tikzcd}
WF(\cdot) \arrow[r, "F^{\mathbb{L}}", "\cong"'] \arrow[dr] & MF(x^p+y^q+xyz) \arrow[d, "z=0"] \\
& MF(x^p+y^q)
\end{tikzcd}$$

(Chain) $x^p+xy^q=1 / G_W$

$G_W \cong \mathbb{Z}/pq$

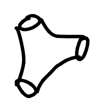
$= \{(\lambda,\mu) \mid \lambda^p=1, \lambda\mu^q=1\}$

$WF\Big($  $\Big)$

$$\begin{tikzcd}
WF(\cdot) \arrow[r, "F^{\mathbb{L}}", "\cong"'] \arrow[dr] & MF(y^q+xyz) \arrow[d, "z=x^{p-1}"] \\
& MF(y^q+yx^p)
\end{tikzcd}$$

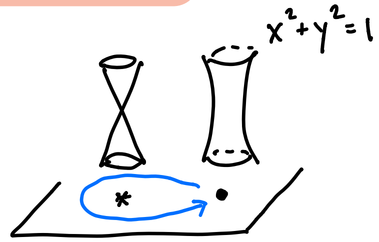
(Loop) $x^py+xy^q=1 / G_W$

$G_W \cong \mathbb{Z}/(pq-1)$

$WF\Big($  $\Big)$

$$\begin{tikzcd}
WF(\cdot) \arrow[r, "(AAEKO)", "\cong"'] \arrow[dr] & MF(xyz) \arrow[d, "z=x^{p-1}+y^{q-1}"] \\
& MF(x^py+xy^q)
\end{tikzcd}$$

Monodromy



Mirror of Milnor fiber quotient

$$
\begin{tikzcd}
WF\left(x^2+y^2=1 / G_W \right) \arrow[rr, "F^{\amalg}", "\cong"'] \arrow[d] \arrow[drr, "\ncong"] & & MF(x^2+y^2+xyz) \arrow[d, "z=0"] \\
\boxed{\mathcal{C}_\Gamma} \arrow[rr, "\cong"'] & & MF(x^2+y^2)
\end{tikzcd}
$$

Isolated singularity

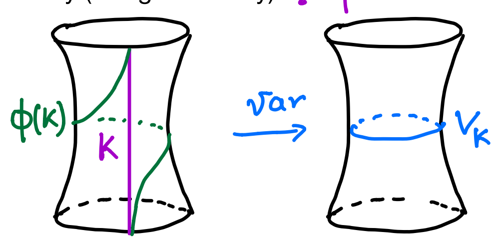
Aim:

(1) Define the new category using monodromy information

(2) Show that this operation is mirror to the restriction to z=g(x,y) hypersurface.

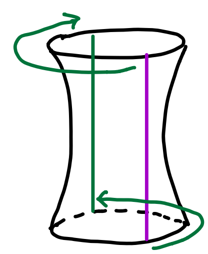
(3) Prove HMS for invertible curve singularities.

Monodromy (fixing boundary) : ϕ



untwist boundary

Monodromy (not fixing boundary) : ψ



Variation operator:

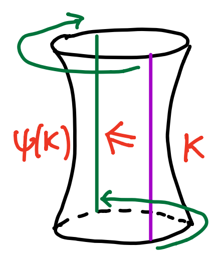
$$\mathrm{var} : H_1(M, \partial M) \longrightarrow H_1(M)$$

$$[K] \longrightarrow (\cancel{id} - \cancel{\phi_*})[K] = V_K$$

$$\phi_* - id$$

$\boxed{\text{Key}}$

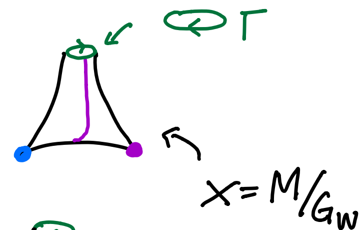
(1) $K \longrightarrow \psi(K)$ is part of G_W-action



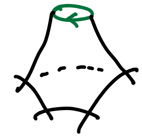
(2) ∂M is contact mfd.

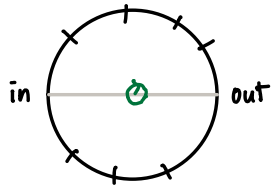
monodromy $\sim$ Reeb flow

In the quotient, we get a Reeb orbit Γ corresponding to the monodromy



Quantum cap action by monodromy orbit on wrapped Fukaya category

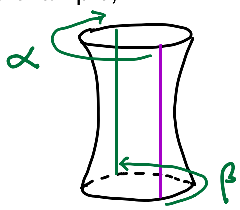




$$\Rightarrow \quad WF(X)_\Delta \xrightarrow{\cap\Gamma} WF(X)_\Delta$$

- A_∞-bimodule map over $WF(X)$
- $WF(X)_\Delta$: diagonal bimodule

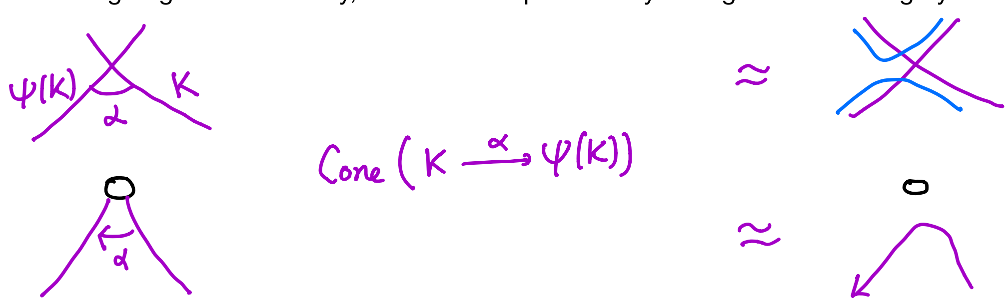
In our baby example,



Cap action by Γ ~ multiplication by $\alpha+\beta$

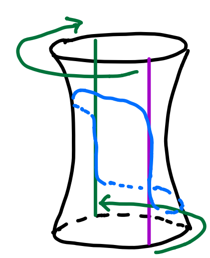
on CW (K,K)

Recall that in Lagrangian Floer theory, cone of a morphism may be regarded as a surgery.



Analogue of variation operator : We take a cone of quantum cap action by Γ

$$WF(X)_\Delta \xrightarrow{\cap\Gamma} WF(X)_\Delta \longrightarrow \mathcal{C}_\Gamma \rightarrow$$



(1) This gives only A_∞-bimodule, not an A_∞-category.
We construct new A_∞-operations on $\mathcal{C}_\Gamma$

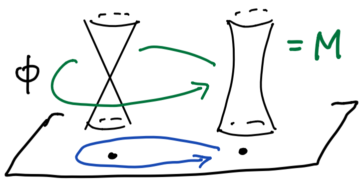
(2) Morphisms in $\mathcal{C}_\Gamma$ are different from morphisms of category of such cones (it is roughly half of it)

(3) The image of cap action vanishes, most of wrapped generators are killed to give finite dim. hom space.

Quick summary:

Variation operator

Monodromy around the origin(fixing boundary).



Monodromy homomorphism

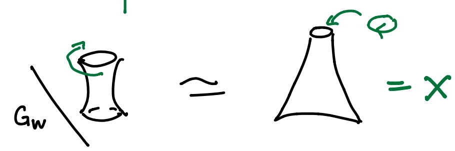
$$\phi_* : H_*(M, \partial M) \longrightarrow H_*(M, \partial M)$$

Variation operator

$$var = \phi_* - id$$

Lagrangian Floer theory

Reeb orbit Γ of the quotient



(not fixing boundary)

Quantum cap action

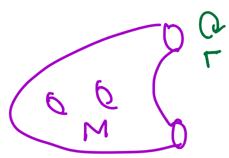
$$\cap\Gamma : WF(X) \longrightarrow WF(X)$$

New A-infinity category with distinguished exact triangle $\mathcal{C}_\Gamma(X)$

$$WF(X) \xrightarrow{\cap\Gamma} WF(X) \longrightarrow \mathcal{C}_\Gamma(X)$$

To define the new category, we first consider the manifold case.

Thm 2. Given a Louville manifold M and an element $\Gamma \in SH^0(M)$, there exist a new A_∞-category $\mathcal{C}_\Gamma(M)$ with such a distinguished triangle as A_∞-bimodules



$\longrightarrow \mathcal{C}_\Gamma(M)$

Construction

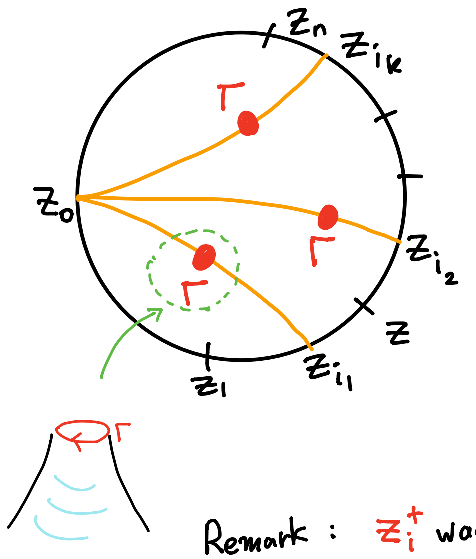
- Objects: Same set of objects as WF(M) : (wrapped Fukaya category)
- Morphisms: $Hom_{\mathcal{C}_\Gamma}(L_1, L_2) := Hom_{WF}(L_1, L_2) \oplus Hom_{WF}(L_1, L_2)\varepsilon$

  $\ni a + b\varepsilon$, ε: $deg = -1$
- To define A_∞-operations:

  we use popsicle structures in the domain, developed by Abouzaid-Seidel and Seidel

  Geometric setting is very different, but algebraic structures as well as desired algebraic effects are similar to the recent work of Seidel (Lefschetz fibration 6)

  Popsicles are discs with interior marked points lying on hyperbolic geodesics



- geodesic l between z_0 and $\{i_1, \dots, i_k\}$ bdy marking
- interior marked points $\{z_1^+, \dots, z_k^+\}$

  z_j^+ on l_{i_j}
- use z_j^+ as an input for Γ
- only one interior marking

  $\Rightarrow$ Quantum Cap-action!

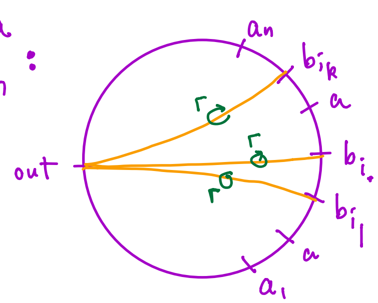
Remark: z_i^+ was not used as an input in Abouzaid-Seidel, Seidel.

Abouzaid-Seidel: subclosed one-form for continuation map $\rightsquigarrow$ Wrapped Fukaya with linear Hamilt.

Seidel: continuation map to kill compact part or intersect fiber at ∞.
$CF(L_0, L_1; -H) \longrightarrow CF(L_0, L_1; 0)$

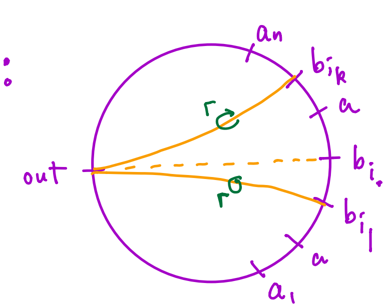
$$M_n(a_1, a_2, \cdots, \varepsilon b_{i_1}, a, \cdots, \varepsilon b_{i_k}, \cdots a_n) = M_n^a + M_n^b \varepsilon$$

① M_n^a:



put Γ-insertion for each $\underline{\underline{\varepsilon b_{i_j}}}$ on a geodesic between z_0 and z_{i_j} for all $i_1 \cdots i_k$

② M_n^b:



put Γ-insertion for each εb_{i_j} for all $i_1, \cdots, i_k$ $\underline{\text{except one}}$

$\square$

Fukaya category for weighted homogeneous polynomial W with diagonal symmetry group G_W

$$W(t^{w_1}z_1, \dots, t^{w_n}z_n) = t^h \cdot W(z_1, \dots, z_n)$$

(Milnor). Characteristic homeomorphism for monodromy of WHS : $\longrightarrow$ $\exists$ Monodromy orbit Γ_W ($\sim$ fund. class of $[Link/G_W]$)

$$(z_1, \dots, z_n) \longrightarrow \left(e^{\frac{2\pi i w_1}{h}} z_1, \dots, e^{\frac{2\pi i w_n}{h}} z_n\right)$$

Thm 3. For a weighted homogeneous polynomial W, and its maximal diagonal symmetry group G_W, take the Milnor fiber quotient $[M_W/G_W] =: X$

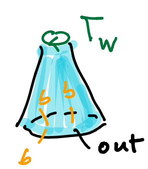
Then (1) we can define the Monodromy Reeb orbit Γ_W in X

(2) New A -category $\mathcal{C}_{\Gamma_W}(X)$ is well-defined

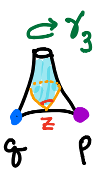
Def For any subgroup $G < G_W$ $\Rightarrow \mathcal{F}(W, G) := \mathcal{C}_{\Gamma_W}(X) \rtimes G^T$, $G^T := Hom(G_W/G, \mathbb{C}^*)$

Monodromy is mirror to the restricting hypersurface.

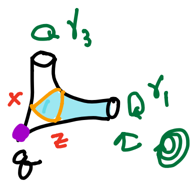
[Amorim-C-Hong-Lau], [Fukaya-Oh-Ohta-Ono],



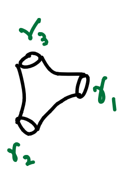
Monodromy for curve singularities $\xrightarrow{\text{Kodaira-Spencer map}}$ $Jac(W_{\amalg})$: output is multiple of unit $\Rightarrow$ take coefficient

Fermat 

$\Gamma_W = \gamma_3$ $\longrightarrow$ z

Chain 

$\Gamma_W = \gamma_1^{p-1} + \gamma_3$ $\longrightarrow$ $z - x^{p-1}$

Loop 

$\Gamma_W = \gamma_3 + \gamma_1^{p-1} + \gamma_2^{q-1}$ $\longrightarrow$ $z - x^{p-1} - y^{q-1}$

Thm 4. Let W be an invertible curve singularity.

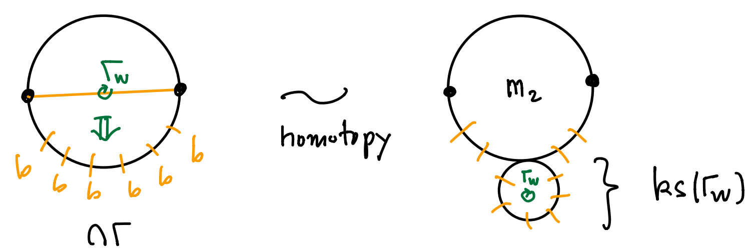
(1) Cap action for monodromy orbit is mirror to the hypersurface restriction. (up to homotopy)

(2) There exist an A_∞-functor which proves Berglund-Hübsch HMS for curve singularities.

$$\begin{tikzcd}
 & WF(M_W/G_W) \arrow[r, "\mathcal{F}^{\mathbb{L}}"] \arrow[d, "\cap \Gamma_W"] \arrow[dr, phantom, "\circlearrowright"] & MF(W_{\mathbb{L}}) \arrow[d, "\text{multiplication by } ks(\Gamma_W) = z - h(x,y)"] \\
 & WF(M_W/G_W) \arrow[r] \arrow[d] & MF(W_{\mathbb{L}}) \arrow[d] \\
\text{BH HMS :} & \mathcal{C}_{\Gamma_W}(M_W/G_W) \arrow[r, "\cong", "\exists\, \mathcal{G}^{\mathbb{L}} \ A_\infty\text{-functor}"'] & MF(W^t)
\end{tikzcd}$$

Proof) Combine popsicles and localized mirror functor.

(1)



$\boxed{\text{Cor}}$ BH HMS in general form holds: $\mathcal{F}(M, G) \cong MF(W^t, G^t)$ for any $G < G_W$.

$G^t := \mathrm{Hom}(G_W/G, \mathbb{C}^*)$

$MF(W^t, G^t)$ ← G^t-equiv. MF.

Thm 5. For an ADE curve singularity W^t,

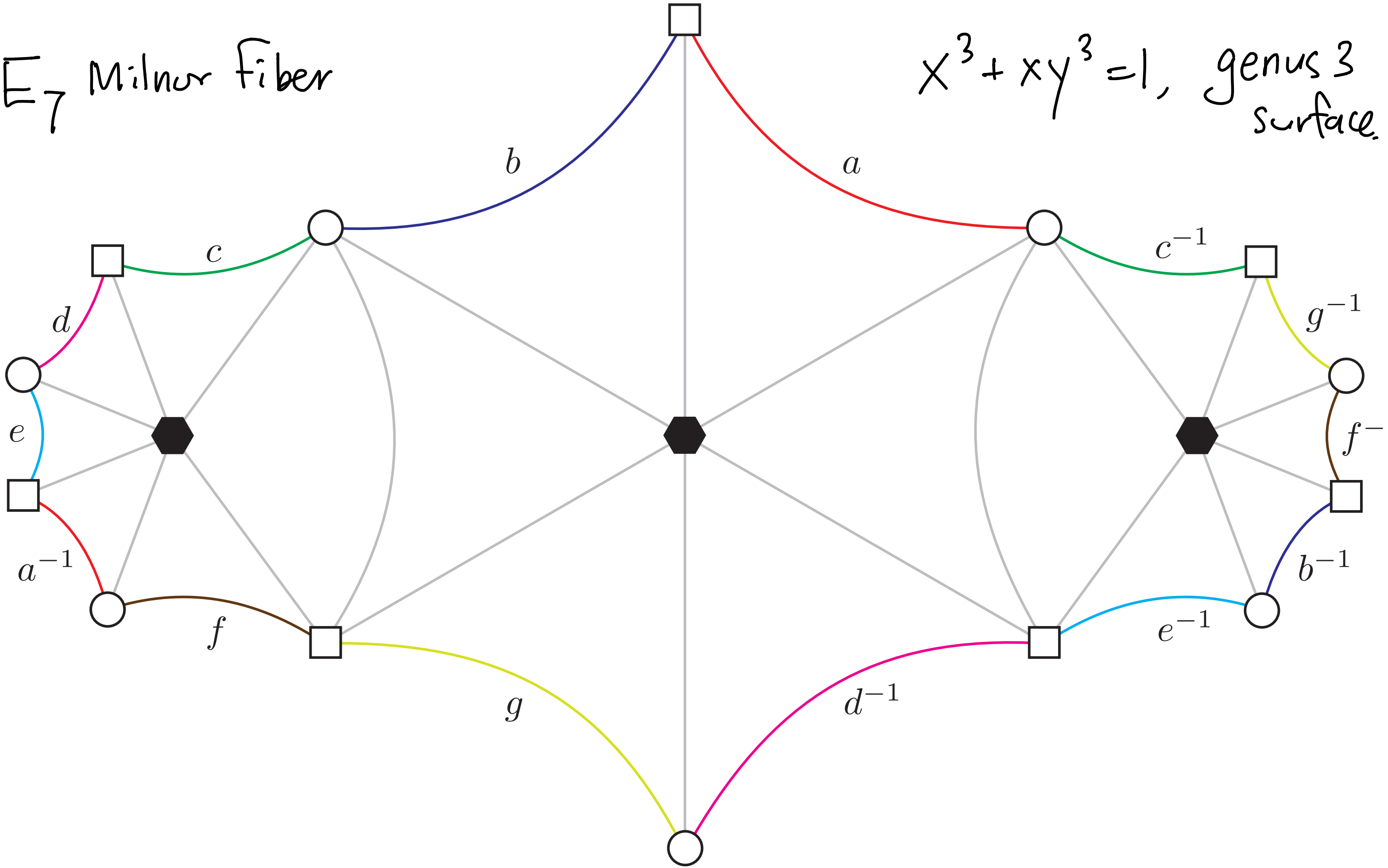
(1) we find all Lagrangians in $[W^{-1}(1)/G_W]$ corresponding to indecomposable MF,

(2) we realize Auslander-Reiten exact sequence for MF as Lagrangian surgeries.

We illustrate this for E7 singularity:

E_7 Milnor Fiber

$x^3 + xy^3 = 1$, genus 3 surface.



Indecomposable MF for E_7 singularity $x^3 + xy^3$

- Yuji Yoshino's textbook (1990):

Maximal Cohen-Macaulay modules over Cohen-Macaulay rings

$$
\begin{tikzcd}
 & & & C \arrow[r, dash, dashed] \arrow[ddd] & D \arrow[d] & & & \\
A \arrow[r] \arrow[dd, dash, dashed] & M_2 \arrow[r] \arrow[ddl] \arrow[dd, dash, dashed] & Y_2 \arrow[rr] \arrow[ddl] \arrow[dd, dash, dashed] & & Y_3 \arrow[r] \arrow[ddll] \arrow[ul] \arrow[ddl, dash, dashed] & Y_1 \arrow[r] \arrow[ddll] \arrow[dd, dash, dashed] & M_1 \arrow[ddl] \arrow[dd, dash, dashed] & \\
 & & & & & & & R \arrow[ul] \\
B \arrow[r] & N_2 \arrow[r] \arrow[uul] & X_2 \arrow[r] \arrow[uul] & X_3 \arrow[rr] \arrow[uul] \arrow[uuur] & & X_1 \arrow[r] \arrow[uul] & N_1 \arrow[uul] \arrow[ur] &
\end{tikzcd}
$$

- A (resp. B) : 1×1 factorization $x \bullet (x^2 + y^3)$ (resp. $(x^2 + y^3) \bullet x$).

- C (resp. D) : 2×2 MF (γ, δ) (resp. (δ, γ))

$$\gamma = \begin{pmatrix} x^2 & xy \\ xy^2 & -x^2 \end{pmatrix}, \delta = \begin{pmatrix} x & y \\ y^2 & -x \end{pmatrix}$$

- . M_j (resp. N_j) : 2×2 MF (ϕ_j, ψ_j) (resp. (ψ_j, ϕ_j)) for $j = 1, 2$.

$$\phi_1 = \begin{pmatrix} x & y \\ xy^2 & -x^2 \end{pmatrix}, \psi_1 = \begin{pmatrix} x^2 & y \\ xy^2 & -x \end{pmatrix}$$

$$\phi_2 = \begin{pmatrix} x & y^2 \\ xy & -x^2 \end{pmatrix}, \psi_2 = \begin{pmatrix} x^2 & y^2 \\ xy & -x \end{pmatrix}$$

- X_j (resp. Y_j): 3×3 MF (ξ_j, η_j) (resp. (η_j, ξ_j)) for $j = 1, 2$.

$$\xi_1 = \begin{pmatrix} xy^2 & -x^2 & -x^2 y \\ xy & -y^2 & x^2 \\ x^2 & xy & xy^2 \end{pmatrix}, \eta_1 = \begin{pmatrix} y & 0 & x \\ -x & xy & 0 \\ 0 & -x & y \end{pmatrix}$$

$$\xi_2 = \begin{pmatrix} x^2 & -y^2 & -xy \\ xy & x & -y^2 \\ xy^2 & xy & x^2 \end{pmatrix}, \eta_2 = \begin{pmatrix} x & 0 & y \\ -xy & x^2 & 0 \\ 0 & -xy & x \end{pmatrix}$$

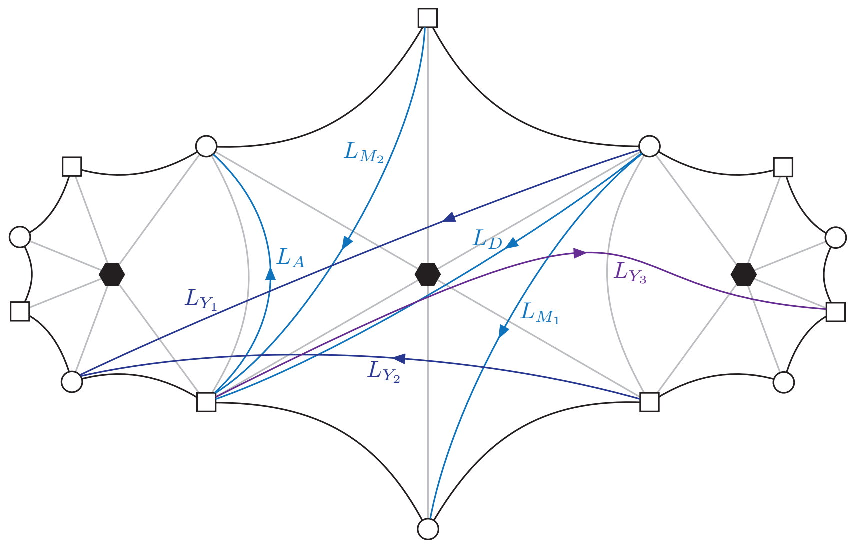
- X_3 (resp. Y_3) : 4×4 MF (ξ_3, η_3) (resp. (η_3, ξ_3)).

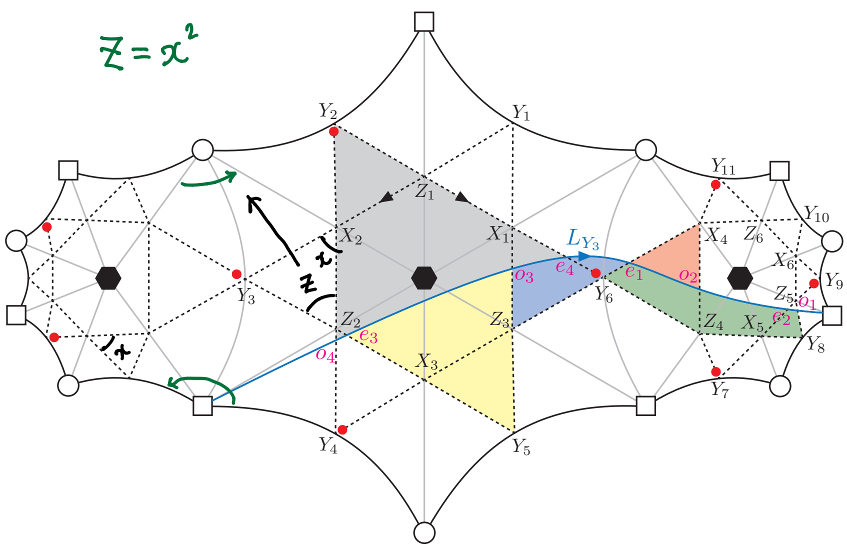
$$\xi_3 = \begin{pmatrix} \gamma & \epsilon \\ 0 & \delta \end{pmatrix}, \eta_3 = \begin{pmatrix} \delta & -\epsilon \\ 0 & \gamma \end{pmatrix} \text{ with } \epsilon = \begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix}.$$

Remark: In fact, in Yoshino's book, X_3, Y_3 is defined with

$$\epsilon = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}$$

We think it is a typo/mistake.





Lemma 0.1. *The following AR exact sequences (and their AR transpose) for E_7 singularity can be realized as Lagrangian surgeries.*

$$0 \to A \to M_2 \to B \to 0 \tag{0.1}$$

$$0 \to D \to Y_3 \to C \to 0 \tag{0.2}$$

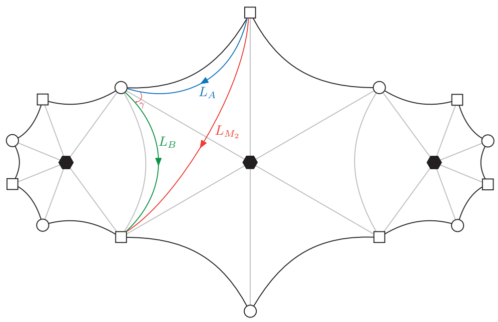
$$0 \to M_1 \to X_1 \to N_1 \to 0 \tag{0.3}$$

$$0 \to M_2 \to B \oplus Y_2 \to N_2 \to 0 \tag{0.4}$$

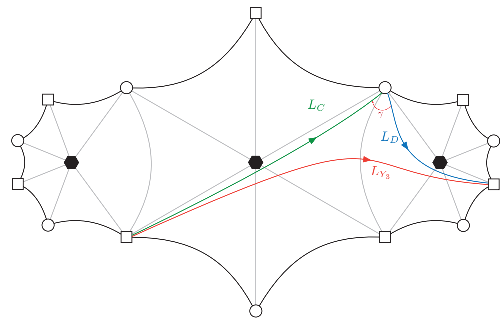
$$0 \to Y_1 \to M_1 \oplus X_3 \to X_1 \to 0 \tag{0.5}$$

$$0 \to Y_2 \to N_2 \oplus Y_3 \to X_2 \to 0 \tag{0.6}$$

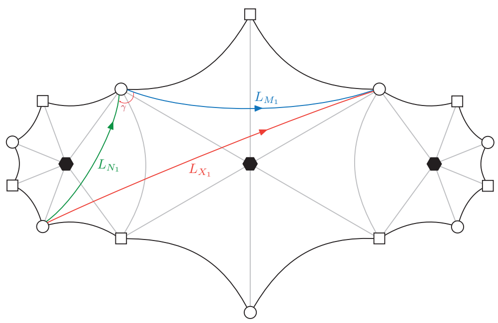
$$0 \to Y_3 \to X_2 \oplus C \oplus Y_1 \to X_3 \to 0 \tag{0.7}$$



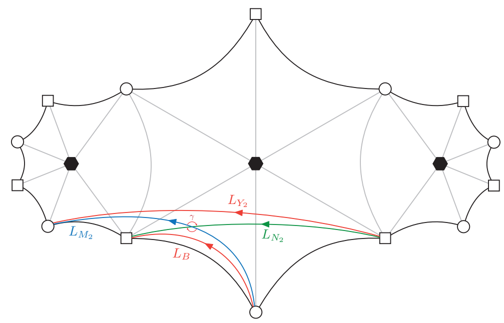
(A) Lagrangian surgery for the exact sequence (0.1)



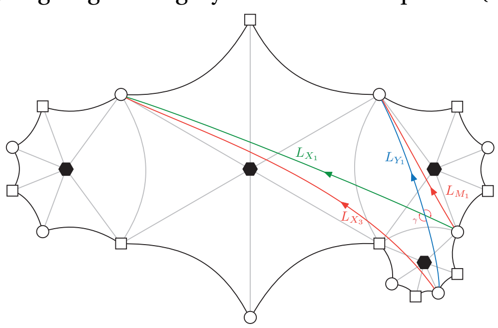
(B) for the exact sequence (0.2)



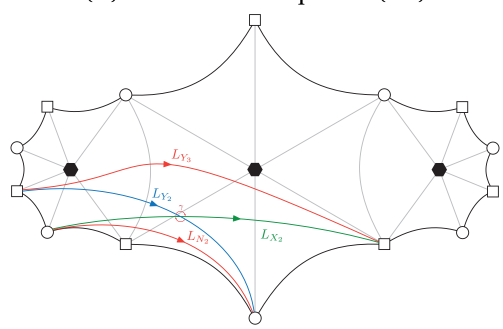
(C) Lagrangian surgery for the exact sequence (0.3)



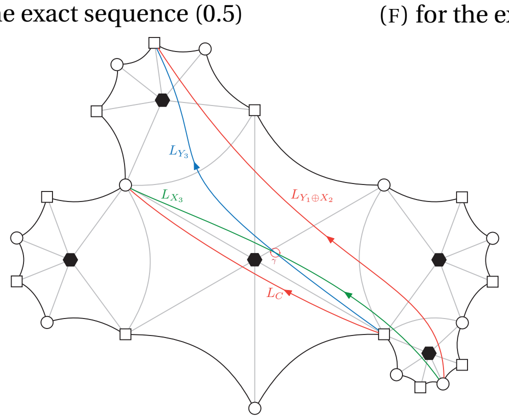
(D) for the exact sequence (0.4)



(E) Lagrangian surgery for the exact sequence (0.5)



(F) for the exact sequence (0.6)



(G) Lagrangian surgery for the exact sequence (0.7)