Real Lagrangian Tori in toric symplectic manifolds

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Symplectic Zoominar

Real Lagrangians

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Its fixed point set $\operatorname{Fix} \sigma$ is Lagrangian (whenever non-empty).

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If $L = \operatorname{Fix}\sigma$ and $\varphi \in \operatorname{Symp}(M, \omega)$ then $\varphi(L) = \operatorname{Fix}(\varphi\sigma\varphi^{-1})$. The notion is invariant under symplectomorphisms.

Real Lagrangians

Let $L \subset (M, \omega)$ be a Lagrangian.

Main Question:

Is L real?

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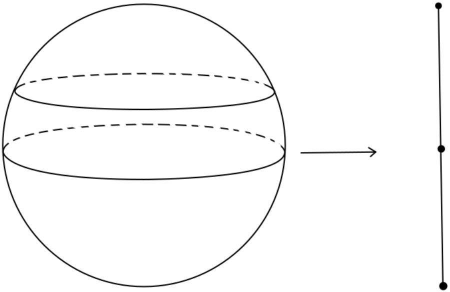
Examples

The equator in (S^2, ω) is real. Other circles of constant height are not real.



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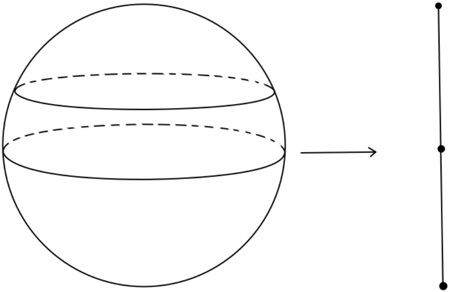

In general: If (M, ω) is monotone and L is real, then L is monotone.

Toric fibres

From now on: (M^{2n}, ω) toric monotone symplectic manifold, i.e. there is a moment map $\mu \colon M \to \mathbb{R}^n$ which generates an effective Hamiltonian T^n-action on M.

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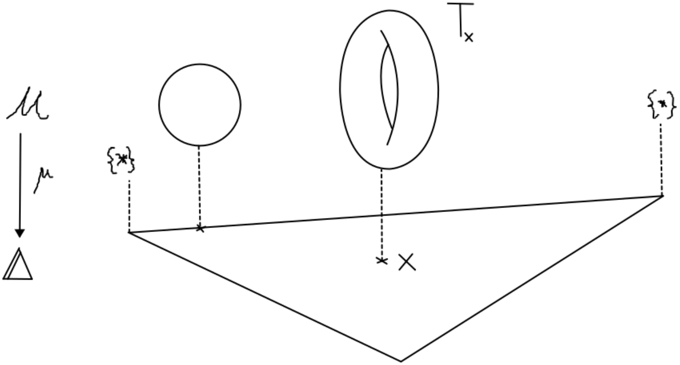

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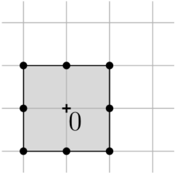
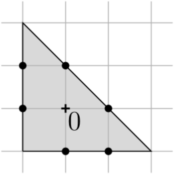
Has been checked for $n \leqslant 9$ by M. Øbro and A. Paffenholz.

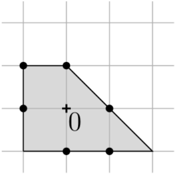
Toric fibres



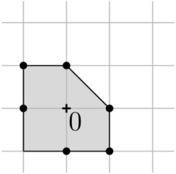
(A) $S^2 \times S^2$



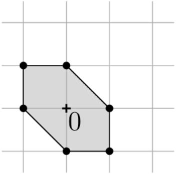
(B) $\mathbb{C}\mathrm{P}^2$



(C) $X_1 = \mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}^2}$



(D) $X_2 = \mathbb{C}\mathrm{P}^2 \# 2\overline{\mathbb{C}\mathrm{P}^2}$



(E) $X_3 = \mathbb{C}\mathrm{P}^2 \# 3\overline{\mathbb{C}\mathrm{P}^2}$

Toric fibres

Theorem: (P. A. Smith '39)

Let $F \subset M$ be the fixed point set of a smooth involution, then

1) $\chi(F) \equiv \chi(M) \pmod 2$
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This excludes $\mathbb{C}P^2$ and $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$ from having real tori already at the topological level.

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The converse is also true! Joint work with J. Kim and J. Moon.

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Whenever $\Delta = -\Delta$, then the Chekanov tori are, however, the fixed point set of a smooth involution.

Sketch of proof

Method: Versal deformations (Chekanov '96; Chekanov–Schlenk '10/'15) Elementary in the sense that the only "hard" result used is the computation of displacement energy of product tori in $\mathbb{C}^n$.

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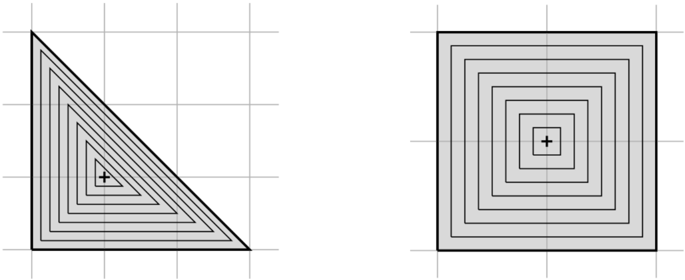

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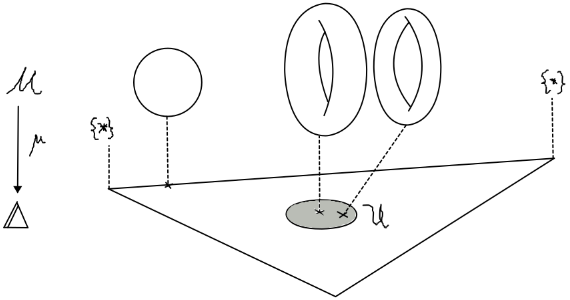

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4) Many Lagrangian neighbours of exotic tori are toric fibres $\Rightarrow$ Theorem B. (One can also distinguish Vianna tori in this way (B.–Chekanov–Schlenk) and prove that they are not real.)

Thank you!

Displacement energy

Definition:

Let $A \subset (M, \omega)$ be a subset. The **displacement energy** of A is defined by

$$e(A) = \inf \left\{ \|H\| \mid H \text{ Hamiltonian with } \varphi_H^1(A) \cap A = \varnothing \right\},$$

where $\| \cdot \|$ is the **Hofer norm** defined by

$$\|H\| = \int_0^1 \left(\max_{p \in M} H_t(p) - \min_{p \in M} H_t(p) \right) dt$$

Example: Let $S^1(a) \subset \mathbb{C}$ be the circle enclosing area a, then

$$e(S^1(a)) = a.$$