Real Lagrangian Tori in toric symplectic manifolds

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Symplectic Zoominar
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**Definition:**

\[\sigma : M \rightarrow M\] is an **antisymplectic involution** if

- \(\sigma \circ \sigma = id\);
- \(\sigma^*\omega = -\omega\).
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- \(\sigma \circ \sigma = id\); 
- \(\sigma^* \omega = -\omega\).

Its fixed point set \(\text{Fix} \sigma\) is Lagrangian (whenever non-empty).
Definition:

A Lagrangian \( L \subset (M, \omega) \) is called \textbf{real} if there is an antisymplectic involution \( \sigma \) such that \( \text{Fix} \, \sigma = L \).
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A Lagrangian $L \subset (M, \omega)$ is called real if there is an antisymplectic involution $\sigma$ such that $\text{Fix} \, \sigma = L$.

If $L = \text{Fix} \, \sigma$ and $\varphi \in \text{Symp}(M, \omega)$ then $\varphi(L) = \text{Fix}(\varphi \sigma \varphi^{-1})$. The notion is invariant under symplectomorphisms.
Real Lagrangians

Let $L \subset (M, \omega)$ be a Lagrangian.

**Main Question:**

Is $L$ real?
Examples

1) In \((\mathbb{C}^n,\omega_0)\):
   \[ L = \mathbb{R}^n \] is real. Product tori \(L = T(a_1, \ldots, a_n)\) are not real.

2) The zero section in \((T^*Q, \omega = d\lambda)\) is real.

3) Real projective space \(\mathbb{R}P^n\) in \((\mathbb{C}P^n, \omega_{FS})\) is real.
   (This example generalizes to all toric manifolds.)
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Examples

The equator in \((S^2, \omega)\) is real. Other circles of constant height are not real.

![Diagram showing the equator in \(S^2\) and other circles of constant height]

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In general: If \((M, \omega)\) is monotone and \(L\) is real, then \(L\) is monotone.
From now on: \((M^{2n}, \omega)\) toric monotone symplectic manifold, i.e. there is a moment map \(\mu : M \to \mathbb{R}^n\) which generates an effective Hamiltonian \(T^n\)-action on \(M\).
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Conjecturally,

$$M \text{ monotone } \Rightarrow \Delta \text{ has property } FS.$$ 

Has been checked for $n \leq 9$ by M. Øbro and A. Paffenholz.
Toric fibres

(A) $S^2 \times S^2$

(B) $\mathbb{CP}^2$

(C) $X_1 = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$

(D) $X_2 = \mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$

(E) $X_3 = \mathbb{CP}^2 \# 3\overline{\mathbb{CP}^2}$
Toric fibres

**Theorem:** (P. A. Smith '39)

Let $F \subset M$ be the fixed point set of a smooth involution, then

1) $\chi(F) \equiv \chi(M) \pmod{2}$

2) $\dim H_*(F, \mathbb{Z}_2) \leq \dim H_*(M, \mathbb{Z}_2)$
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**Theorem:** (P. A. Smith '39)

Let \( F \subset M \) be the fixed point set of a smooth involution, then

1) \( \chi(F) \equiv \chi(M) \pmod{2} \)

2) \( \text{dim} \, H^*(F, \mathbb{Z}_2) \leq \text{dim} \, H^*(M, \mathbb{Z}_2) \)

This excludes \( \mathbb{C}P^2 \) and \( \mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2} \) from having real tori already at the topological level.
Theorem A: (B.)

If the central fibre $T_0$ is real, then $\Delta$ is centrally symmetric, i.e. $\Delta = -\Delta$. 

Joint work with J. Kim and J. Moon.
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The converse is also true! Joint work with J. Kim and J. Moon.
Exotic tori

What about exotic tori?

Theorem: (J. Kim '19)

The Chekanov torus in $S^2 \times S^2$ is not real.

Theorem: (J. Kim '20)

If $T \subset S^2 \times S^2$ is real, then it is Hamiltonian isotopic to the Clifford torus.

Theorem B: (B. '20)

There is an exotic Chekanov torus in every toric monotone symplectic manifold and it is not real.

Whenever $\Delta = -\Delta$, then the Chekanov tori are, however, the fixed point set of a smooth involution.
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Sketch of proof

Method: Versal deformations (Chekanov ’96; Chekanov–Schlenk ’10/’15) Elementary in the sense that the only ”hard” result used is the computation of displacement energy of product tori in $\mathbb{C}^n$. 

1) Determine the displacement energy of toric fibres. (Using Property FS, McDuff’s probes and Symplectic reduction) 

$e(T \times x) = \text{dist}(x, \partial \Delta)$
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(A) $M = \mathbb{CP}^2$

(B) $M = S^2 \times S^2$
Sketch of proof

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S_{T_0} : \mathcal{U} \to \mathbb{R} \cup \{\infty\}; \quad x \mapsto e(T_x)
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3) Antisymplectic involutions preserve displacement energy and hence obtain

**Lemma:**

If $L$ is real, then its displacement energy germ satisfies

$$S_L \circ (-id) = S_L.$$ 

$\Rightarrow$ Theorem A
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If $L$ is real, then its displacement energy germ satisfies

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$\Rightarrow$ Theorem A

4) Many Lagrangian neighbours of exotic tori are toric fibres $\Rightarrow$ Theorem B. (One can also distinguish Vianna tori in this way (B.–Chekanov–Schlenk) and prove that they are not real.)
Thank you!
**Definition:**

Let $A \subset (M, \omega)$ be a subset. The **displacement energy** of $A$ is defined by

$$e(A) = \inf \{ \|H\| \mid H \text{ Hamiltonian with } \varphi^1_H(A) \cap A = \emptyset \},$$

where $\| \cdot \|$ is the **Hofer norm** defined by

$$\|H\| = \int_0^1 \left( \max_{p \in M} H_t(p) - \min_{p \in M} H_t(p) \right) dt$$

Example: Let $S^1(a) \subset \mathbb{C}$ be the circle enclosing area $a$, then

$$e(S^1(a)) = a.$$